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Roy Cerqueti

Claudio Lupi

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Total positivity for a class of non-exchangeable copulas

Roy Cerqueti[a,*], Claudio Lupi[b]

[a] *University of Macerata, Dept. of Economics and Law. Via Crescimbeni, 20. I-62100 Macerata, Italy*
[b] *University of Molise, Dept. of Economics. Via De Sanctis. I-86100 Campobasso, Italy*

Abstract

This note contributes to the development of the theory of stochastic dependence by employing the general concept of copula. In particular, it deals with the construction of a new family of non-exchangeable copulas characterizing the multivariate total positivity of order 2 (MTP2) dependence.

Keywords: Copulas, MTP2 dependence, Non-exchangeability

1. Introduction

In copula theory, exchangeability (see e.g. Ghiselli Ricci [10]) represents an important feature which describes the symmetry of some classes of copulas. Archimedean copulas (see Alsina *et al.* [1], Durante *et al.* [9], Schweizer & Sklar [17] for some recent results and extensive surveys and [13] for a detailed discussion of the necessary and sufficient condition for an Archimedean copula generator to generate a d-dimensional copula), semilinear copulas (see Durante [6], Durante *et al.* [8]), and elliptical copulas (see e.g. Cambanis *et al.* [2], Schmidt [16]) are among the most relevant examples of exchangeable copulas. Despite its mathematical relevance, exchangeability may represent a strong requirement to be fulfilled by a set of random variables. In this respect, it is worth paying attention to the concept of Archimedean asymmetric copulas, which represent a non-exchangeable counterpart of the Archimedean ones (see Liebscher [11, 12]).

*Corresponding author.
Email addresses: roy.cerqueti@unimc.it (Roy Cerqueti), lupi@unimol.it (Claudio Lupi)

In this note, we move from Liebscher [11, 12] and construct a family of asymmetric copulas generated by a one-dimensional function which is able to characterize multivariate total positivity of order 2 (MTP2). In doing this, we extend Müller & Scarsini [14] to the case of non-exchangeability property for the considered set of dependent random variables.

The rest of the paper is organized as follows. Section 2 provides the necessary preliminaries, notation and statistical concepts. The main results are offered in Section 3. Some concluding remarks are given in Section 4.

2. Preliminaries and notation

For an easier reading of the model, we introduce the vectorial notation:

Notation 2.1. *Fix $m = 1, 2, \ldots$. The following notations are introduced:*

$$\boldsymbol{W} = (w_1, \ldots, w_m); \boldsymbol{X} = (x_1, \ldots, x_m); \boldsymbol{Y} = (y_1, \ldots, y_m); \boldsymbol{U} = (u_1, \ldots, u_m).$$

We now recall the definition of the dependence concept we deal with.

Definition 2.2. *Let f be the joint density function of the m-variate random vector $\boldsymbol{W}$. The components of $\boldsymbol{W}$ are said to be MTP2 if and only if, for each $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\mathbb{R}^m$, it results:*

$$f(\boldsymbol{X}) \cdot f(\boldsymbol{Y}) \geqslant f(\min\{\boldsymbol{X}, \boldsymbol{Y}\}) \cdot f(\max\{\boldsymbol{X}, \boldsymbol{Y}\})$$

where the min *and* max *operators have to be intended componentwise.*

Indeed, it is customary to represent the (linear) dependence among individual w's in $\boldsymbol{W}$ by a non-diagonal variance-covariance matrix $\boldsymbol{\Sigma} = (\sigma_{i,j})$ with $i, j = 1, \ldots, m$. Hence, it is natural to guess the existence of a relationship between the value of the covariances and the subsistence of MTP2. In this respect, it is useful to recall a further definition of dependence among random variables:

Definition 2.3. *The random components of $\boldsymbol{W}$ are associated if*

$$\text{Cov}\left[g(\boldsymbol{W}), h(\boldsymbol{W})\right] \geqslant 0,$$

for any coordinatewise nondecreasing functions g and h for which this covariance exists.

