Università degli Studi del Molise
Facoltà di Economia
Dipartimento di Scienze Economiche, Gestionali e Sociali
Via De Sanctis, I-86100 Campobasso (Italy)

ECONOMICS & STATISTICS DISCUSSION PAPER
No. 27/05

Asymptotic convergence of weighted random matrices: nonparametric cointegration analysis for I(2) processes

by

Roy Cerqueti
University "La Sapienza", Rome, Faculty of Economics, Dept. of Mathematics

and

Mauro Costantini
University "La Sapienza", Rome, Faculty of Economics, Dept. of Public Economics
and
ISAE, Rome

Asymptotic convergence of weighted random matrices: nonparametric cointegration analysis for I(2) processes

Roy Cerqueti
University "La Sapienza" Rome,
Faculty of Economics, Department of Mathematics
e.mail: roy.cerqueti@uniroma1.it

Mauro Costantini
University "La Sapienza" Rome
Faculty of Economics, Department of Public Economics,
Institute for Studies and Economic Analysis (ISAE)
e.mail: mcostan@dep.eco.uniroma1.it

Abstract

The aim of this paper is to provide a new perspective on the nonparametric co-integration analysis for integrated processes of the second order. Our analysis focus on a pair of random matrices related to such integrated process. Such matrices are constructed by introducing some weight functions. Under asymptotic conditions on such weights, convergence results in distribution are obtained. Therefore, a generalized eigenvalue problem is solved. Differential equations and stochastic calculus theory are used.

Keywords: *Co-integration, Nonparametric, Differential equations, Asymptotic properties*

JEL Classification: *C14, C32, C65*

1 Introduction

The importance of notion of co-integration introduced by Engle and Granger (1987) stimulated a vast amount of theoretical and empirical works. A wide number of studies have investigated the properties of co-integrated system of order one, I(1), (Engle and Yoo, 1987; Johansen, 1988, 1991; Johansen and Juselius, 1994; Stock and Watson, 1988; Phillips, 1991). Some economic series like prices, wages, money balances etc., seems to be more smooth and more changing than what is usually observed for $I(1)$ variables. Such process are called I(2). Co-integration among I(2) process has been recently analyzed in a growing number of theoretical and empirical studies (Johansen, 1995; Paruolo, 1996; Jorgensen *et al.*, 1996; Rahbek *et al.*, 1999; Banerjee *et al.*, 2001; Nielsen, 2002; Kongsted, 2003; Kongsted and Nielsen, 2004).

Johansen (1995) discusses inference for I(2) variables in a VAR model. A two reduced rank regressions estimation procedure is proposed and the asymptotic distribution of the estimators is provided. A multivariate test for the existence of I(2) variables is applied to U.K. and foreign price and interest rates as well as the exchange rate. Paruolo (1996) proposed estimators of the number of common components integrated of a given order in a VAR system. The analysis is based on the Johansen (1995) approach and allows for a determinist component in a VAR system. A joint test for the presence of a linear trend is presented. Rahbek *et al.* define a VAR model for I(2) process which allows for trend-stationary components and restricts the determinist part of the process to be at most linear. A two step statistical analysis of the model. An application for UK monetary data illustrates the approach proposed. Banerjee it et al. empirically study the proposition that inflation and murk-up are related in the long run in the sense proposed by Engel and Granger (1987) and that higher inflation is associated with a lower murk up and viceversa in a I(2) system. The findings show that levels of prices and costs are characterized as integrated process of order 2 and that a linear combination of the levels cointegrated with price inflation. In addition, a long run-relationship where higher inflation is associated with a lower markup and viceversa is found. Nielsen (2002) studies the long-run and short-run structure in the price and quantity formation of Danish manufactured exports by applying a multivariate cointegration model for I(2) variables. The data evidence a level shift in the Danish Market share. To take into account this effect, Nielsen includes a step dummy in the model and restrict it to allow for level shifts in all directions.

