Infinitesimal Generators on the Quantum Group $SU_q(2)$ in the Classical and Anti-classical Limit

Michael Skeide*

Lehrstuhl für Wahrscheinlichkeitstheorie und Statistik
Brandenburgische Technische Universität Cottbus
Postfach 10 13 44
D–03013 Cottbus
Germany

December 1997

Abstract

In [5] we found an explicit formula for the infinitesimal generators of white noises on the quantum group $SU_q(2)$ in the case $q \in (-1, 1)$. In [7] we compared the case $q \in (-1, 1)$ with the classical case $q = 1$ which corresponds to the infinitesimal generators of Lévy processes on $SU(2)$. We pointed out that our formula is the perfect analogue of Hunt's formula which describes the classical infinitesimal generators. In the first part of these notes we find Hunt's formula for $SU_q(2)$ in the anti-classical case $q = -1$. This completes our theory of infinitesimal generators on $SU_q(2)$ for all $q \in [-1, 1]$.

In the second part of these notes we show that $SU_q(2)$ is a 'good' quantization of $SU(2)$. We show that not only states on the algebra of functions on $SU(2)$, but even infinitesimal generators may be approximated in a suitable sense by states and infinitesimals generators on $SU_q(2)$ for $q \in (-1, 1)$. Simultaneously, we show that a similar result holds true for $SU_{-1}(2)$.

1 Introduction

Lévy processes on Lie groups are classified by their infinitesimal generators. In [3] Hunt found his famous formula for the explicit form of the infinitesimal generators. In a series of papers Schürmann has generalized the notion of Lévy processes on a Lie group to Lévy processes on $*$–bialgebras; see [4] for a detailed list of references. Schürmann found an algebraic characterization of the infinitesimal generators by a weakened positivity condition; see below. An explicit formula like Hunt's formula in the general case is yet missing.

In [5] we found an explicit formula for the infinitesimal generators on Woronowicz's quantum group $SU_q(2)$ [10] in the case $q \neq \pm 1$. In [7] we pointed out that our formula is the perfect analogue of Hunt's formula by comparing it with the classical case $q = 1$. In these notes we do the same for the anti-classical case $q = -1$. In other words, we reprove Hunt's formula in the special case of the algebra of functions on $SU(2)$ (which is more or less $SU_q(2)$ for $q = 1$) in form which is suitable for an adaptation to the case $q = -1$.

After some general general remarks on quantization in Section 4 we show in Section 5 that states and in Section 6 that infinitesimal generators on $SU_{\pm 1}(2)$ may *approximated* by states and infinitesimal generators on $SU_q(2)$ ($|q| < 1$), respectively.

In Appendix A we provide a self-contained introduction into the necessary notions of q–analysis. In Appendix B we introduce the q–coherent states and express them in terms of a Hilbert space of analytic functions.

Like the papers [5, 7, 8] the material in these notes is based on our PHD-thesis [6].

*E-mail: skeide@math.tu-cottbus.de

Unless stated otherwise explicitly, we use the following conventions. Algebras are complex and unital $*$–algebras. Ideals are $*$–ideals (and, of course, usually non-unital). Representations are non-degenerate $*$–representations and cocycles are 1–cocycles. Functionals, always are linear functionals. For notational convenience we use Dirac's notation.

2 Basics and the classical case

$*$–Bialgebras, in the first place, are (unital $*$–)algebras with a character (i.e. a $*$–homomorphism into $\mathbf{C}$) δ (the counit). Let $\mathcal{A}$ be an algebra and $\delta : \mathcal{A} \to \mathbf{C}$ a character. Let $K_1 = \ker(\delta)$ be the kernel of δ. We call a (linear) hermitian functional ψ on $\mathcal{A}$ *conditionally positive* (with respect to δ), if it is positive on the ($*$–)ideal K_1, i.e. if

$$\psi(a^*a) \geq 0 \quad \text{for all} \quad a \in K_1.$$

Schürmann has shown (see [4]) that the set of infinitesimal generators of Lévy processes on a $*$–bialgebra consists precisly of those conditionally positive functionals which vanish at the unit. In the remainder of these notes we will consider the slightly more general set of conditionally positive functionals. Given a conditionally positive functional ψ, we obtain an infinitesimal generator by $\psi - \psi(\mathbf{1})\delta$

Let $\mathcal{D}$ be a pre–Hilbert space and π a ($*$–)representation of $\mathcal{A}$ acting on $\mathcal{D}$. By a *(1–)cocycle* with respect to π we mean a linear mapping $\eta : \mathcal{A} \to \mathcal{D}$ such that

$$\eta(ab) = \pi(a)\eta(b) + \eta(a)\delta(b) \quad \text{for all} \quad a, b \in \mathcal{A}. \tag{2.1}$$

Let $K_2 = \text{lin}(K_1 \cdot K_1)$ be the ideal which is spanned by all products of elements of K_1. By GNS–construction one shows the following

Theorem 2.1 (Schürmann [4]) *For an arbitrary conditionally positive functional ψ there is a triplet $(\mathcal{D},\pi,\eta)$ consisting of a pre–Hilbert space $\mathcal{D}$, a representation π on $\mathcal{D}$ and a cocycle η with respect to this representation such that the values of ψ on K_2 are given by*

$$\psi(ab) = \langle \eta(a^*) | \eta(b) \rangle \quad \textit{for all} \quad a, b \in K_1. \tag{2.2}$$

The restriction of π to the invariant subspace $\eta(\mathcal{A})$ of $\mathcal{D}$ is determined up to unitary equivalence.

(Of course, the GNS–construction is well-known. The importance of Theorem 2.1 lies in the fact that the ingedients ψ, η, π play a crucial role in the reconstruction of a Lévy process from its infinitesimal generator. The reconstruction also is the place where the comultiplication of the $*$–bialgebra structure enters. In these notes we do not need the comultipication.)

By Theorem 2.1 we reduce the problem of finding all conditionally positive functionals to that of finding all representations, all cocycles with respect to these representations, and checking for which of them we can find a conditionally positive functional satisfying (2.2).

We define for any representation π on a pre–Hilbert space $\mathcal{D}$ and any vector $\eta \in \mathcal{D}$ the mappings $\langle \eta | \pi | \eta \rangle : \mathcal{A} \to \mathbf{C}$ and $\pi\eta : \mathcal{A} \to \mathcal{D}$ by

$$\langle \eta | \pi | \eta \rangle(a) = \langle \eta | \pi(a) | \eta \rangle \text{ and } (\pi\eta)(a) = \pi(a)\eta, \tag{2.3}$$

respectively. The mapping $Id - \delta\mathbf{1} : \mathcal{A} \to \mathcal{A}$ defined by

$$(Id - \delta\mathbf{1})(a) = a - \delta(a)\mathbf{1},$$

is a canonical projection onto K_1. We immediately see that

$$(\pi\eta) \circ (Id - \delta\mathbf{1}) \text{ and } \langle \eta | \pi | \eta \rangle \circ (Id - \delta\mathbf{1})$$

are a cocycle and a conditionally positive functional respectively, satisfying (2.2). In cohomology theory such a cocycle is called a *coboundary* and the functional is a positive multiple of $\varphi - \delta$, where φ is a *state* on $\mathcal{A}$. In [5] we have shown that in the case when $\mathcal{A}$ is a C^*–algebra all cocycles and infinitesimal generators have the above form. This corresponds to the fact that in the classical case of the algebra of functions on a Lie group the infinitesimal generators, in general, are defined only on functions which are twice differentiable at the neutral element.

From *Leibniz rule* we obtain the following

Proposition 2.2 *Let δ_φ be a family of homomorphisms $\delta_\varphi : \mathcal{A} \to \mathbf{C}$ which are pointwise continuous in φ and such that $\delta_0 = \delta$. Then we have*

(i) If δ_φ is pointwise differentiable with respect to φ at $\varphi = 0$, then δ_0' is a cocycle with respect to δ.

(ii) If δ_φ is pointwise twice differentiable with respect to φ at $\varphi = 0$, then δ_0' is a cocycle with respect to δ and $\frac{\delta_0''}{2}$ is a conditionally positive functional satisfying (2.2).

We recall the definition of the algebra structure of $SU_q(2)$.

