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A new family of nonexchangeable copulas for positive dependence

Roy Cerqueti and **Claudio Lupi**

Abstract

This note contributes to the development of the theory of stochastic dependence by employing the general concept of copula. In particular, it deals with the construction of a new family of non-exchangeable copulas characterizing the multivariate total positivity of order 2 (MTP2) dependence.

Keywords: Copulas, MTP2 dependence, Non-exchangeability.

1. Introduction

In copula theory, exchangeability (see, e.g., Ghiselli Ricci 2013) represents an important feature which describes the symmetry of some classes of copulas. Archimedean copulas (for some recent results and extensive surveys we refer to Schweizer and Sklar 2005; Durante, Quesada-Molina, and Sempi 2007) and semilinear ones (see, e.g., Durante 2007; Durante, Kolesárová, Mesiar, and Sempi 2008) are among the most relevant examples of exchangeable copulas. Despite its mathematical relevance, exchangeability may represent a strong requirement to be fulfilled by a set of random variables. In this respect, it is worth paying attention to the concept of Archimedean asymmetric copulas, which represent a non-exchangeable counterpart of the Archimedean ones (see Liebscher 2008, 2011).

In this note, we move from Liebscher (2008, 2011) and construct a family of copulas of asymmetric Archimedean type which is able to characterize multivariate total positivity of order 2 (MTP2). In doing this, we extend Müller and Scarsini (2005) to the case of non-exchangeability property for the considered set of dependent random variables.

The rest of the paper is organized as follows. Section 2 provides the necessary preliminaries, notation and statistical concepts. The main results are offered in Section 3. A discussion and some concluding remarks are given in Section 4.

2. Preliminaries and notation

For an easier reading of the model, we introduce the vectorial notation:

Notation 2.1. *Fix $m = 1, 2, \ldots$. The following notations are introduced:*

$$\boldsymbol{w} = (w_1, \ldots, w_m); \boldsymbol{x} = (x_1, \ldots, x_m); \boldsymbol{y} = (y_1, \ldots, y_m); \boldsymbol{u} = (u_1, \ldots, u_m).$$

We now recall the definition of the dependence concept we deal with.

Definition 2.2. *Let f be the joint density function of the m-variate random variable $\boldsymbol{w}$. The components of $\boldsymbol{w}$ are said to be MTP2 if and only if, for each $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^m$, it results:*

$$f(\boldsymbol{x}) \cdot f(\boldsymbol{y}) \geqslant f(\min\{\boldsymbol{x}, \boldsymbol{y}\}) \cdot f(\max\{\boldsymbol{x}, \boldsymbol{y}\})$$

where the min *and* max *operators have to be intended componentwise.*

Indeed, it is customary to represent the (linear) dependence among individual w's in $\boldsymbol{w}$ by a non-diagonal variance-covariance matrix $\boldsymbol{\Sigma} = (\sigma_{i,j})$ with $i, j = 1, \ldots, m$. Hence, it is natural to guess the existence of a relationship between the value of the covariances and the subsistence of MTP2. In this respect, it is useful to recall a further definition of dependence among random variables:

Definition 2.3. *The random components of $\boldsymbol{w}$ are associated if*

$$\mathrm{Cov}\left[g(\boldsymbol{w}), h(\boldsymbol{w})\right] \geqslant 0,$$

for any coordinatewise nondecreasing functions g and h for which this covariance exists.

A classical result states that random variables which are MTP2 are also associated. Therefore, by using Definition 2.3 with $g(\boldsymbol{w}) = w_i$ and $h(\boldsymbol{w}) = w_j$, we can also say that if $\{w_1, \ldots, w_m\}$ are MTP2, then $\sigma_{i,j} \geqslant 0$, for each $i, j = 1, \ldots, m$. This fact implies that if there exists a couple (w_i, w_j), with $i \neq j$ and $i, j = 1, \ldots, m$, such that $\sigma_{i,j} < 0$, then $\{w_1, \ldots, w_m\}$ are not MTP2.