Kongsted (2003) developed a sequential procedure for testing the hypothesis that a common second order stochastic trends loads into a set of variables in known proportions. The procedure is applied to the analysis of small import price determination for the Danish data over the period 1975-1995.

Kongsted and Nielsen (2004) provide a general and formal characterization of the partly differencing approach. Specifically, they derive the properties of a transformed vector process obtained by partly differencing an I(2) process. The transformation eliminates the I(2) trends while retaining a possible cointegrating relationship between the variables.

This paper proposes an extension of the Bierens (1997) nonparametric approach to the $I(2)$ process. Bierens (1997) proposes new consistent cointegration tests and estimators of a basis of the space co-integrating vectors that do not depend on the specification of the data-generating process. The tests proposed are conducted analogously to Johansen's (1988, 1991) tests, inclusive of the test for parametric restrictions on the co-integrating vectors.

The analysis is carried out on the construction of two random matrices. To this end, we define a collection of weight functions involving the data generating process. By imposing asymptotic conditions on such weights, convergence results for the stochastic matrices are obtained.

The existence of such weights is assured by solving a differential equation and their functional shape is provided. Thus, a generalized eigenvalue problem is solved and nonparametric co-integration results for $I(2)$ process are given. [1]

The paper is organized as follows. Section 2 describes the data generating process. In section 3 the random matrices are defined and their asymptotic behavior is studied.

In section 4, non parametric results via convergence theorems are obtained. Section 5 concludes.

2 Preliminary assumptions on the $I(2)$ process

In this section we analyze the data generating process. We provide several definitions, assumptions and results in order to let this work be self-containing. We start from the definition of a multivariate integrated process of order $d \in \mathbf{N}$ with drift.

Definition 1 *Given $p, d \in \mathbf{N}$, Y_t is a discrete time p-variate integrated process of order d with drift $\mu \in \mathbf{R}^p$ if and only if*

$$Y_t = \mu + \nabla^{-d}\epsilon_t = \mu + (1-L)^{-d}\epsilon_t, \tag{1}$$

[1] The same problem has been addressed in Cerqueti and Costantini (2005). A different approach has been used to obtain nonparametric convergence results for $I(d)$ processes.

with $Y_t = (Y_t^1, \dots, Y_t^p)$*,* L *is the lag difference operator, i.e.* $L\epsilon_t := \epsilon_{t-1}$*, and* $\epsilon_t = (\epsilon_t^1, \dots, \epsilon_t^p)$ *is a zero mean stationary process.*

We denote Y an integrate process of order 2 as $Y_t \sim I(2)$ (the presence of the drift will be omitted in the notation).

We don't lose of generality assuming $Y_0 = 0$, and we make this assumption for the rest of the paper. In fact, it is easy to verify that for each $d \in \mathbf{N}$

$$Y_t \sim I(d) \;\;\Rightarrow\;\; Y_t - Y_0 \sim I(d). \tag{2}$$

Furthermore, denoting the difference operator $\Delta := 1 - L$, we have

$$Y_t \sim I(2) \;\;\Rightarrow\;\; \Delta Y_t = Y_t - Y_{t-1} \sim I(1); \tag{3}$$

$$Y_t = \Delta Y_t + \Delta Y_{t-1} + \ldots + \Delta Y_1. \tag{4}$$

By (3) and (4), we have that the t-th realization of an $I(2)$ process can be viewed as the sum of the first t realizations of an $I(1)$ process.

Let us now describe the data generating process. We state some conditions on the zero-mean process ϵ.

We assume that the Wold decomposition theorem for the process ϵ hold. Due to this fact and fixed $t = 1, \dots, n$, we can write

$$\epsilon_t = \sum_{j=0}^{\infty} C_j \rho_{t-j} =: C(L)\rho_t, \tag{5}$$

where ρ_t is a p-variate stationary white noise process and $C(L)$ is a p-squared matrix of lag polynomials in the operator L.