Definition 2.1 (Woronowicz [10]) *For a real number $q \in [-1, 1]$ we denote by $\mathcal{A}_q$ the unital algebra generated by α, γ, with the following relations:*

$$\begin{array}{lrcl} \text{(a)} & \alpha\gamma & = & q\gamma\alpha \\ \text{(b)} & \alpha\gamma^* & = & q\gamma^*\alpha \\ \text{(c)} & \gamma^*\gamma & = & \gamma\gamma^* \\ \text{(d)} & \alpha\alpha^* - \alpha^*\alpha & = & (1-q^2)\gamma^*\gamma \\ \text{(e)} & \gamma^*\gamma + \alpha^*\alpha & = & \mathbf{1}. \end{array} \tag{2.4}$$

By Relation (e) all representation operators are bounded. Henceforth, we assume all representations to act on a Hilbert space.

We recall the form of the irreducible representations for $q \neq \pm 1$.

Theorem 2.3 (Vaksman, Soibelman [9], cf. also [8]) *Let $q \neq \pm 1$. Let h_0 be a Hilbert space with an ONB $\{e_k\}_{k \in \mathbf{N}_0}$. The following equations*

(i)

$$\begin{array}{rcll} \rho_\varphi(\alpha)e_k & = & \sqrt{1-q^{2k}}\, e_{k-1}, & k \in \mathbf{N} \\ \rho_\varphi(\alpha)e_0 & = & 0, & k = 0 \\ \rho_\varphi(\gamma)e_k & = & e^{i\varphi}q^k e_k, & k \in \mathbf{N}_0 \end{array}$$

(ii)

$$\begin{array}{rcl} \delta_\varphi(\alpha) & = & e^{i\varphi} \\ \delta_\varphi(\gamma) & = & 0 \end{array}$$

define inequivalent irreducible representations $\rho_\varphi : \mathcal{A}_q \to \mathcal{B}(h_0)$ and $\delta_\varphi : \mathcal{A}_q \to \mathbf{C}$ of $\mathcal{A}_q$ for any $\varphi \in [0, 2\pi)$.

Any irreducible representation must be unitarily equivalent to one of these representations.

The homomorphism δ is just δ_0. Clearly, δ_φ evaluated at a fixed algebra element a is an analytic function of φ. We use the notation of Proposition 2.2 and ommit the subscript $\varphi = 0$.

Now we investigate the sets K_1 and K_2. Clearly, if we introduce

$$\beta = \alpha - \mathbf{1},$$

the set $\{\mathbf{1}, \beta, \beta^*, \gamma, \gamma^*\}$ generates the whole algebra. Henceforth, since we have $\beta, \beta^*, \gamma, \gamma^* \in K_1$ and $\mathbf{1} \notin K_1$, the set

$$G = \{\beta, \beta^*, \gamma, \gamma^*\} \tag{2.5}$$

generates K_1. Relations (2.4), expressed in terms of β and γ, transform into

$$
\begin{array}{llcl}
(\widetilde{\text{a}}) & \beta\gamma & = & q\gamma\beta - (1-q)\gamma \\
(\widetilde{\text{b}}) & \beta\gamma^* & = & q\gamma^*\beta - (1-q)\gamma^* \\
(\widetilde{\text{c}}) & \gamma^*\gamma & = & \gamma\gamma^* \\
(\widetilde{\text{d}}) & \beta\beta^* - \beta^*\beta & = & (1-q^2)\gamma^*\gamma \\
(\widetilde{\text{e}}) & \gamma^*\gamma + \beta^*\beta + \beta^* + \beta & = & 0.
\end{array}
\tag{2.6}
$$

An arbitrary element a of $\mathcal{A}_q$ can be written in the form

$$
a \;=\; c_1\mathbf{1} \;+\; \sum_{g\in G} c_g g \;+\; c
$$

where c_1, c_g are complex numbers and $c \in K_2$. If $q \neq 1$, we see from Relations $(\widetilde{\text{a}})$, $(\widetilde{\text{b}})$, and $(\widetilde{\text{e}})$ that the elements γ, γ^*, and $\beta + \beta^*$ can be expressed as sums of products of elements of K_1, hence are elements of K_2. In other words, any $a \in \mathcal{A}_q$ can be written as

$$
a \;=\; c_1\mathbf{1} + c_2\frac{\beta - \beta^*}{2i} + c \;=\; c_1\mathbf{1} + c_2\frac{\alpha - \alpha^*}{2i} + c \tag{2.7}
$$

where c_1, c_2 are complex numbers and $c \in K_2$, in at least one way.

Proposition 2.4 *Decomposition (2.7) is unique for any $a \in \mathcal{A}_q$.*

The mapping $\mathcal{P} : \mathcal{A}_q \longrightarrow \mathcal{A}_q$, $a \longmapsto a - \delta(a)\mathbf{1} - \delta'(a)\frac{\alpha-\alpha^*}{2i}$ is a projection onto K_2. That means $\mathcal{P}(\mathcal{A}_q) = K_2$ and $\mathcal{P}^2 = \mathcal{P}$. Moreover, $\mathcal{P}(a^*) = \mathcal{P}(a)^*$.

Proof We apply δ and δ' to (2.7). By definition δ is 0 on K_1 and K_2, hence $\delta(a) = c_1$. Using the factorization property of δ, we obtain by an application of *Leibniz rule* that δ' vanishes on K_2 as well as it does on $\mathbf{1}$. Hence, $\delta'(a) = c_2\delta'\left(\frac{\alpha-\alpha^*}{2i}\right) = c_2$. Therefore, the numbers c_1, c_2 are determined by a, and so is c by (2.7). The properties of $\mathcal{P}$ are obvious. ■

In the sequel, we have to separat from a conditionally positive functional its gaußian part in the classical case $q = 1$ and its anti-gaußian part in the anti-classical case $q = -1$. A conditionally positive functional is called gaußian, if it is associated with a representation of the form $\delta\mathbf{1}$. It is called anti-gaußian, if it is associated with a representation of the form $\delta_\pi\mathbf{1}$. Similarly, gaußian and anti-gaußian cocycles are cocycles with respect to $\delta\mathbf{1}$ and $\delta_\pi\mathbf{1}$, respectively. It is easy to see that a subspace of the representation space of a representation π, where α acts as $\mathbf{1}$ or $-\mathbf{1}$, is an invariant subspace. (In fact, by Relation (e) we see that on this subspace γ must act as 0.) Clearly, the decomposition turns over to the cocycles. Unfortunately, this must not be true for the conditionally positive functionals. Fortunately, in our cases the remaining direct summand of π will always be such that we can guarantee existence of an associated conditionally positive functional. In this case also the other summand has no choice and must be associated with conditionally positive functional; see [7] for a detailed argument.

A key-lemma in [5] asserts that a cocycle η is determined uniquely by its value $\eta(\alpha^*)$ (which equals $\eta(\beta^*)$, because our representations are non-degenerate). We adapt this to the cases $q = \pm 1$.

Lemma 2.5 *Let π denote a representation of $\mathcal{A}_1$ and $\mathcal{A}_{-1}$ and assume that there is no gaußian and anti-gaußian part, respectively. Then a cocycle η is determined uniquely by its value $\eta(\alpha^*)$.*

Proof We have $q^2 = 1$ so that β is normal by Relation $(\widetilde{\text{d}})$. In particular, $\pi(\beta)\eta(\beta^*) = \pi(\beta^*)\eta(\beta)$. Hence, if $\pi(\beta^*)$ is injective (in other words, if π does not contain a gaußian part), then $\eta(\beta)$ is determined by $\eta(\beta^*)$. So let us assume that π does not contain a gaußian part. By Relations $(\widetilde{\text{a}})$ and $(\widetilde{\text{b}})$ we find $\pi(\beta + (1-q)\mathbf{1})\eta(\gamma^{(*)}) = q\pi(\gamma^{(*)})\eta(\beta)$. Hence, $\eta(\gamma^{(*)})$ is determined by $\eta(\beta)$ (henceforth, by $\eta(\beta^*)$), if $\pi(\beta + (1-q)\mathbf{1})$ is injective. In the case $q = 1$ this is true, because we assumed π to have no gaußian part, and we are finished. In the case $q = -1$ we find that $\pi(\beta + (1-q)\mathbf{1})$ is injective, if π has no anti-gaußian part. The only case which still has to be shown is $q = -1$ and π purely gaußian, i.e. $\pi = \delta$. Assume $\eta(\gamma^{(*)}) \neq 0$. Then by Relation $(\widetilde{\text{a}})$ we have $\delta(\alpha + \mathbf{1})\eta(\gamma^{(*)}) = 0$ contradicting $\delta(\alpha + \mathbf{1}) = 2$. ■

In the remainder of this section we briefly discuss the classical case $q = 1$ in a way which is adaptable also to the following section where we discuss the ant-classical case $q = -1$. A general discussion of the classical case of functions on a compact Lie group can be found in [7].