A classical result states that random variables which are MTP2 are also associated. Therefore, by using Definition 2.3 with $g(\boldsymbol{W}) = w_i$ and $h(\boldsymbol{W}) = w_j$, we can also say that if $\{w_1, \ldots, w_m\}$ are MTP2, then $\sigma_{i,j} \geqslant 0$, for each $i, j = 1, \ldots, m$. This fact implies that if there exists a couple (w_i, w_j), with $i \neq j$ and $i, j = 1, \ldots, m$, such that $\sigma_{i,j} < 0$, then $\{w_1, \ldots, w_m\}$ are not MTP2.

A rather general way to capture the stochastic dependence structure among random variables is the introduction of the concept of *multivariate copula* or, simply, *copula* (we refer the reader to Nelsen [15] for a detailed discussion). In particular, Sklar's Theorem [18] highlights how multivariate copulas model the dependence structure among random variables (see, e.g., [15, Section 2.3]).

An important feature of some classes of copulas which turns out to be relevant in our context is *exchangeability* (an accurate characterization of exchangeable copulas can be found in Ghiselli Ricci [10]):

Definition 2.4. *The copula $C : [0,1]^m \to [0,1]$ is exchangeable if, for each $\boldsymbol{U} \in [0,1]^m$ and a permutation ϱ of $\{1, \ldots, m\}$, one has:*

$$C(\boldsymbol{U}) = C(u_{\varrho(1)}, \ldots, u_{\varrho(m)}).$$

When Definition 2.4 is not satisfied, then copula C is said to be nonexchangeable.

3. Main result

The family of nonexchangeable copulas we deal with is generated by a one-dimensional function, and represents a generalization of the usual Archimedean copulas. We formalize it in the following:

Definition 3.1. *Fix $J \in \mathbb{N}$ and a set of $m \times J$ functions*

$$h_{jk} : [0,1] \to [0,1], \qquad j = 1, \ldots, J; \;\; k = 1, \ldots, m \tag{1}$$

such that:
(C3.1.i) h_{jk} is differentiable in $(0,1)$ and strictly increasing in $[0,1]$, for each j, k;
(C3.1.ii) $h_{jk}(0) = 0$ and $h_{jk}(1) = 1$, for each j, k;
(C3.1.iii) $\frac{1}{J} \sum_{j=1}^{J} h_{jk}(x) = x$, for each $k = 1, \ldots, m$ and $x \in [0,1]$.
Moreover, define

$$\psi : [0,1] \to [0,1] \tag{2}$$

such that:
(C3.1.iv) ψ *is* $m+2$ *times differentiable in* $(0,1)$*;*
(C3.1.v) $\psi^{(i)} > 0$ *in* $(0,1)$*, for* $i = 1, \ldots, m$*;*
(C3.1.vi) $\psi(0) = 0$ *and* $\psi(1) = 1$*.*
We define an asymmetric copula generated by a one-dimensional function as $C_{AS}^{\psi} : [0,1]^m \to [0,1]$ *such that:*

$$C_{AS}^{\psi}(\boldsymbol{U}) = \psi^{-1}\left(\frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h_{jk}(\psi(u_k))\right). \tag{3}$$

Remark 3.2. *The case of symmetric copula is a subcase of the asymmetric setting proposed in (3). To achieve the symmetric copula, it is sufficient to set* $h_{jk}(x) = x$*, for each* $j = 1, \ldots, J$*,* $k = 1, \ldots, m$ *and* $x \in [0,1]$*.*

Copula (3) has been first introduced and explored in Liebscher [11, 12]. It is worth noting that, as far as the copula's definition is concerned, conditions (C3.1.iv) and (C3.1.v) could be weakened. However, our mildly stronger version is required to prove the following general result:

Theorem 3.3. *Assume that* h_{jk} *is twice differentiable in* $(0,1)$*, with*

$$h''_{jk}(x) \geqslant 0, \qquad x \in (0,1),\ j = 1, \ldots, J,\ k = 1, \ldots, m, \tag{4}$$

and that

$$\begin{cases} \displaystyle\prod_{k=k_1,k_2}\Big[h''_{jk}(\psi(u_k))\,(\psi'(u_k))^2 + h'_{jk}(\psi(u_k))\psi''(u_k)\Big] \geqslant \Big[h''_{jk_1}(\psi(u_{k_1}))\,\times \\ \qquad \times\,(\psi'(u_{k_1}))^2 + h'_{jk_1}(\psi(u_{k_1}))\psi''(u_{k_1})\Big] \times h'_{jk_2}(\psi(u_{k_2}))\psi'(u_{k_2}) \\ \displaystyle\prod_{k=k_1,k_2} h'_{jk}(\psi(u_k))\psi'(u_k) \geqslant \Big[h''_{jk_1}(\psi(u_{k_1}))\,(\psi'(u_{k_1}))^2 + \\ \qquad + h'_{jk_1}(\psi(u_{k_1}))\psi''(u_{k_1})\Big] \times h'_{jk_2}(\psi(u_{k_2}))\psi'(u_{k_2}), \end{cases} \tag{5}$$

holds for each $j = 1, \ldots, J$*,* $k_1, k_2 = 1, \ldots, m$*,* $k_1 \neq k_2$*.*
Moreover, suppose that

$$\left(\psi^{-1}\right)^{(m+2)}\left(\psi^{-1}\right)^{(m)} - \left[\left(\psi^{-1}\right)^{(m+1)}\right]^2 \geqslant 0, \qquad \text{in } (0,1). \tag{6}$$

If the dependence among the components of the m*-variate random vector* $\boldsymbol{W}$ *is described by copula (3), then* $\boldsymbol{W}$ *is MTP2.*

Proof. In virtue of Müller & Scarsini [14], it is sufficient to check that the density f of C^{ψ}_{AS} is log-supermodular, that is equivalent to say that

$$\log(f(\boldsymbol{U})) := \log\left(\frac{\partial^m}{\partial u_1 \dots \partial u_m} C^{\psi}_{AS}(\boldsymbol{U})\right) \tag{7}$$

is supermodular.
By (3) we have

$$\begin{aligned} f(\boldsymbol{U}) &= \frac{\partial^m}{\partial u_1 \dots \partial u_m} C^{\psi}_{AS}(\boldsymbol{U}) \\ &= \left(\psi^{-1}\right)^{(m)} \left(\frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h_{jk}(\psi(u_k))\right) \times \\ &\quad \times \frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h'_{jk}(\psi(u_k))\psi'(u_k)\,. \end{aligned} \tag{8}$$

By (8) we can write

$$\begin{aligned} \log(f(\boldsymbol{U})) &= \log\left[\left(\psi^{-1}\right)^{(m)} \left(\frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h_{jk}(\psi(u_k))\right)\right] + \\ &\quad + \log\left[\frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h'_{jk}(\psi(u_k))\psi'(u_k)\right] \\ &=: A(\boldsymbol{U}) + B(\boldsymbol{U})\,, \end{aligned} \tag{9}$$

where the terms $A(\cdot)$ and $B(\cdot)$ are an intuitive shorthand for the two $\log[\cdot]$ terms.
The supermodularity of $\log(f(\boldsymbol{U}))$ is equivalent to the following condition:

$$\frac{\partial^2}{\partial u_{k_1}\partial u_{k_2}}\left[A(\boldsymbol{U}) + B(\boldsymbol{U})\right] \geqslant 0, \tag{10}$$

for each $k_1, k_2 \in \{1, \dots, m\}$, and $(u_1, \dots, u_m) \in [0,1]^m$. For an easier notation, we will pose hereafter

$$\xi := \frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m} h_{jk}(\psi(u_k)). \tag{11}$$