We state now some assumptions on the process. The first one concerns the matrix $C(L)$.

Assumption 2 *The process ϵ_t can be written as (5), where ρ_t are i.i.d. zero mean p-variate gaussian variables with variance I_p, and there exist $C_1(L)$ and $C_2(L)$, p-squared matrices of lag polynomials in the lag operator L such that all the roots of* $\det C_1(L)$ *are outside the complex unit circle and $C(L) = C_1(L)^{-1}C_2(L)$.*

$L = 1$ is a root of the lag polynomial equation $C(L) - C(1) = 0$. Therefore, there exists a lag polynomial D defined as

$$D(L) = \sum_{k=0}^{\infty} D_k L^k$$

such that $C(L) - C(1) = (1 - L)D(L)$.

We can write

$$\epsilon_t = C(1)\rho_t + D(L)(1 - L)\rho_t. \tag{6}$$

Let us define $\zeta_t := D(L)\rho_t$. Then, substituting ζ_t into (6), we get

$$\epsilon_t = C(1)\rho_t + \zeta_t - \zeta_{t-1}. \tag{7}$$

(7) implies that, given $Y_t \sim I(2)$, by recursive calculation we obtain

$$\Delta Y_t = \Delta Y_0 + \zeta_t - \zeta_0 + \mu t + C(1)\sum_{j=1}^{t} \rho_j. \tag{8}$$

If $\mathrm{rank}(C(1)) = p - r < p$, then the process ΔY_t is co-integrated with r linear independent co-integrating vectors $\gamma_1, \dots, \gamma_r$.

Remark 3 *By Assumption 2, we get that $C(L)\rho_t$ and $D(L)\rho_t$ are well-defined stationary processes and the series*

$$\sum_{k=0}^{\infty} C_k, \qquad \sum_{k=0}^{\infty} C_k C_k^T, \qquad \sum_{k=0}^{\infty} D_k, \qquad \sum_{k=0}^{\infty} D_k D_k^T$$

converge.

Assumption 4 *Let us consider R_r the matrix of the eigenvectors of $C(1)C(1)^T$ corresponding to the r zero eigenvalues. Then the matrix $R_r^T D(1)D(1)^T R_r$ is nonsingular.*

We formalize now the assumption related to an orthogonality condition between the eigenvectors matrix and the drift vector.

Assumption 5

$$R_r^T \mu = 0.$$

The last assumption is trivial in the case of $\mu = 0$, in absence of drift.

3 The stochastic matrices and their asymptotic behavior

In this section we provide the tools used to achieve nonparametric results. A pair of stochastic matrices related to the data generating process are defined. Let us introduce an integer $m \geq p$, an index $k = 1, \ldots, m$ and the real functions F_k and G_k.

We define the stochastic matrices as follows.

$$A_m = \sum_{k=1}^{m} a_{n,k} a_{n,k}^T; \tag{9}$$

$$B_m = \sum_{k=1}^{m} b_{n,k} b_{n,k}^T, \tag{10}$$

with

$$a_{n,k} = \frac{M_n^{Y,\Delta Y}/\sqrt{n}}{\sqrt{\int \int F_k(x) F_k(y) \min\{x, y\} \mathrm{d}x \mathrm{d}y}} \tag{11}$$

and

$$b_{n,k} = \frac{\sqrt{n} M_n^{\Delta^2 Y}}{\sqrt{\int F_k(x)^2 \mathrm{d}x}}, \tag{12}$$

where

$$M_n^{Y,\Delta Y} = \frac{1}{n}\sum_{t=1}^{n} G_k(t/n)Y_t + \frac{1}{n}\sum_{t=1}^{n} F_k(t/n)\Delta Y_t; \tag{13}$$

$$M_n^{\Delta^2 Y} = \frac{1}{n}\sum_{t=1}^{n} F_k(t/n)\Delta^2 Y_t. \tag{14}$$

Remark 6 *The introduction of a pair of stochastic matrices allows for a distinguishing between stationary and nonstationary part of the data generating process. A_m and B_m are related, respectively, to the non stationary part of the process and to the stationary one. This fact is implied by the presence in A_m of Y_t and ΔY_t, and in B_m of the second difference $\Delta^2 Y_t$.*

Now we prove an important convergence result.