An element of $U \in SU(2)$ is given by a unitary matrix $U = (u_{ij})_{i,j=1,2}$ with unit determinant. Consider the unital $*$–algebra $\mathcal{A}_{\rm f}$, which is generated by the coefficient functions $f_{ij} : U \mapsto u_{ij}$ on $SU(2)$. We use the parametrization of $SU(2)$ defined by

$$(f_{ij}(\varphi, x, y))_{ij} \quad = \quad \begin{pmatrix} \sqrt{1-x^2-y^2}e^{i\varphi} & -(x-iy) \\ x+iy & \sqrt{1-x^2-y^2}e^{-i\varphi} \end{pmatrix},$$

with $\varphi \in [-\frac{\pi}{2}, \frac{3\pi}{2})$ and $x^2 + y^2 \leq 1$, which is non-singular at the neutral element I and at $-I$. It is easy to see that the irreducible representations of $\mathcal{A}_{\rm f}$ are given by $\rho_U(f) = f(U)$, where U can be any point in $SU(2)$, i.e. they are in one-to-one correspondence with the group elements. We introduce the usual supremum norm on $\mathcal{A}_{\rm f}$ by $\|f\| = \sup_{U \in SU(2)} |f(U)|$. By an application of *Stone-Weierstrass theorem* $\mathcal{A}_{\rm f}$ is dense in $C(SU(2))$, the $*$–algebra of continuous functions on $SU(2)$, which, therefore, is the completion of $\mathcal{A}_{\rm f}$.

The generators f_{ij} of $\mathcal{A}_{\rm f}$ satisfy $f_{11} = f_{22}^*$, $f_{12} = -f_{21}^*$, and $f_{11}^* f_{11} + f_{21}^* f_{21} = \mathbf{1}$. Thus, by

$$\begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \quad \longmapsto \quad \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

we define a $*$–algebra homomorphism from $\mathcal{A}_1$ onto $\mathcal{A}_{\rm f}$. We define a three-parameter family $\delta_{\varphi xy}$ representations of $\mathcal{A}_1$ by setting

$$\delta_{\varphi xy} \begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \quad = \quad \varrho_U \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \quad = \quad \begin{pmatrix} \sqrt{1-x^2-y^2}e^{i\varphi} & -(x-iy) \\ x+iy & \sqrt{1-x^2-y^2}e^{-i\varphi} \end{pmatrix},$$

where φ, x, y describes $U \in SU(2)$. The general representation π_E of $\mathcal{A}_1$ is given by

$$\pi_E \quad = \quad \int_{SU(2)} \delta_{\varphi xy} \, dE_{\varphi xy},$$

where dE is an arbitrary spectral measure on $SU(2)$. Notice that $\pi_E(a) = \int_{SU(2)} f(U) \, dE_{\varphi xy}$, if f is the funtion in $\mathcal{A}_{\rm f}$ corresponding to $a \in \mathcal{A}_1$. $\mathcal{A}_1$ has a faithful representation; see Woronowicz [10]. Therefore, $\| \bullet \| = \sup_{\pi} \|\pi(\bullet)\|$ defines a C^*–norm on $\mathcal{A}_1$. Clearly, this norm coincides with the norm on $\mathcal{A}_{\rm f}$. Thus, $\mathcal{A}_{\rm f}$ and $\mathcal{A}_1$ are isometrically isomorphic pre–C^*–algebras, and $C(SU(2))$ can be identified with the C^*–completion of $\mathcal{A}_1$.

Now we come to the cocycles and conditionally positive functionals on $\mathcal{A}_1$. First, we investigate the gaußian parts. Consider the three mappings $\delta'^{\varphi} = \partial_\varphi \delta_{000}$, $\delta'^{x} = \partial_x \delta_{000}$, and $\delta'^{y} = \partial_y \delta_{000}$. These mappings are cocycles with respect to $\delta = \delta_{000}$. Since

$$\delta'^{\varphi} \begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \quad = \quad i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\delta'^{x} \begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \quad = \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\delta'^{y} \begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \quad = \quad i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we see that the three cocycles are linearly independent. In particular, we see that any linear combination of δ'^{x} and δ'^{y} is a cocycle different from 0 but vanishing on α^*. Let $r \in \mathbf{R}^3$ be a vector with components (φ, x, y). Setting

$$\delta'^{r} \quad = \quad \left. \frac{d\delta_{(t\varphi)(tx)(ty)}}{dt} \right|_{t=0}, \tag{2.8}$$

we define a three parameter family of cocycles with respect to δ which consists of all real linear combinations of δ'^{φ}, δ'^{x}, and δ'^{y}. Setting

$$\frac{\delta''^{r}}{2} \quad = \quad \frac{1}{2} \left. \frac{d^2\delta_{(t\varphi)(tx)(ty)}}{dt^2} \right|_{t=0},$$

we obtain a conditionally positive functional fulfilling (2.2).

Now we see by Relations (2.6) that any gaußian cocycle, i.e. a cocycle η_δ with respect to a representation of the form $\delta\mathbf{1}_{H_\delta}$ on a Hilbert space H_δ, is defined by its values on $\alpha - \alpha^*$, γ, and γ^* which, on the other hand, can be chosen arbitrarily. In other words, we obtain

Theorem 2.6 *By*

$$\eta_\delta \quad = \quad \delta'^{\varphi}\eta_\varphi + \delta'^{x}\eta_x + \delta'^{y}\eta_y, \tag{2.9}$$

we establish a one-to-one correspondence between gaußian cocycles η_δ and triplets $(\eta_\varphi, \eta_x, \eta_y)$ of vectors $\eta_\varphi, \eta_x, \eta_y \in H_\delta$.

Since η_δ must vanish on K_2, we have $\alpha - \alpha^*, \gamma, \gamma^* \notin K_2$ and any basis of K_2 can be extended by $\mathbf{1}$, $\frac{\alpha-\alpha^*}{2i}$, $\frac{\gamma+\gamma^*}{2}$, $\frac{\gamma-\gamma^*}{2i}$ to a basis of $\mathcal{A}_1$. Setting

$$\mathcal{P}_1 \quad = \quad Id - \delta\mathbf{1} - \delta'^{\varphi}\frac{\alpha-\alpha^*}{2i} - \delta'^{x}\frac{\gamma+\gamma^*}{2} - \delta'^{y}\frac{\gamma-\gamma^*}{2i},$$

we obtain a projection onto K_2.

The form (2.9) of the gaußian cocycles is not yet suitable to see the form of the gaußian conditionally positive functionals. In the following theorem we find a more practicable one. Moreover, it turns out that not all cocycles determine the values of a conditionally positive functional on K_2.

Theorem 2.7 *For any gaußian cocycle η_δ which determines the values of a conditionally positive functional on K_2 there are three orthonormal vectors $\eta_1, \eta_2, \eta_3 \in H_\delta$ and three vectors $r_1, r_2, r_3 \in \mathbf{R}^3$ such that*

$$\eta_\delta \quad = \quad \delta'^{r_1}\eta_1 + \delta'^{r_2}\eta_2 + \delta'^{r_3}\eta_3. \tag{2.10}$$

Choosing an arbitrary real number r_0 and an arbitrary vector $r \in \mathbf{R}^3$ we obtain all conditionally positive functionals ψ_δ, fulfilling (2.2), in the form

$$\psi_\delta \quad = \quad r_0\delta + \delta'^{r} + \frac{\delta''^{r_1}}{2} + \frac{\delta''^{r_2}}{2} + \frac{\delta''^{r_3}}{2}.$$

Proof If a cocycle η_δ has the form (2.10), the form of the conditionally positive functionals follows straightforwardly. Thus, it remains to show that a cocycle which determines the values of a conditionally positive functional on K_2 has to be of the form (2.10).