A rather general way to capture the stochastic dependence structure among random variables is the introduction of the concept of *multivariate copula* (or, simply, *copula*). We now provide the definition of this concept, referring the reader to Nelsen (2006) for a detailed discussion:

Definition 2.4. *The function $C : [0,1]^m \to [0,1]$ is a copula if and only if:*
(C2.4.i) $C(\boldsymbol{u}) = 0$ if $u_1 \times \ldots \times u_m = 0$;
(C2.4.ii) $C(\boldsymbol{u}) = u_{\bar{k}}$ if $u_k = 1$, for each $k \neq \bar{k}$;
(C2.4.iii) Given the m-dimensional rectangle $[a_1, b_1] \times \ldots \times [a_m, b_m] \subseteq [0,1]^m$, then

$$\sum_{i_1=1}^{2} \ldots \sum_{i_m=1}^{2} (-1)^{i_1+\ldots+i_m} C(u_{1,i_1}, \ldots, u_{m,i_m}) \geqslant 0,$$

where $u_{j,1} = a_j$ and $u_{j,2} = b_j$.

Sklar's Theorem (Sklar 1959) highlights how multivariate copulas introduced in Definition 2.4 model the dependence structure among random variables. For the sake of completeness, we report here the enunciation of the theorem, adapted to our case:

Theorem 2.5 (Sklar, 1959)**.** *Let F be the joint distribution function of the m-ple $\boldsymbol{w}$. Define the margins as $F_1, \ldots, F_m$. Then there exists a m-variate copula C such that, for each $\boldsymbol{x} \in \mathbb{R}^m$,*

$$F(\boldsymbol{x}) = C(F_1(x_1), \ldots, F_m(x_m)). \tag{1}$$

If the margins $F_1, \ldots, F_m$ are continuous, then the copula C is unique. Conversely, if C is a m-variate copula and $F_1, \ldots, F_m$ are distribution functions, then the function F defined in (1) is a m-dimensional distribution function with margins $F_1, \ldots, F_m$.

Theorem 2.5 points out that, given a set of random variables, the relationship between joint and marginal distributions is stated through copulas. An important feature of some classes of copulas which turns out to be relevant in our context is *exchangeability* (an accurate characterization of exchangeable copulas can be found in Ghiselli Ricci 2013):

Definition 2.6. *The copula* $C : [0,1]^m \to [0,1]$ *is exchangeable if, for each* $\boldsymbol{u} \in [0,1]^m$ *and a permutation* ϱ *of* $\{1,\dots,m\}$*, one has:*

$$C(\boldsymbol{u}) = C(u_{\varrho(1)},\dots,u_{\varrho(m)}).$$

When Definition 2.6 is not satisfied, then copula C is said to be nonexchangeable.

3. Main result

The general family of nonexchangeable copulas we deal with are of the Archimedean type. We formalize it in the following:

Definition 3.1. *Fix* $J \in \mathbb{N}$ *and a set of* $m \times J$ *functions*

$$h_{jk} : [0,1] \to [0,1], \qquad j = 1,\dots,J; \;\; k = 1,\dots,m \tag{2}$$

such that:
(C3.1.i) h_{jk} *is differentiable in* $(0,1)$ *and strictly increasing in* $[0,1]$*, for each* j,k*;*
(C3.1.ii) $h_{jk}(0) = 0$ *and* $h_{jk}(1) = 1$*, for each* j,k*;*
(C3.1.iii) $\frac{1}{J}\sum_{j=1}^{J} h_{jk}(x) = x$*, for each* $k = 1,\dots,m$ *and* $x \in [0,1]$*.*
Moreover, define

$$\psi : [0,1] \to [0,1] \tag{3}$$

such that:
(C3.1.iv) ψ *is* $m+2$ *times differentiable in* $(0,1)$*;*
(C3.1.v) $\psi^{(i)} > 0$ *in* $(0,1)$*, for* $i = 1,\dots,m$*;*
(C3.1.vi) $\psi(0) = 0$ *and* $\psi(1) = 1$*.*
We define an (Archimedean) asymmetric copula as $C_{AS}^{\psi} : [0,1]^m \to [0,1]$ *such that:*

$$C_{AS}^{\psi}(\boldsymbol{u}) = \psi^{-1}\left(\frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h_{jk}(\psi(u_k))\right). \tag{4}$$