Theorem 7 *Assume that Assumption 2 and the following properties for the functions F_k and G_k hold.*

$$\lim_{n\to+\infty} n \cdot \max_{1\le t\le n} \left| \frac{t(t+1)}{2} G_k(t/n) - tF_k(t/n) \right| = 0; \tag{15}$$

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} F_k(t/n) = o(1); \tag{16}$$

$$\frac{1}{n\sqrt{n}}\sum_{t=1}^{n} tF_k(t/n) = o(1); \tag{17}$$

$$\int\int F_i(x)F_j(y)\min\{x,y\}\mathrm{d}x\mathrm{d}y = 0, \qquad i \neq j; \tag{18}$$

$$\int F_i(x)\int_0^x F_j(y)\mathrm{d}x\mathrm{d}y = 0, \qquad i \neq j; \tag{19}$$

$$\int F_i(x)F_j(x)\mathrm{d}x = 0, \qquad i \neq j. \tag{20}$$

Then we have the following convergence in distribution:

$$\begin{pmatrix} M_n^{Y,\Delta Y}(F_k, G_k)/\sqrt{n} \\ \\ M_n^{\Delta^2 Y}(F_k)\sqrt{n} \end{pmatrix} \to \begin{pmatrix} C(1)\int F_k(x)W(x)\mathrm{d}x \\ \\ C(1)(F_k(1)W(1) - \int f_k(x)W(x)\mathrm{d}x) \end{pmatrix}, \tag{21}$$

where W is a p-variate standard Wiener process and f_k is the derivative of F_k.

Proof. By (2), (3) and (4), we can write

$$M_n^{Y,\Delta Y} = \frac{1}{n}\sum_{t=1}^{n} G_k(t/n)\Big[\sum_{j=0}^{t-1}(j+1)\epsilon_{t-j}\Big] + \frac{1}{n}\sum_{t=1}^{n} F_k(t/n)\Delta Y_t. \qquad (22)$$

Using the definition of the p-variate normal random variable ϵ_t, we get

$$\Delta Y_t = \sum_{j=1}^{t} \epsilon_j + \mu t \sim t\epsilon_1 + \mu t. \qquad (23)$$

Therefore, by (23), (22) can be rewritten as

$$\frac{1}{\sqrt{n}} \cdot M_n{}^{Y,\Delta Y} = \frac{\epsilon_1}{n\sqrt{n}} \cdot \Big[\sum_{t=1}^{n} \Big(G_k(t/n)\frac{t(t+1)}{2} + tF_k(t/n)\Big)\Big] + \frac{\mu}{n\sqrt{n}}\sum_{t=1}^{n} tF_k(t/n). \qquad (24)$$

By hypothesis (15) it results, for each $t = 1, \dots, n$,

$$G_k(t/n)\frac{t(t+1)}{2} \sim tF_k(t/n), \qquad (25)$$

as $n \to +\infty$.

Then, by Proposition 3, hypotheses (16)-(20) and by the theory on standard Wiener measure calculus, we prove the theorem.[2] ■

The weight functions defining the matrices A_m and B_m exist. Moreover, it is possible to provide the functional shape of F_k and G_k, in order to be the conditions (15)-(20) fulfilled. First of all, we can easily prove that the weights F_k are trigonometric functions. We formalize this fact in the following proposition.

Proposition 8 *Let us consider, for each k,*

$$F_k(x) = cos(2k\pi x).$$

Then the conditions (16), (17), (18), (19) and (20) hold.

[2]For the details on the Wiener measure calculus see Bierens (1994), Billingsley (1968) and Phillips (1987).