We analyze the terms $A(\cdot)$ and $B(\cdot)$ separately.
First notice that

$$\frac{\partial A(\boldsymbol{U})}{\partial u_{k_1}} = \frac{\left(\psi^{-1}\right)^{(m+1)}(\xi)}{\left(\psi^{-1}\right)^{(m)}(\xi)} \times \frac{1}{J}\sum_{j=1}^{J} h'_{jk_1}(\psi(u_{k_1}))\psi'(u_{k_1}) \prod_{k\neq k_1}\left[h_{jk}(\psi(u_k))\right]$$

and

$$\begin{aligned}
\frac{\partial^2 A(\boldsymbol{U})}{\partial u_{k_1}\partial u_{k_2}} = & \frac{1}{\left\{\left(\psi^{-1}\right)^{(m)}\right\}^2} \times \left\{\left(\frac{1}{J}\sum_{j=1}^{J} h'_{jk_1}(\psi(u_{k_1}))\psi'(u_{k_1}) \prod_{k\neq k_1}\left[h_{jk}(\psi(u_k))\right]\right) \times \right. \\
& \times \left(\frac{1}{J}\sum_{j=1}^{J} h'_{jk_1}(\psi(u_{k_2}))\psi'(u_{k_2}) \prod_{k\neq k_2}\left[h_{jk}(\psi(u_k))\right]\right) \times \\
& \times \left[\left(\psi^{-1}\right)^{(m+2)}(\xi) \times \left(\psi^{-1}\right)^{(m)}(\xi) - \left[\left(\psi^{-1}\right)^{(m+1)}(\xi)\right]^2\right] + \\
& + \left(\psi^{-1}\right)^{(m)}(\xi) \times \left(\psi^{-1}\right)^{(m+1)}(\xi) \times \\
& \times \frac{1}{J}\sum_{j=1}^{J} h'_{jk_1}(\psi(u_{k_1}))\psi'(u_{k_1}) h'_{jk_2}(\psi(u_{k_2}))\psi'(u_{k_2}) \times \\
& \left. \times \prod_{k\neq k_1,k_2}\left[h_{jk}(\psi(u_k))\right]\right\} . \qquad (12)
\end{aligned}$$

Hence, under Condition (C3.1.v) and hypothesis (6), we have

$$\frac{\partial^2 A(\boldsymbol{U})}{\partial u_{k_1}\partial u_{k_2}} \geqslant 0. \qquad (13)$$

Let us now turn to $B(\cdot)$:

$$\begin{aligned}
\frac{\partial B(\boldsymbol{U})}{\partial u_{k_1}} = & \frac{1}{\sum_{j=1}^{J}\prod_{k=1}^{m} h'_{jk}(\psi(u_k))\psi'(u_k)} \times \\
& \times \left\{\sum_{j=1}^{J}\left[h''_{jk_1}(\psi(u_{k_1}))\left(\psi'(u_{k_1})\right)^2 + h'_{jk_1}(\psi(u_{k_1}))\psi''(u_{k_1})\right] \times \right. \\
& \left. \times \prod_{k\neq k_1} h'_{jk}(\psi(u_k))\psi'(u_k)\right\} , \qquad (14)
\end{aligned}$$

hence we have:

$$
\begin{aligned}
\frac{\partial^2 B(\boldsymbol{U})}{\partial u_{k_1}\partial u_{k_2}} = & \frac{1}{\left[\sum_{j=1}^{J}\prod_{k=1}^{m} h'_{jk}(\psi(u_k))\psi'(u_k)\right]^2} \times \\
& \left\{\left[\sum_{j=1}^{J}\prod_{k\neq k_1,k_2} h'_{jk}(\psi(u_k))\psi'(u_k)\times\right.\right. \\
& \left.\times \prod_{k=k_1,k_2}[h''_{jk}(\psi(u_k))(\psi'(u_k))^2 + h'_{jk}(\psi(u_k))\psi''(u_k)]\right] \times \\
& \times \sum_{j=1}^{J}\prod_{k=1}^{m} h'_{jk}(\psi(u_k))\psi'(u_k) - \\
& -\left[\sum_{j=1}^{J}\prod_{k\neq k_1} h'_{jk}(\psi(u_k))\psi'(u_k)\times\right. \\
& \left.[h''_{jk_1}(\psi(u_{k_1}))(\psi'(u_{k_1}))^2 + h'_{jk_1}(\psi(u_{k_1}))\psi''(u_{k_1})]\right] \times \\
& \times\left[\sum_{j=1}^{J}\prod_{k\neq k_2} h'_{jk}(\psi(u_k))\psi'(u_k)\times\right. \\
& \left.\left.[h''_{jk_2}(\psi(u_{k_2}))(\psi'(u_{k_2}))^2 + h'_{jk_2}(\psi(u_{k_2}))\psi''(u_{k_2})]\right]\right\}. \qquad (15)
\end{aligned}
$$