Copula (4) has been first introduced and explored in Liebscher (2008, 2011). It is worth noting that, as far as the copula's definition is concerned, conditions (C3.1.iv) and (C3.1.v) could be weakened. However, our mildly stronger version is required to prove the following general result:

Theorem 3.2. *Assume that:*

$$\left(\psi^{-1}\right)^{(m+2)}\left(\psi^{-1}\right)^{(m)} - \left[\left(\psi^{-1}\right)^{(m+1)}\right]^2 \geqslant 0, \qquad in (0,1) \tag{5}$$

and

$$\psi(u_k) = h_{jk}^{-1}\left(\frac{e^{u_k}-1}{e-1}\right), \qquad j = 1,\dots,J; \;\; k = 1,\dots,m. \tag{6}$$

If the dependence among the components of the m-variate random vector $\boldsymbol{w}$ *is described by copula (4), then* $\boldsymbol{w}$ *is MTP2.*

Proof. In virtue of Müller and Scarsini (2005), it is sufficient to check that the density f of C^{ψ}_{AS} is log-supermodular, that is equivalent to say that

$$\log(f(\boldsymbol{u})) := \log\left(\frac{\partial^m}{\partial u_1 \dots \partial u_m} C^{\psi}_{AS}(\boldsymbol{u})\right) \tag{7}$$

is supermodular.
By (4) we have

$$\begin{aligned} f(\boldsymbol{u}) &= \frac{\partial^m}{\partial u_1 \dots \partial u_m} C^{\psi}_{AS}(\boldsymbol{u}) \\ &= \left(\psi^{-1}\right)^{(m)} \left(\frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h_{jk}(\psi(u_k))\right) \times \\ &\quad \times \frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h'_{jk}(\psi(u_k))\psi'(u_k)\,. \end{aligned} \tag{8}$$

By (8) we can write

$$\begin{aligned} \log(f(\boldsymbol{u})) &= \log\left[\left(\psi^{-1}\right)^{(m)} \left(\frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h_{jk}(\psi(u_k))\right)\right] + \\ &\quad + \log\left[\frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h'_{jk}(\psi(u_k))\psi'(u_k)\right] \\ &=: A(\boldsymbol{u}) + B(\boldsymbol{u})\,, \end{aligned} \tag{9}$$

where the terms $A(\cdot)$ and $B(\cdot)$ are an intuitive shorthand for the two $\log[\cdot]$ terms.
The supermodularity of $\log(f(\boldsymbol{u}))$ is equivalent to the following condition:

$$\frac{\partial^2}{\partial u_{k_1} \partial u_{k_2}} [A(\boldsymbol{u}) + B(\boldsymbol{u})] \geqslant 0, \tag{10}$$

for each $k_1, k_2 \in \{1, \dots, m\}$, and $(u_1, \dots, u_m) \in [0,1]^m$. For an easier notation, we will pose hereafter

$$\xi := \frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h_{jk}(\psi(u_k)). \tag{11}$$

We analyze the terms $A(\cdot)$ and $B(\cdot)$ separately.
First notice that

$$\frac{\partial A(\boldsymbol{u})}{\partial u_{k_1}} = \frac{\left(\psi^{-1}\right)^{(m+1)}(\xi)}{\left(\psi^{-1}\right)^{(m)}(\xi)} \times \frac{1}{J}\sum_{j=1}^{J} h'_{jk_1}(\psi(u_{k_1}))\psi'(u_{k_1}) \prod_{k\neq k_1}\left[h_{jk}(\psi(u_k))\right]$$