From Proposition (8), we obtain that the weight G_k exist. The following result hold:

Proposition 9 *Assume that, for each k, F_k is as in Proposition 8. Moreover, assume that*

$$G_k(t/n) = \frac{2}{n} \cdot \frac{k\pi(t/n)+1}{t/n+1/n} + \gamma, \qquad \gamma \in \mathbf{R}. \tag{26}$$

and

$$n \cdot \max\Big\{\Big|\frac{1}{2}G_k(1/n) - cos(\frac{2k\pi}{n})\Big|, \Big|\frac{n(n+1)}{2}G_k(1) - n\Big|\Big\} = o(\frac{1}{n}). \tag{27}$$

Then the condition (15) holds.

Proof. We can replace in (15) the functions F_k with their explicit expression provided in Proposition 8. We get

$$\lim_{n\to+\infty} n \cdot \max_{1\le t\le n}\Big|\frac{t(t+1)}{2}G_k(t/n) - tcos(\frac{2k\pi t}{n})\Big| = 0. \tag{28}$$

Then there exists $\epsilon > 0$ such that

$$\max_{1\le t\le n}\Big|\frac{t(t+1)}{2}G_k(t/n) - tcos(\frac{2k\pi t}{n})\Big| \sim \frac{1}{n^{1+\epsilon}}. \tag{29}$$

Let us define

$$f(t) := \frac{t(t+1)}{2}G_k(t/n) - tcos(\frac{2k\pi t}{n}). \tag{30}$$

The weight functions G_k can be found by imposing a growth condition on the auxiliary function f and an asymptotic condition on the extremal values $f(1),\ f(n)$. At this aim, we provide an estimate of the first derivative of f.

$$f'(t) := \frac{2t+1}{2}G_k(t/n) + \frac{t(t+1)}{2n}\frac{\partial}{\partial t}G_k(t/n) - cos(\frac{2k\pi t}{n}) + \frac{2k\pi t}{n}sin(\frac{2k\pi t}{n}) >$$

$$> \frac{2t+1}{2}G_k(t/n) + \frac{t(t+1)}{2n}\frac{\partial}{\partial t}G_k(t/n) - 1 - \frac{2k\pi t}{n} = 0. \tag{31}$$

The last term of (31) is a differential equation on G_k. Its solution is

$$G_k(t/n) = \frac{2}{n} \cdot \frac{k\pi(t/n) + 1}{t/n + 1/n} + \gamma, \qquad \gamma \in \mathbf{R}.$$

Due to the fact that f is increasing with respect to t, the condition (27) implies that (15) holds. ■

4 Nonparametric results

In this section we show the implications of the choices of the functions F_k and G_k in the construction of the stochastic matrices. Substantially, we analyze the consequences of Theorem 7.

From Theorem 7 the following result holds.

Theorem 10 *Assume that the hypotheses of Theorem 7 hold. Then we have the following convergence in distribution:*

$$\begin{pmatrix} M_n^{Y,\Delta Y}(F_k, G_k)/\sqrt{n} \\ \\ M_n^{\Delta^2 Y}(F_k)\sqrt{n} \end{pmatrix} \rightarrow \begin{pmatrix} C(1)X_k\sqrt{\int\int F_k(x)F_k(y)\min\{x,y\}\mathrm{d}x\mathrm{d}y} \\ \\ C(1)Y_k\sqrt{\int F_k(x)^2\mathrm{d}x} \end{pmatrix}, \tag{32}$$

for each k, where X_k and Y_k are independent p-variate standard normally distributed random vectors such that

$$X_k = \frac{\int F_k(x)W(x)\mathrm{d}x}{\sqrt{\int\int F_k(x)F_k(y)\min\{x,y\}\mathrm{d}x\mathrm{d}y}}, \tag{33}$$

$$Y_k = \frac{F_k(1)W(1) - \int f_k(x)W(x)\mathrm{d}x}{\int F_k(x)^2\mathrm{d}x}. \tag{34}$$

By the hypotheses of the previous theorem and the Assumption 4, we have the following result.