Let η_δ be an arbitrary gaußian cocycle, which determines the values of a conditionally positive functional on K_2, given in the form (2.9). Obviously, the component $\langle\eta_0|\eta_\delta\rangle$ of η_δ in the direction of an arbitrary vector η_0 itself is a gaußian cocycle (with values in $\mathbf{C}$). There is nothing to prove, if all vectors $\eta_\varphi, \eta_x, \eta_y$ are 0. Therefore, we assume, without loss of generality, that $\eta_x \neq 0$ and choose $\eta_1 = \frac{\eta_x}{\|\eta_x\|}$. It is easy to conclude by commutativity of $\frac{\alpha-\alpha^*}{2i}$, $\frac{\gamma+\gamma^*}{2}$, $\frac{\gamma-\gamma^*}{2i}$ that the numbers $\langle\eta_1|\eta_\varphi\rangle, \langle\eta_1|\eta_y\rangle$ are real numbers. In other words, we can find $r_1 \in \mathbf{R}^3$, such that $\langle\eta_1|\eta_y\rangle = \delta'^{r_1}$. We know that this cocycle defines a conditionally positive functional. Therefore, also its 'orthogonal complement' defined by $\tilde{\eta}_\delta = \eta_\delta - \eta_1\delta'^{r_1}$ must define a conditionally positive functional. Furthermore, there are vectors $\tilde{\eta}_\varphi, \tilde{\eta}_y$ orthogonal to η_1, such that $\tilde{\eta}_\delta = \delta'^{\varphi}\tilde{\eta}_\varphi + \delta'^{y}\tilde{\eta}_y$. If $\tilde{\eta}_y \neq 0$, we continue our argument in the same manner, by setting $\eta_2 = \frac{\tilde{\eta}_y}{\|\tilde{\eta}_y\|}$. If this is not so, then either also $\tilde{\eta}_\varphi = 0$ (and we are finished choosing arbitrary η_2, η_3 such that $\eta_1 \perp \eta_2 \perp \eta_3 \perp \eta_1$ and $r_2 = r_3 = 0$) or $\tilde{\eta}_\varphi \neq 0$ (and we may choose $\eta_2 = \frac{\tilde{\eta}_\varphi}{\|\tilde{\eta}_\varphi\|}$, arbitrary η_3 with $\eta_1 \perp \eta_3 \perp \eta_2$ and $r_3 = 0$). ■

In the sequel, we will restrict ourselves to representations without gaußian part, i.e. $\pi(\beta^*)$ is injective. Such a representation is given by

$$\pi \quad = \quad \int_{SU(2)} \delta_{\varphi xy}\, dE_{\varphi xy} \quad = \quad \lim_{\epsilon\to 0}\int_{SU(2)\setminus U_\epsilon(I)} \delta_{\varphi xy}\, dE_{\varphi xy},$$

where $U_\epsilon(I)$ denotes an ϵ–neighbourhood of the neutral element and the limit is strong.

Motivated by [5] we define mapping $\mathcal{O}$ from $\mathcal{A}_1$ into the measurable functions on $SU(2)$ by setting

$$[\mathcal{O}(a)](U) \quad = \quad \frac{\delta_{\varphi xy}(a) - \delta(a)}{\sqrt{1 - x^2 - y^2 e^{-i\varphi}} - 1}.$$

Observe that $\mathcal{O}(a)$, in general, is unbounded around the neutral element. (Insert any non-vanishing linear combination of γ and γ^*, set $\varphi = 0$ and let $|x + iy|$ go to zero.) On the other hand, $[\mathcal{O}(\alpha)](U)$ is a continuous function on $SU(2)\backslash\{I\}$. Therefore, by

$$\mathcal{O}_\pi(a) \quad = \quad \int_{SU(2)} [\mathcal{O}(a)](U)\, dE_{\varphi xy}$$

we define a possibly unbounded operator with dense domain. Clearly, (interpreting a as an element of $\mathcal{A}_\mathrm{f}$) $\mathcal{O}(a)$ fulfills

$$\mathcal{O}(ab) = a\mathcal{O}(b) + \mathcal{O}(a)\delta(b).$$

By this cocycle property we conclude that a maximal common dense domain $\mathcal{D}$ of all the operators $\mathcal{O}_\pi(\mathcal{A}_1)$ is given by

$$\mathcal{D} \quad = \quad \mathcal{D}_\gamma \quad = \quad \mathcal{D}_{\gamma^*},$$

where $\mathcal{D}_{\gamma^{(*)}}$ denote the domains of $\pi(\gamma^{(*)})$. Obviously, $\mathcal{D}$ consists of all vectors η, for which

$$\int_{SU(2)} \left|[\mathcal{O}(\gamma^{(*)})](U)\right|^2 d\langle\eta|E_{\varphi xy}|\eta\rangle \quad = \quad \int_{SU(2)} \frac{x^2 + y^2}{|\sqrt{1 - x^2 - y^2 e^{-i\varphi}} - 1|^2}\, d\langle\eta|E_{\varphi xy}|\eta\rangle \;<\; \infty.$$

Clearly, for any vector $\eta_{\alpha^*} \in \mathcal{D}$ we define a cocycle η by setting

$$\eta \quad = \quad \mathcal{O}_\pi \eta_{\alpha^*} \tag{2.11}$$

which fulfills $\eta(\alpha^*) = \eta_{\alpha^*}$.

On the other hand, if η is a given cocycle assuming the value η_{α^*} on α^*, it follows immediately from Relations (2.6) that

$$\eta(\gamma^{(*)}) \quad = \quad \lim_{\epsilon \to 0} \int_{SU(2)\backslash U_\epsilon(I)} [\mathcal{O}(\gamma^{(*)})](U)\, dE_{\varphi xy}\, \eta_{\alpha^*},$$

i.e. $\eta_{\alpha^*} \in \mathcal{D}$. Thus, we obtain

Theorem 2.8 *Let π be a representation of $\mathcal{A}_1$ on a Hilbert space H without a gaußian part, and $\mathcal{D}$ the (dense) subspace of H as defined above. By (2.11) we establish a one-to-one corresondence between cocycles with respect to π and vectors $\eta_{\alpha^*} \in \mathcal{D}$.*

Proceeding as in [5], we define

$$\mathcal{T}_\pi(a) \quad = \quad \int_{SU(2)} [\mathcal{T}(a)](U)\, dE_{\varphi xy},$$

where the function $\mathcal{T}(a)$ on $SU(2)$ is given by

$$[\mathcal{T}(a)](U) \quad = \quad \frac{\delta_{\varphi xy} \circ \mathcal{P}_1(a)}{|\sqrt{1 - x^2 - y^2 e^{-i\varphi}} - 1|^2}.$$

Clearly, $\mathcal{T}_\pi$ and $\mathcal{O}_\pi$ fulfill

$$\mathcal{T}(ab) = \mathcal{O}(a^*)^* \mathcal{O}(b) \qquad (a, b \in K_1).$$

Therefore, we obtain that any cocycle (2.11) defines via (2.2) the values of a conditionally positive functional $\psi = \langle\eta_{\alpha^*}|\mathcal{T}_\pi|\eta_{\alpha^*}\rangle$ on K_2. (Notice that the domain of $\mathcal{T}_\pi$ is smaller than $\mathcal{D}$. However, it is obvious that this domain can be extended to $\mathcal{D}$ if we interprete $\mathcal{T}_\pi$ as mapping into $\mathcal{D}^*$ being the dual of $\mathcal{D}$.) By

$$d\mu_{\varphi xy} \quad = \quad \frac{d\langle\eta_{\alpha^*}|E_{\varphi xy}|\eta_{\alpha^*}\rangle}{|\sqrt{1 - x^2 - y^2 e^{-i\varphi}} - 1|^2}$$

we define a positive regular not necessarily finite measure on $SU(2)$. We obtain Hunt's formula for $SU(2)$.