and

$$\begin{aligned}\frac{\partial^2 A(\boldsymbol{u})}{\partial u_{k_1}\partial u_{k_2}} &= \frac{1}{\left\{\left(\psi^{-1}\right)^{(m)}\right\}^2}\times\left\{\left(\frac{1}{J}\sum_{j=1}^{J}h'_{jk_1}(\psi(u_{k_1}))\psi'(u_{k_1})\prod_{k\neq k_1}\left[h_{jk}(\psi(u_k))\right]\right)\times\right.\\ &\times\left(\frac{1}{J}\sum_{j=1}^{J}h'_{jk_1}(\psi(u_{k_2}))\psi'(u_{k_2})\prod_{k\neq k_2}\left[h_{jk}(\psi(u_k))\right]\right)\times\\ &\times\left[\left(\psi^{-1}\right)^{(m+2)}(\xi)\times\left(\psi^{-1}\right)^{(m)}(\xi)-\left[\left(\psi^{-1}\right)^{(m+1)}(\xi)\right]^2\right]+\\ &+\left(\psi^{-1}\right)^{(m)}(\xi)\times\left(\psi^{-1}\right)^{(m+1)}(\xi)\times\\ &\times\frac{1}{J}\sum_{j=1}^{J}h'_{jk_1}(\psi(u_{k_1}))\psi'(u_{k_1})h'_{jk_2}(\psi(u_{k_2}))\psi'(u_{k_2})\times\\ &\left.\times\prod_{k\neq k_1,k_2}\left[h_{jk}(\psi(u_k))\right]\right\}. \end{aligned} \tag{12}$$

Hence, under Condition (C3.1.v) and hypothesis (5), we have

$$\frac{\partial^2 A(\boldsymbol{u})}{\partial u_{k_1}\partial u_{k_2}}\geqslant 0. \tag{13}$$

Let us now turn to $B(\cdot)$:

$$\begin{aligned}\frac{\partial B(\boldsymbol{u})}{\partial u_{k_1}} &= \frac{1}{\sum_{j=1}^{J}\prod_{k=1}^{m}h'_{jk}(\psi(u_k))\psi'(u_k)}\times\\ &\times\left\{\sum_{j=1}^{J}\left[h''_{jk_1}(\psi(u_{k_1}))\left(\psi'(u_{k_1})\right)^2+h'_{jk_1}(\psi(u_{k_1}))\psi''(u_{k_1})\right]\times\right.\\ &\left.\times\prod_{k\neq k_1}h'_{jk}(\psi(u_k))\psi'(u_k)\right\}, \end{aligned} \tag{14}$$

hence we have:

$$\begin{aligned}
\frac{\partial^2 B(\boldsymbol{u})}{\partial u_{k_1}\partial u_{k_2}} &= \frac{1}{\left[\sum_{j=1}^{J}\prod_{k=1}^{m} h'_{jk}(\psi(u_k))\psi'(u_k)\right]^2}\times \\
&\left\{\left[\sum_{j=1}^{J}\prod_{k\neq k_1,k_2} h'_{jk}(\psi(u_k))\psi'(u_k)\times\right.\right. \\
&\left.\times\prod_{k=k_1,k_2}[h''_{jk}(\psi(u_k))(\psi'(u_k))^2+h'_{jk}(\psi(u_k))\psi''(u_k)]\right]\times \\
&\times\sum_{j=1}^{J}\prod_{k=1}^{m} h'_{jk}(\psi(u_k))\psi'(u_k)- \\
&-\left[\sum_{j=1}^{J}\prod_{k\neq k_1} h'_{jk}(\psi(u_k))\psi'(u_k)\times\right. \\
&\left.[h''_{jk_1}(\psi(u_{k_1}))(\psi'(u_{k_1}))^2+h'_{jk_1}(\psi(u_{k_1}))\psi''(u_{k_1})]\right]\times \\
&\times\left[\sum_{j=1}^{J}\prod_{k\neq k_2} h'_{jk}(\psi(u_k))\psi'(u_k)\times\right. \\
&\left.\left.[h''_{jk_2}(\psi(u_{k_2}))(\psi'(u_{k_2}))^2+h'_{jk_2}(\psi(u_{k_2}))\psi''(u_{k_2})]\right]\right\}. \qquad (15)
\end{aligned}$$