Theorem 11 *Assume that the hypotheses of Theorem 7 hold and there exist r linear independent cointegrating vectors (thus, we get the existence of the* *matrix R_r defined as in Assumption 4) such that Assumption 5 holds. Then we have the following joint in $k = 1, \ldots, n$ convergence in distribution:*

$$\begin{pmatrix} R_r^T M_n^{Y,\Delta Y}(F_k, G_k)\sqrt{n} \\ \\ R_r^T M_n^{\Delta^2 Y}(F_k) n \end{pmatrix} \to \begin{pmatrix} R_r^T D(1) Y_k \sqrt{\int F_k(x)^2 \mathrm{d}x} \\ \\ F_k(1) R_r^T D_* Z \end{pmatrix}, \qquad (35)$$

where the Y_k's and Z are independent p-variate standard gaussian, Z independent on F_k, Y_k defined as in Theorem 10 and D_* is such that

$$\sum_{k=0}^{\infty} D_k D_k^T = D_* D_*^T.$$

Theorem 12 *Let us assume* $\mathrm{rank}C(1) = p - r$. *The following convergence in distribution results are satisfied:*

$$\begin{pmatrix} I_{p-r} & 0 \\ \\ 0 & nI_r \end{pmatrix} R^T A_m R \begin{pmatrix} I_{p-r} & 0 \\ \\ 0 & nI_r \end{pmatrix} = \begin{pmatrix} R_{p-r}^T A_m R_{p-r} & nR_{p-r}^T A_m R_r \\ \\ nR_r^T A_m R_{p-r} & n^2 R_r^T A_m R_r \end{pmatrix} \to$$

$$\to \begin{pmatrix} R_{p-r}^T C(1) \sum_{k=1}^m X_k X_k^T C(1)^T R_{p-r} & R_{p-r}^T C(1) \sum_{k=1}^m \gamma_k X_k Y_k^T D(1)^T R_r \\ \\ R_r^T D(1) \sum_{k=1}^m \gamma_k Y_k X_k^T C(1)^T R_{p-r} & R_r^T D(1) \sum_{k=1}^m \gamma_k^2 Y_k Y_k^T D(1)^T R_r \end{pmatrix} \qquad (36)$$

and

$$\begin{pmatrix} I_{p-r} & 0 \\ \\ 0 & \sqrt{n}I_r \end{pmatrix} R^T B_m R \begin{pmatrix} I_{p-r} & 0 \\ \\ 0 & \sqrt{n}I_r \end{pmatrix} = \begin{pmatrix} R_{p-r}^T B_m R_{p-r} & \sqrt{n}R_{p-r}^T B_m R_r \\ \\ \sqrt{n}R_r^T B_m R_{p-r} & nR_r^T B_m R_r \end{pmatrix} \to$$

$$\to \begin{pmatrix} R_{p-r}^T C(1) \sum_{k=1}^m Y_k Y_k^T C(1)^T R_{p-r} & R_{p-r}^T C(1) \sum_{k=1}^m \delta_k Y_k Z^T D_*^T R_r \\ \\ R_r^T D_* \sum_{k=1}^m \delta_k Z Y_k^T C(1)^T R_{p-r} & R_r^T D_* \sum_{k=1}^m \delta_k^2 Z Z^T D_*^T R_r \end{pmatrix} \qquad (37)$$

where X_j, Y_i and Z are the same defined in Theorems 10 and 11. Furthermore, the following convergence in distribution holds:

$$\frac{R^T A_m^{-1} R}{n^2} \to \begin{pmatrix} 0 & 0 \\ \\ 0 & V_{r,m}^{-1} \end{pmatrix} \qquad (38)$$

with

$$V_{r,m} = R_r^T D(1) \sum_{k=1}^{m} \gamma_k^2 Y_k Y_k^T D(1)^T R_r - \Big(R_r^T D(1) \sum_{k=1}^{m} \gamma_k Y_k X_k^T C(1)^T R_{p-r} \Big) \cdot$$

$$\cdot \Big(R_{p-r}^T C(1) \sum_{k=1}^{m} X_k X_k^T C(1)^T R_{p-r} \Big)^{-1} \cdot \Big(R_{p-r}^T C(1) \sum_{k=1}^{m} \gamma_k X_k Y_k^T D(1)^T R_r \Big).$$

By Assumption 4, we deduce immediately that the matrix $V_{r,m}$ is not singular a.s..