Theorem 2.9 *The formula*

$$\psi \quad = \quad \psi_\delta + \int_{SU(2)} \delta_{\varphi xy} \circ \mathcal{P}_1 \, d\mu_{\varphi xy}$$

establishes a one-to-one correspondence between conditionally positive functionals on $\mathcal{A}_1$, and pairs (ψ_δ, μ) consisting of a gaußian part ψ_δ and a positive regular measure μ on $SU(2)$, fulfilling

$$\int_{SU(2)} (x^2+y^2)\, d\mu_{\varphi xy} \;<\; \infty \quad \textit{and} \quad \int_{SU(2)} |\sqrt{1-x^2-y^2}e^{-i\varphi}-1|^2\, d\mu_{\varphi xy} \;<\; \infty.$$

3 The case $q=-1$

Now we investigate the anti-classical case, where $q=-1$. We obtain a result looking very similar to Theorem 2.9. Contrary to the classical case, not the gaußian part, but, the anti-gaußian part is the complicated one. In both parts of Hunt's formula we have to replace, more or less, the family $\delta_{\varphi xy}$ of states by suitable states of the form Tr $\widehat{m}(\varphi,x,y)\hat{\delta}_{\varphi xy}$ where $\hat{\delta}_{\varphi xy}$ is a family of two-dimensional representations and $\widehat{m}(\varphi,x,y)$ is a measurable function on $SU(2)$ with values in the positive 2×2–matrices of unit trace.

First, let us agree on some notation. Let π be a representation of $\mathcal{A}_q$. By $\underline{\pi}$ we denote the representation defined by

$$\underline{\pi}(\alpha) \;=\; -\,\pi(\alpha) \quad \text{and} \quad \underline{\pi}(\gamma) \;=\; -\,\pi(\gamma).$$

Clearly, we again have a one-parameter family δ_φ of homomorphisms, mapping α to $e^{i\varphi}$ and γ to 0, and the derivatives δ' and δ''. By the same arguments as for $q\in(-1,1)$ it follows that K_2 is of codimension 1 in K_1 and the projection $\mathcal{P}$, extended to $q=-1$, is again a projection onto K_2. We denote this projection by $\mathcal{P}_{-1}$. As an immediate consequence we obtain by Lemma 2.5 the form of the gaußian part is given like in Proposition 2.2.

The main difference compared to the classical case becomes apparent, if we have a look at $\underline{\delta}=\delta_\pi$. A representation $\underline{\delta}\mathbf{1}_{H_{\underline{\delta}}}$ on a Hilbert space $H_{\underline{\delta}}$, a cocycle with respect to such a representation, and a conditionally positive functional associated with such a cocycle we call anti-gaußian. Obviously, for any choice of $\eta_{\alpha^*},\eta_\gamma,\eta_{\gamma^*}\in H_{\underline{\delta}}$ we define an anti-gaußian cocycle η, by setting

$$\eta(\alpha^*) \;=\; \eta(\alpha) \;=\; \eta_{\alpha^*} \quad , \quad \eta(\gamma^{(*)}) \;=\; \eta_{\gamma^{(*)}},$$

and $\eta(a)=0$ for $a\in K_2$. Hence, an anti-gaußian cocycle is not determined by its value on α^* alone.

We define a family of representations $\hat{\delta}_{\varphi xy}$ on $\mathbf{C}^2$, which is labeled by elements of $SU(2)$, by setting

$$\begin{aligned} \hat{\delta}_{\varphi xy}(\alpha) &= \sqrt{1-x^2-y^2}e^{i\varphi}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \hat{\delta}_{\varphi xy}(\gamma) &= (x+iy)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

It is easy to check that two different members $\hat{\delta}_{\varphi xy},\hat{\delta}_{\varphi' x' y'}$ of the family are unitarily equivalent, if and only if either $\varphi'-\varphi=(2n+1)\pi$ or $x'=-x$, $y'=-y$ or both. (A unitary equivalence transform leaves invariant the determinant. Therefore, in two dimensions the factors in front of the matrices can only differ by sign. On the other hand, choosing the unitary transforms $u=\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $u=\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ we indeed obtain the sign changes of $\hat{\delta}_{\varphi xy}(\gamma)$ and $\hat{\delta}_{\varphi xy}(\alpha)$, respectively.)

The representation belonging to the neutral element we denote by

$$\hat{\delta} \;=\; \hat{\delta}_{000}.$$

The partial derivatives of $\hat{\delta}_{\varphi xy}$ at 0 are defined in the same manner as those of $\delta_{\varphi xy}$ in the preceding section. Notice that $\hat{\delta}=\delta\oplus\underline{\delta}$. We denote a basis of $\mathbf{C}^2$ by $\hat{e}_1=(1,0)$ and $\hat{e}_2=(0,1)$. Clearly, we have $\hat{\delta}\hat{e}_1=\hat{e}_1\delta$ and $\hat{\delta}\hat{e}_2=\hat{e}_2\underline{\delta}$. Therefore, for any $r\in\mathbf{R}^3$ with components (φ,x,y),

$$a \quad\longmapsto\quad \langle \hat{e}_2|\hat{\delta}'^r(a)|\hat{e}_1\rangle$$

defines a cocycle with respect to $\underline{\delta}$. Clearly, $\langle \hat{e}_1|\hat{\delta}'^r|\hat{e}_1\rangle$ is a cocycle with respect to δ, hence, must be given by a multiple of δ'. By evaluating at α, we obtain $\langle \hat{e}_1|\hat{\delta}'^r|\hat{e}_1\rangle = \varphi\delta'$, with φ being the first component of r. Furthermore, we have for $a, b \in K_1$ that

$$\begin{aligned} \langle \hat{e}_1|\hat{\delta}''^r(ab)|\hat{e}_1\rangle &= 2\sum_{i=1}^{2}\langle \hat{e}_1|\hat{\delta}'^r(a)|\hat{e}_i\rangle\langle \hat{e}_i|\hat{\delta}'^r(b)|\hat{e}_1\rangle \\ &= 2\langle \hat{e}_2|\hat{\delta}'^r(a^*)|\hat{e}_1\rangle^*\langle \hat{e}_2|\hat{\delta}'^r(b)|\hat{e}_1\rangle \ + \ \varphi^2\delta''(ab). \end{aligned} \tag{3.12}$$

In other words,

$$\frac{\langle \hat{e}_1|\hat{\delta}''^r|\hat{e}_1\rangle}{2} - \varphi^2\frac{\delta''}{2}$$

is a conditionally positive functional fulfilling (2.2).

In the classical case the cocycles corresponding to $r = (1,0,0), (0,1,0), (0,0,1)$ are linearly independent. However, we immediately see that in the anti-classical case the cocycle $\langle \hat{e}_2|\hat{\delta}'^\varphi|\hat{e}_1\rangle$ is identically 0. We can obtain a third linearly independent one-dimensional cocycle with respect to $\underline{\delta}$ by

$$\underline{\delta}\circ(Id - \delta\mathbf{1}) \ = \ \langle \hat{e}_2|\hat{\delta}|\hat{e}_2\rangle\circ(Id-\delta\mathbf{1}) \ = \ \underline{\delta} - \delta.$$

We obtain the analogue of Theorem 2.6.

Theorem 3.1 *By*

$$\eta_{\underline{\delta}} \ = \ \underline{\delta}\circ(Id-\delta\mathbf{1})\eta_\varphi + \langle \hat{e}_2|\hat{\delta}'^x|\hat{e}_1\rangle\eta_x + \langle \hat{e}_2|\hat{\delta}'^y|\hat{e}_1\rangle\eta_y,$$

we establish a one-to-one correspondence between anti-gaußian cocycles $\eta_{\underline{\delta}}$ and triplets $(\eta_\varphi, \eta_x, \eta_y)$ of vectors $\eta_\varphi, \eta_x, \eta_y \in H_{\underline{\delta}}$.