By (15) we obtain that a sufficient condition for being $\partial^2 B_s(\boldsymbol{u})/\left(\partial u_{k_1}\partial u_{k_2}\right)=0$ is:

$$h''_{jk_2}(\psi(u_{k_2}))(\psi'(u_{k_2}))^2+h'_{jk_2}(\psi(u_{k_2}))\psi''(u_{k_2})=h'_{jk_2}(\psi(u_{k_2}))\psi'(u_{k_2}). \qquad (16)$$

We then need to solve the ODE in (16), with the initial condition given by (C3.1.vi). Note that (16) is equivalent to

$$\frac{h''_{jk_2}(\psi(u_{k_2}))\psi'(u_{k_2})}{h'_{jk_2}(\psi(u_{k_2}))}=-\frac{\psi''(u_{k_2})-\psi'(u_{k_2})}{\psi'(u_{k_2})},$$

which leads to:

$$\log(h'_{jk_2}(\psi(u_{k_2})))=-\int\frac{\psi''(u_{k_2})-\psi'(u_{k_2})}{\psi'(u_{k_2})}\mathrm{d}u_{k_2}.$$

Then

$$h'_{jk_2}(\psi(u_{k_2}))=\frac{H_1\mathrm{e}^{u_{k_2}}}{\psi'(u_{k_2})},\qquad H_1\in(0,+\infty). \qquad (17)$$

(17) can be rewritten as follows:

$$h'_{jk_2}(\psi(u_{k_2}))\cdot\psi'(u_{k_2})=H_1\mathrm{e}^{u_{k_2}},$$

which immediately leads to

$$h_{jk_2}(\psi(u_{k_2})) = H_1 e^{u_{k_2}} + H_2, \qquad H_2 \in \mathbb{R}\,.$$

Since h_{jk_2} is invertible, by imposing the initial conditions in (C3.1.vi) we obtain:

$$\psi(u_{k_2}) = h_{jk_2}^{-1}\left(\frac{e^{u_{k_2}} - 1}{e - 1}\right),$$

that is exactly condition (6) when $k = k_2$.
The result is proved, by the arbitrariness of k_2. □

4. Discussion and concluding remarks

Dealing with dependence among random variables is of paramount importance, mainly in the context of multivariate analysis. In this respect, a relevant role is played by the concept of copulas. This paper contributes to this field of research by studying the relationship between copulas and the MTP2 dependence property in the case of non-exchangeability.
A copula-based framework seems to be preferable to a setting adopting the explicit definition of MTP2. The reason is twofold:

- it is generally hard to check if the dependence structure among groups of random variables is of MTP2-type by using the formal definition. Therefore, a characterization of the MTP2 property through more affordable and useful mathematical tools is needed. The concept of copula meets this need in that it is rather general and easy to be treated;

- recently, Durante, Foschi, and Sarkoci (2010) introduced the concept of *distorted copulas*, which represents a new family of copulas obtained by applying an isomorphism to a given copula. They showed that bivariate distorted copulas retain the *total positivity of order 2* (TP2) property under some mild conditions on the isomorphism. This result allows us to interpret copulas as generators of joint distributions whose couples of variables exhibit TP2 dependence through distortion of a reference TP2 copula.

Furthermore, there are several potential benefits from employing copulas in describing the dependence structure of a group of random variables:

- Sklar's Theorem (Sklar 1959) allows one to obtain the joint distribution of a random vectors test through copulas by the knowledge of the marginals of its components. This is particularly relevant in several context, a remarkable example being the exploration of the distribution of the multiple test and the p-values in the field of multivariate statistics;

- dependence structures described by copulas are quite easy to be identified. When the distribution functions of the univariate and multivariate random quantities are available, then appropriate best fit procedures allow one to determine the analytical form of the underlying copula.

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