Let us denote

$$X_k^* = \Big(R_{p-r}^T C(1) C(1)^T R_{p-r} \Big)^{\frac{1}{2}} R_{p-r}^T C(1) X_k,$$

$$Y_k^* = \Big(R_{p-r}^T C(1) C(1)^T R_{p-r} \Big)^{\frac{1}{2}} R_{p-r}^T C(1) Y_k. \tag{39}$$

By Anderson *et al.* (1983), Bierens (1997) and using Theorem 12, we prove the following result.

Theorem 13 *Let us consider* $\hat{\lambda}_{1,m} \geq \ldots \geq \hat{\lambda}_{p,m}$ *the ordered solutions of the generalized eigenvalue problem*

$$\det\Big[A_m - \lambda (B_m + n^{-2} A_m^{-1}) \Big] = 0, \tag{40}$$

and let us consider $\lambda_{1,m} \geq \ldots \geq \lambda_{p-r,m}$ *the ordered solutions of the generalized eigenvalue problem*

$$\det\Big[\sum_{k=1}^{M} X_k^* X_k^{*T} - \lambda \sum_{k=1}^{M} Y_k^* Y_k^{*T} \Big] = 0, \tag{41}$$

where the X_i^**'s and* Y_j^**'s are i.i.d. random variables following a* $N_{p-r}(0, I_{p-r})$ *distribution. If* z_t *is co-integrated with* r *linear independent co-integrating vectors, then Assumptions I, II and III assure that we have the following convergence in distribution:*

$$(\hat{\lambda}_{1,m}, \ldots, \hat{\lambda}_{p,m}) \rightarrow (\lambda_{1,m}, \ldots, \lambda_{p-r,m}, 0, \ldots, 0)$$

Let us define now

$$Y_k^{**} = (R_r^T D(1) D(1)^T R_r)^{-\frac{1}{2}} R_r^T D(1) Y_k. \tag{42}$$

We have that $Y_k^{**} \sim N_r(0, I_r)$.

Moreover, we can write the matrix $V_{r,m}$ as

$$V_{r,m} = (R_r^T D(1) D(1)^T R_r)^{\frac{1}{2}} V_{r,m}^* (R_r^T D(1) D(1)^T R_r)^{\frac{1}{2}}, \tag{43}$$

where

$$V_{r,m}^* = \Big(\sum_{k=1}^m \gamma_k^2 Y_k^{**} Y_k^{**T}\Big) - \Big(\sum_{k=1}^m \gamma_k Y_k^{**} X_k^{*T}\Big)\Big(\sum_{k=1}^m X_k^* X_k^{*T}\Big)^{-1}\Big(\sum_{k=1}^m \gamma_k X_k^* Y_k^{**T}\Big). \tag{44}$$

Thus, we get the following result.

Theorem 14 *Let us consider $\lambda_{1,m}^* \geq \ldots \geq \lambda_{r,m}^*$ the ordered solutions of the generalized eigenvalue problem*

$$\det\Big[V_{r,m}^* - \lambda (R_r^T D(1) D(1)^T R_r)^{-1}\Big] = 0, \tag{45}$$

with $V_{r,m}^*$ defined as in (44) and the X_i^*'s and Y_j^{**}'s are i.i.d. random variables following respectively a $N_{p-r}(0, I_{p-r})$ and $N_r(0, I_r)$ distribution.
Under the hypotheses of Theorem 13, we have the following convergence in distribution

$$n^2(\hat{\lambda}_{p-r+1,m}, \ldots, \hat{\lambda}_{p,m}) \to (\lambda_{1,m}^{*2}, \ldots, \lambda_{r,m}^{*2})$$

In Theorem 14, the relationship between data generating process and stochastic matrices is provided. In this way we formalize the theoretical nonparametric approach to give an estimate of an $I(2)$ process with drift.