By (15) we obtain that sufficient conditions for being $\partial^2 B_s(\boldsymbol{U})/(\partial u_{k_1}\partial u_{k_2}) \geqslant 0$ are given by relations in (4) and (5).
The result is proved, by the arbitrariness of k_1 and k_2. □

It is important to note that the set of copulas described by Definition 3.1 and Theorem 3.3 is not empty, and contains a rather large number of elements. We elaborate this point in the following example.

Example 3.4. *Consider $J \times m$ positive real numbers $\alpha_{11}, \ldots, \alpha_{1m}, \ldots, \alpha_{Jm}$ such that*

$$\sum_{j=1}^{J}\alpha_{jk} = J, \qquad \forall\, k = 1, \ldots, m,$$

and define $h_{jk}(x) = \alpha_{jk}x$*, for each* $j = 1, \ldots, J$*,* $k = 1, \ldots, m$ *and* $x \in [0, 1]$*. Moreover, define*

$$\psi(u_k) = \frac{e^{u_k} - 1}{e - 1}, \tag{16}$$

for each $k = 1, \ldots, m$*.*
Functions h's and ψ *satisfy the regularity conditions required by Definition 3.1 and Theorem 3.3. Moreover, it results*

$$\frac{1}{J}\sum_{j=1}^{J} h_{jk}(x) = x, \qquad \forall\, k = 1, \ldots, m,\, x \in [0, 1],$$

and $h_{jk}(0) = \psi(0) = 0$*,* $h_{jk}(1) = \psi(1) = 1$*.*
It is also easy to see that

$$\psi^{(i)}(x) = \frac{e^x}{e - 1} > 0 \qquad \forall\, i = 1, \ldots, m,\, x \in (0, 1),$$

and

$$\left(\psi^{-1}\right)^{(m+2)}(x)\left(\psi^{-1}\right)^{(m)}(x) - \left[\left(\psi^{-1}\right)^{(m+1)}(x)\right]^2 = 0, \qquad \forall\, x \in (0, 1).$$

By definition of the h's, condition (5) becomes

$$\psi'(x) = \psi''(x), \qquad \forall\, x \in (0, 1),$$

which is trivially satisfied by function ψ *in (16). Then, the h's and* ψ *satisfy the set of conditions listed in Definition 3.1 and the assumptions (4)–(6) of Theorem 3.3. This implies that copula*

$$C^{\psi}_{AS}(\boldsymbol{U}) = \log\left[(e - 1)\left(\frac{1}{J}\sum_{j=1}^{J}\prod_{k=1}^{m}\frac{\alpha_{jk}\left(e^{u_k} - 1\right)}{e - 1}\right) + 1\right] \tag{17}$$

is associated to a MTP2 stochastic dependence.

4. Concluding remarks

Dealing with dependence among random variables is of paramount importance, mainly in the context of multivariate analysis. In this respect, a relevant role is played by the concept of copulas. This paper contributes to this field of research by studying the relationship between copulas and the MTP2 dependence property in the case of non-exchangeability.
A copula-based framework seems to be preferable to a setting adopting the explicit definition of MTP2. The reason is twofold:

- it is generally hard to check if the dependence structure among groups of random variables is of MTP2-type by using the formal definition. Therefore, a characterization of the MTP2 property through more affordable and useful mathematical tools is needed. The concept of copula meets this need in that it is rather general and easy to be treated;

- recently, Durante *et al.* [7] introduced the concept of *distorted copulas*, which represents a new family of copulas obtained by applying an isomorphism to a given copula. They showed that bivariate distorted copulas retain the *total positivity of order 2* (TP2) property under some mild conditions on the isomorphism. This result allows us to interpret copulas as generators of joint distributions whose couples of variables exhibit TP2 dependence through distortion of a reference TP2 copula.

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