In the case of a one-dimensional cocycle we again find by commutativity of γ, γ^* that the complex numbers η_x and η_y must have the same phase factor if $\eta_{\underline{\delta}}$ determines the values of a conditionally positive functional on K_2. Now let $r = (\varphi, x, y)$ be an element of $\mathbf{C}\times\mathbf{R}^2$. By straightforward verification on elements of K_1 we see that

$$\left\langle \hat{e}_2 \left| \hat{\delta}_{0(tx)(ty)} \right| \frac{\hat{e}_1 + t\varphi\hat{e}_2}{t} \right\rangle \circ (Id - \delta\mathbf{1}) \quad \longrightarrow \quad \underline{\eta}_r$$

as $t > 0$ tends to 0 where $\underline{\eta}_r$ is the cocycle $\eta_{\underline{\delta}}$ having $\eta_\varphi = \varphi \in \mathbf{C}$ and $\eta_{x/y} = x/y \in \mathbf{R}$. Notice that

$$\left\langle \hat{e}_1 \left| \hat{\delta}_{0(tx)(ty)} \right| \frac{\hat{e}_1 + t\varphi\hat{e}_2}{t} \right\rangle \circ (Id - \delta\mathbf{1}) \ \longrightarrow \ \langle \hat{e}_1|\hat{\delta}'^{(0,x,y)}|\hat{e}_1\rangle + \varphi\langle \hat{e}_1|\hat{\delta}|\hat{e}_2\rangle\circ(Id-\delta\mathbf{1}) \ = \ 0$$

for $t \to 0$. Therefore, we find by computations similar to (3.12) that

$$\underline{\psi}_r \ = \ \lim_{t\to 0}\left\langle \frac{\hat{e}_1 + t\varphi\hat{e}_2}{t} \left| \hat{\delta}_{0(tx)(ty)} \right| \frac{\hat{e}_1 + t\varphi\hat{e}_2}{t} \right\rangle \circ \mathcal{P}_{-1}$$

defines a conditionally positive functional fulfilling (2.2). We obtain by a proof completely similar to that of Theorem 2.7

Theorem 3.2 *For any anti-gaußian cocycle $\eta_{\underline{\delta}}$ which determines the values of a conditionally positive functional on K_2 there are three orthonormal vectors $\eta_1, \eta_2, \eta_3 \in H_{\underline{\delta}}$ and three vectors $r_1, r_2, r_3 \in \mathbf{C}\times\mathbf{R}^2$ such that*

$$\eta_{\underline{\delta}} \ = \ \underline{\eta}_{r_1}\eta_1 + \underline{\eta}_{r_2}\eta_2 + \underline{\eta}_{r_3}\eta_3.$$

Choosing arbitrary real numbers r_0, r_α we obtain all conditionally positive functionals fulfilling (2.2) in the form

$$\psi_{\underline{\delta}} \ = \ r_0\delta + r_\alpha\delta' + \underline{\psi}_{r_1} + \underline{\psi}_{r_2} + \underline{\psi}_{r_3}.$$

In [8] we proved the following representation theorem.

Theorem 3.3 *Let π be any $*$–representation of $\mathcal{A}_{-1}$ on a Hilbert space H. There is a spectral measure $dE_{\varphi xy}$ on $SU(2)$ with values in $\mathcal{B}(H)$, such that the representation $\widehat{\pi}$ on $\mathbf{C}^2 \otimes H$ defined by*

$$\widehat{\pi} \quad = \quad \int_{SU(2)} \hat{\delta}_{\varphi xy} \otimes dE_{\varphi xy}$$

is unitarily equivalent to $\pi \oplus \underline{\pi}$. In other words, there is a projection E in the commutant of $\widehat{\pi}(\mathcal{A}_{-1})$ such that the representation $E\widehat{\pi}$ on $E(\mathbf{C}^2 \otimes H)$ is unitarily equivalent to π.

Remark 3.1 *Consider the C^*–algebra $C(SU(2), M_{2\times 2})$ of continous functions on $SU(2)$ with values in $M_{2\times 2}$, equipped with the supremum norm $\|f\| = \sup_{SU(2)} \|f(U)\|$. Clearly, the $*$–subalgebra $\mathcal{A}_{-\mathrm{f}}$ of $C(SU(2), M_{2\times 2})$ generated by $f_\alpha(U) = \hat{\delta}_{\varphi xy}(\alpha)$ and $f_\gamma(U) = \hat{\delta}_{\varphi xy}(\gamma)$ can be identified with $\mathcal{A}_{-1}$ equipped with the norm $\| \bullet \| = \sup_{\pi} \|\pi(\bullet)\|$.*

Let $\eta = \sum_{i=1}^{2} \hat{e}_i \otimes \eta_i \in E(\mathbf{C}^2 \otimes H)$. We obtain

$$\begin{aligned} \langle \eta|\pi|\eta\rangle &= \sum_{i,j=1}^{2} \left\langle \hat{e}_i \otimes \eta_i \left| \int_{SU(2)} \hat{\delta}_{\varphi xy} \otimes dE_{\varphi xy} \right| \hat{e}_j \otimes \eta_j \right\rangle \\ &= \mathrm{Tr} \int_{SU(2)} \hat{\delta}_{\varphi xy}\, d\hat{\nu}_{\varphi xy}, \end{aligned}$$

where we introduced the measure valued, self-adjoint matrix $(d\hat{\nu}_{\varphi xy})_{ij} = d\langle \eta_j|dE_{\varphi xy}|\eta_i\rangle$. Notice that the non-diagonal entries are complex. By Cauchy-Schwartz inequality we have

$$|(d\hat{\nu}_{\varphi xy})_{12}| \;\leq\; \sqrt{(d\hat{\nu}_{\varphi xy})_{11}(d\hat{\nu}_{\varphi xy})_{22}} \;\leq\; \frac{1}{2}((d\hat{\nu}_{\varphi xy})_{11} + (d\hat{\nu}_{\varphi xy})_{22}).$$

Therefore, the matrix entries $(d\hat{\nu}_{\varphi xy})_{ij}$ are all absolutely continuous with respect to the (positive, regular, finite) measure $d\nu_{\varphi xy} = (d\hat{\nu}_{\varphi xy})_{11} + (d\hat{\nu}_{\varphi xy})_{22}$. By an application of the *Radon-Nikodym theorem* we can write

$$d\hat{\nu} \quad = \quad \hat{n}\, d\nu,$$

where $\hat{n}$ is a ν–measurable function on $SU(2)$ with values in the positive 2×2–matrices of unit trace. We summarize.

Theorem 3.4 *Any state φ on $\mathcal{A}_{-1}$ can be expressed as*

$$\varphi \quad = \quad \int_{SU(2)} \mathrm{Tr}\ \hat{n}(\varphi, x, y)\hat{\delta}_{\varphi xy}\, d\nu_{\varphi xy}, \tag{3.13}$$

where ν is a probability measure on $SU(2)$ and $\hat{n}$ is a ν–measurable function on $SU(2)$ with values in the positive 2×2–matrices of unit trace.

It is possible to specify conditions under which the correspondence $\eta \longleftrightarrow \hat{n}, \nu$ is one-to-one; see [6, 8].

Now we proceed precisely as in the foregoing section. Assume that a given representation π has no gaußian and anti-gaußian part. We introduce the mapping $\hat{\mathcal{O}}$ from $\mathcal{A}_{-1}$ into the measurable functions on $SU(2)$ with values in the 2×2–matrices by setting

$$\begin{aligned} [\hat{\mathcal{O}}(a)](U) &= (\hat{\delta}_{\varphi xy}(a) - \delta(a)\mathbf{1}_{M_{2\times 2}}) \left[\sqrt{1-x^2-y^2}e^{-i\varphi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]^{-1} \\ &= \frac{\hat{\delta}_{\varphi xy}(a) - \delta(a)\mathbf{1}_{M_{2\times 2}}}{(1-x^2-y^2)e^{-2i\varphi} - 1} \begin{pmatrix} 1 + \sqrt{1-x^2-y^2}e^{-i\varphi} & 0 \\ 0 & 1 - \sqrt{1-x^2-y^2}e^{-i\varphi} \end{pmatrix} \end{aligned}$$

and define the possibly unbounded operator

$$\hat{\mathcal{O}}_\pi(a) \quad = \quad \int_{SU(2)} [\hat{\mathcal{O}}(a)](U) \otimes dE_{\varphi xy}.$$

Similar to the classical case, a vector $\eta_{\alpha^*} = \sum_{i=1}^{2} \hat{e}_i \otimes \eta_{\alpha^* i} \in E(\mathbf{C}^2 \otimes H)$ defines a cocycle with respect to π, assuming this vector on α^*, by setting

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \quad = \quad \hat{\mathcal{O}}_\pi \begin{pmatrix} \eta_{\alpha^* 1} \\ \eta_{\alpha^* 2} \end{pmatrix},$$

if and only if the corresponding vector η_{α^*} is an element of $\widehat{\mathcal{D}}$, the domain of $\hat{\mathcal{O}}_\pi(\gamma^{(*)})$. This condition reads

$$\int_{SU(2)} \frac{x^2+y^2}{|\sqrt{1-x^2-y^2}e^{-i\varphi}-1|^2}\,(d\hat{\nu}_{\varphi xy})_{11} + \int_{SU(2)} \frac{x^2+y^2}{|\sqrt{1-x^2-y^2}e^{-i\varphi}+1|^2}\,(d\hat{\nu}_{\varphi xy})_{22} \quad < \quad \infty.$$