5 Conclusions

Co-integration among $I(2)$ process has been recently analyzed in a growing number of theoretical and empirical studies. This paper proposes a new approach for co-integration analysis of $I(2)$ process. The differential equations approach is used to provide an explicit shape of the parameters involved in the model. A generalized eigenvalue problem is solved and nonparametric convergence results are obtained.

References

Anderson, S.A., H.K. Brons & S.T. Jensen (1983) Distribution of eigenvalues in multivariate statistical analysis. *Annals of Statistics* **11**, 392-415.

Banerjee, A., L. Cockerell & B. Russel (2001) An $I(2)$ Analysis of Inflation and the Murkup. *Journal of Applied Econometrics* **16**, 221-240.

Billingsley, P. (1968) *Convergence of Probability Measure*, Chicago.

Bierens, J.H. (1997) Nonparametric cointegration analysis. *Journal of Econometrics*, **77**, 379-404.

Bierens, J. H. (1994) *Topics in Advanced Econometrics: Estimation, Testing, and Specification of Cross-Section and Time Series Models*, Cambridge: Cambridge University Press.

Cerqueti, R. & M. Costantini (2005) Generalization of a nonparametric cointegration analysis for multivariate integrated processes of an integer order. *Selected for Unit Roots and Cointegration Testing Conference, Faro, Portugal.*

Engle, R.F. & C.W.J. Granger (1987) Co-integration and error correction: Representation, estimation and testing. *Econometrica* **55**, 251-276.

Engle, R.F. & S.B. Yoo (1987) Forecasting and testing in cointegrated system. *Journal of Econometrics* **35**, 143-159.

Johansen, S. (1988) Statistical Analysis of cointegration vectors. *Journal of Economic Dynamics and Control* **12**, 231-254.

Johansen, S. (1991) Estimation and Hypothesis Testing of Cointegration Vectors in Gaussian Vector Autoregressive Models. *Econometrica* **6**, 1551-1580.

Johansen, S. (1995) A statistical analysis of cointegration for $I(2)$ variables. *Econometric Theory* **11**, 25-59.

Johansen, S. & S. Juseliues (1994) Identification of the long run and the short-run structure. An application to the ISLM model. *Journal of Econometrics* **63**, 7-36.

Kongsted, H.C. (2003) An $I(2)$ cointegration analysis of small country import price determination. *Econometrics Journal* **6**, 53-71.

Kongsted, H.C. & H.B. Nielsen (2004) Analysing $I(2)$ Systems by Transformed Vector Autoregressions. *Oxford Bulletin of Economics and Statistics* **66**, 379-397.

Nielsen, H.B. (2002) An $I(2)$ Cointegration Analysis of Price and Quantity Formation in Danish Manufactured Exports. *Oxford Bulletin of Economics and Statistics* **64**, 449-472.

Paruolo, P. (1996) On the determination of integration indices in $I(2)$ system. *Journal of Econometrics* **72**, 313-356.

Phillips, P.C.B. (1987) Time series regression with unit roots. *Econometrica* **55**, 277-302.

Phillips, P.C.B. (1991) Optimal inference in cointegrated system. *Econometrica* **59**, 283-306.

Rahbek, A., Jørgensen, C. & H.C. Kongsted (1999) Trend-stationary in the $I(2)$ cointegration model. *Journal of Econometrics* **90**, 265-289.

Stock, J.H. & M.W. Watson (1988) Testing for common trends. *Journal of the American Statistical Association* **83**, 1097-1107.