By

$$\psi \quad = \quad \left\langle \begin{pmatrix} \eta_{\alpha^* 1} \\ \eta_{\alpha^* 2} \end{pmatrix} \middle| \int_{SU(2)} [\hat{\mathcal{T}}(a)](U) \otimes dE_{\varphi xy} \middle| \begin{pmatrix} \eta_{\alpha^* 1} \\ \eta_{\alpha^* 2} \end{pmatrix} \right\rangle,$$

where we defined

$$\begin{aligned}[\hat{\mathcal{T}}(a)](U) \quad = \quad & \begin{pmatrix} 1+\sqrt{1-x^2-y^2}e^{i\varphi} & 0 \\ 0 & 1-\sqrt{1-x^2-y^2}e^{i\varphi} \end{pmatrix} \\ & \frac{\hat{\delta}_{\varphi xy} \circ \mathcal{P}_{-1}(a)}{|(1-x^2-y^2)e^{-2i\varphi}-1|^2} \begin{pmatrix} 1+\sqrt{1-x^2-y^2}e^{-i\varphi} & 0 \\ 0 & 1-\sqrt{1-x^2-y^2}e^{-i\varphi} \end{pmatrix},\end{aligned}$$

we find a conditionally positive functional satisfying (2.2).

We introduce the measure valued matrix $d\hat{\mu}_{\varphi xy}$ by

$$\begin{aligned}(d\hat{\mu}_{\varphi xy})_{11} \quad &= \quad \frac{(d\hat{\nu}_{\varphi xy})_{11}}{|1-\sqrt{1-x^2-y^2}e^{-i\varphi}|^2} \\ (d\hat{\mu}_{\varphi xy})_{22} \quad &= \quad \frac{(d\hat{\nu}_{\varphi xy})_{22}}{|1+\sqrt{1-x^2-y^2}e^{-i\varphi}|^2} \\ (d\hat{\mu}_{\varphi xy})_{12} \quad &= \quad \frac{(d\hat{\nu}_{\varphi xy})_{12}}{(1-\sqrt{1-x^2-y^2}e^{-i\varphi})(1+\sqrt{1-x^2-y^2}e^{i\varphi})} \\ (d\hat{\mu}_{\varphi xy})_{21} \quad &= \quad \overline{(d\hat{\mu}_{\varphi xy})_{12}}.\end{aligned}$$

Notice that the entries of $d\hat{\mu}_{\varphi xy}$ also fulfill Cauchy-Schwartz inequality. Therefore, we obtain again a (positive, regular) measure μ and a μ–measurable function $\widehat{m}$ with values in the positive 2×2–matrices of unit trace, such that $d\hat{\mu} = \widehat{m} d\mu$, and by Equation (3.13)

$$\psi \quad = \quad \int_{SU(2)} \operatorname{Tr}\, \widehat{m}(\varphi,x,y) \hat{\delta}_{\varphi xy} \circ \mathcal{P}_{-1}\, d\mu_{\varphi xy}.$$

This yields the analogue of Hunt's formula.

Theorem 3.5 *For an arbitrary conditionally positive functional on $\mathcal{A}_{-1}$ we can find a quadruple $(\psi_\delta, \psi_{\underline{\delta}}, \mu, \widehat{m})$ consisting of a gaußian part ψ_δ, an anti-gaußian part $\psi_{\underline{\delta}}$, a (not necessarily finite) measure μ on $SU(2)$, and a μ–measurable, positive 2×2–matrix valued function $\widehat{m}$ of unit trace, fulfilling*

$$\int_{SU(2)} (x^2+y^2)((\widehat{m}(\varphi,x,y))_{11} + (\widehat{m}(\varphi,x,y))_{22})\, d\mu_{\varphi xy} \quad < \quad \infty,$$

$$\int_{SU(2)} |\sqrt{1-x^2-y^2}e^{-i\varphi}-1|^2(\widehat{m}(\varphi,x,y))_{11}\,d\mu_{\varphi xy} \;<\; \infty$$

$$\int_{SU(2)} |\sqrt{1-x^2-y^2}e^{-i\varphi}+1|^2(\widehat{m}(\varphi,x,y))_{22}\,d\mu_{\varphi xy} \;<\; \infty$$

such that

$$\psi \;=\; \psi_\delta + \psi_{\underline{\delta}} + \int_{SU(2)} \operatorname{Tr}\, \widehat{m}(\varphi,x,y)\hat{\delta}_{\varphi xy} \circ \mathcal{P}_{-1}\,d\mu_{\varphi xy}.$$

4 Thoughts on quantization

Usually, algebras of classical observables are algebras of real functions on some (measurable, usually polish) phase space P. The expectation value of an observable $f\colon P \to \mathbf{R}$ is determined by a probability measure μ on P via $E_\mu f = \int_P f(x)\mu(dx)$. We say the system with phase space P is in the state E_μ. If μ is concentrated on a single point $x_0 \in P$, we recover the deterministic situation where any observable f takes the value $f(x_0)$ (with probability 1).

Usually, the algebra of classical observables is generated (algebraically, for simplicity) by a finite number of basic observables. For instance, the phase space of a classical particle in one dimension is $\mathbf{R}^2$. The basic observable 'position' of this particle is the function $q(x,y)=x$. And the basic observable 'momentum' of this particle is the function $p(x,y)=y$. Henceforth, the algebra of classical observables for this particle is the (unital) algebra of functions on $\mathbf{R}^2$ which is generated by the two functions q and p. Obviously, this algebra is isomorphic to the (non-commutative unital) algebra generated by two indeterminates q' and p' divided out by the relation $q'p'-p'q' = 0$. (Therefore, we will identify the functions q,p with the indeterminates q',p'.)

Most algebras of classical observables can be considered in this way, i.e. as algebras generated by (finitely many) indeterminates with (finitely many) relations. The relations have to asure at least that the algebra is commutative. However, there may be also other relations, if the generators are not (like q,p) generators of a free commutative algebra. (For instance, suppose that for some reason sombody wants to consider the subalgebra of polynomials in q,p with degree greater than 1. Such a person could define the commutative algebra generated by indeterminates $qq, qp, pp, qqq, qqp, qpp, ppp$. Of course, a couple of relations among these generators has to be divided out.)

A quantization of such an algebra of classical observables is obtained by modifying some of the relations. This, in general leads to non-commutative algebras. For instance, the one-dimensional particle is quantized by replacing $qp-pq=0$ with $qp-pq=i\hbar$. Immediately, we see that we leave the framework of real algebras. (The attempt to use real algabras comes from the wish that observables should take expectation values in the real numbers. However, formally there is no reason why we should not consider the algebra of complex observables obtained by complexification. In fact, it is well-known that, for instance, in the study of the harmonic oszillator the pair $q+ip, q-ip$ is very useful.) Henceforth, by algebra $\mathcal{A}$ we mean a unital $*$–algebra, i.e. a complex unital algebra with an involution $*$. Since in the non-commutative context there is no longer a probability measure, the expectation E_μ is replaced by a state φ, i.e. a normalized ($\varphi(\mathbf{1})=1$) positive ($\varphi(a^*a)\geq 0 \forall a \in \mathcal{A}$) linear functional on $\mathcal{A}$.

Usually, the quantization of an algebra of classical observables depends on a quantization parameter (e.g. $\hbar$) and the relations to be diveded out (e.g. $qp-pq=i\hbar$) from a freely generated algebra depend 'continuously' on this parameter. In order to have the right to speak of a quantization of a classical algebra, the quantized algebra should 'converge' in some sense to the classical algebra. What does this mean? In Definition 4.1 we explain what we understand by convergence of functionals on the quantization (which, actually, is a family of quantizations parametrized by the quantization parameter) to a functional on the classical algebra. In a 'good' quantization it should be possible to approximate a state on the classical algebra by states on the quantized version.

In Section 5 we show that in the above sense Woronowicz's quantized $SU(2)$ [10] is, indeed, a quantization of the compact Lie group $SU(2)$. Like in the case of the particle with observables q,p where the classical pure states may be approximated by coherent states on the quantization, pure states on the algebra of functions on $SU(2)$ may be approximated by q–coherent states. In Section 6 we point out that the uniform approximation of classical states by states on the quantization turns over to the approximation of classical infinitesimal generators by infinitesimal generators on the quantization. This means that not only expectations at a fixed time, but the whole dynamics of a white noise time evolution on $SU(2)$ may be approximated by white noise time evolutions on the quantization $SU_q(2)$.

Definition 4.1 *Let $n, m \in \mathbf{N}$. Let $\mathcal{F}$ denote the free unital algebra generated by the set $\mathcal{G} = \{g_1, \ldots, g_n\}$ of indeterminates. Let Q denote a subset of $\mathbf{R}$. For $q \in Q$ let $\mathcal{R}_q = \{R_1^q, \ldots, R_m^q\}$ denote a subset of $\mathcal{F}$. Let $\mathcal{I}_q$ denote the ideal generated by $\mathcal{R}_q$ and let $\mathcal{A}_q$ denote the quotient algebra $\mathcal{F}/\mathcal{I}_q$.*

We say the family $\mathcal{A}_q$ is continuous, if for all $i = 1, \ldots, m$ the mapping $q \mapsto R_i^q$ is weakly continuous. (In other words, there are continuous functions $f_i^g : Q \to \mathbf{C}$ for any $i = 1, \ldots, m$ and g a monomial in $\mathcal{F}$, such that for fixed i and q we have $f_i^g(q) \neq 0$ only for a finite number of g and $R_i^q = \sum_g f_i^g(q) g$.)

Let φ_q be a family of linear functionals on $\mathcal{A}_q$. Denote by $\mathcal{E}_q : \mathcal{F} \to \mathcal{A}_q$ the canonical mapping. We say φ_q is weakly continuous, if the family $\Phi_q = \varphi_q \circ \mathcal{E}_q$ of functionals on $\mathcal{F}$ depends weakly continuous on q.

Proposition 4.1 *Let the family $\mathcal{A}_q$ be continuous and let $q_0 \in Q$. Assume that the family φ_q of functionals on $\mathcal{A}_q$ ($q \in Q \backslash \{q_0\}$) is weakly continuous on $q \in Q \backslash \{q_0\}$. Furthermore, assume that the limit $\Phi_{q_0} = \lim_{q \to q_0} \Phi_q$ exists pointwise. Then Φ_{q_0} vanishes on $\mathcal{I}_q$, hence, gives rise to a functional φ_{q_0} on $\mathcal{A}_{q_0}$. Moreover, the family φ_q ($q \in Q$) is weakly continuous.*

Proof The remaining statements being obvious, we only show that Φ_{q_0} vanishes on $\mathcal{I}_{q_0}$. It is sufficient to show this on elements of the form $a R_i^{q_0} b$ ($a, b \in \mathcal{F}; 1 \leq i \leq m$), because $\mathcal{I}_{q_0}$ is spanned by such elements. Of course, there exists a neighbourhood of q_0 on which f_i^g is different from zero only for finitely many monomials g. We have

$$0 = \Phi_q(a R_i^q b) = \Phi_q(a R_i^{q_0} b) + \Phi_q(a R_i^q b - a R_i^{q_0} b) = \Phi_q(a R_i^{q_0} b) + \sum_g (f_i^g(q) - f_i^g(q_0)) \Phi_q(agb).$$

Our assertion follows from the fact that the first summand converges to $\Phi_{q_0}(a R_i^{q_0} b)$ and the second summand converges to 0. ■

Remark 4.1 *Obviously, properties of the family φ_q on $Q \backslash \{q_o\}$ like positivity or conditional positivity (with respect to fixed character δ on $\mathcal{F}$ vanishing on all $\mathcal{I}_q$) are preserved in the limit φ_{q_0}.*

5 Approximation of states

Let λ, μ be complex numbers. From $|\lambda - \overline{\mu}|^2 > 0$, if and only if $\lambda \neq \overline{\mu}$, we conclude that

$$\lim_{t \to \infty} \frac{e^{t\lambda\overline{\mu}} e^{t\overline{\lambda}\mu}}{e^{t|\lambda|^2} e^{t|\mu|^2}} = \begin{cases} 1 & \text{for } \lambda = \overline{\mu} \\ 0 & \text{otherwise.} \end{cases}$$

We prove a q–analogue (see Appendix A).

Proposition 5.1 *For $\lambda, \mu \in U_1(0)$ the q–exponential fulfills*

$$\lim_{q \to 1} \frac{e_q^{\lambda\overline{\mu}} e_q^{\overline{\lambda}\mu}}{e_q^{|\lambda|^2} e_q^{|\mu|^2}} = \begin{cases} 1 & \textit{for } \lambda = \overline{\mu} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof For $\lambda = \overline{\mu}$ the statement is clear. Thus, let $\lambda \neq \overline{\mu}$. By the product representation of e_q^z we obtain

$$\frac{e_q^{\lambda\overline{\mu}} e_q^{\overline{\lambda}\mu}}{e_q^{|\lambda|^2} e_q^{|\mu|^2}} = \prod_{k=0}^{\infty} \frac{(1 - q^k|\lambda|^2)(1 - q^k|\mu|^2)}{(1 - q^k\lambda\overline{\mu})(1 - q^k\overline{\lambda}\mu)} = \prod_{k=0}^{\infty} \left(1 - \frac{q^k(\lambda - \mu)\overline{(\lambda - \mu)}}{(1 - q^k\lambda\overline{\mu})(1 - q^k\overline{\lambda}\mu)}\right). \tag{5.1}$$

All factors lie in $[0, 1)$. Therefore, so does the product. On the other hand, for $q \to 1$ the factors converge to

$$1 - \frac{(\lambda - \mu)\overline{(\lambda - \mu)}}{(1 - \lambda\overline{\mu})(1 - \overline{\lambda}\mu)}$$

which is less than 1. Since the product contains an arbitrary number of factors close to this limit, if only q is sufficiently close to 1, the product will be smaller than any positive number. ■

Roughly speaking, the unit vectors

$$\hat{e}_{q^2}(\lambda) \quad = \quad \frac{e_{q^2}(\lambda)}{\sqrt{e_{q^2}^{|\lambda|^2}}}$$

(see Equation (B.1)) 'become orthogonal' if q tends to 1. We use this property in order to approximate irreducible states on $\mathcal{A}_{\pm 1}$.

Proposition 5.2 *Let (φ, x, y) be a point in $SU(2)$ such that $0 < x^2+y^2 < 1$. Furthermore, let $c = (c_1, c_2)$ be a unit vector in $\mathbf{C}^2$. Denote by $c \odot \hat{e}_{q^2}(\lambda)$ the unitvector*

$$c \odot \hat{e}_{q^2}(\lambda) \quad = \quad \frac{c_1 \hat{e}_{q^2}(\lambda) + c_2 \hat{e}_{q^2}(-\lambda)}{\sqrt{1 + (\overline{c_1} c_2 + \overline{c_2} c_1) \frac{e_{q^2}^{-|\lambda|^2}}{e_{q^2}^{|\lambda|^2}}}}$$

in h_0 where $\lambda = \sqrt{1 - x^2 - y^2} e^{i\varphi}$, $\chi = \arg(x + iy)$. Then we have

(i)

$$\lim_{q \to 1} \langle \hat{e}_{q^2}(\lambda) | \rho_\chi | \hat{e}_{q^2}(\lambda) \rangle \quad = \quad \delta_{\varphi x y}.$$

(ii)

$$\lim_{q \to -1} \langle c \odot \hat{e}_{q^2}(\lambda) | \rho_\chi | c \odot \hat{e}_{q^2}(\lambda) \rangle \quad = \quad \langle c | \hat{\delta}_{\varphi x y} | c \rangle.$$

Proof Assume, for the moment, that $\chi = 0$. Then, since ρ_0 does not distinguish between γ and γ^* and by hermiticity, it follows that it is sufficient to prove the statement for $\gamma^n \alpha^m$ with $n, m \in \mathbf{N}_0$.

We have $\rho_0(\alpha) e_{q^2}(\lambda) = \lambda e_{q^2}(\lambda)$ and $\rho_0(\gamma) e_{q^2}(\lambda) = e_{q^2}(q\lambda)$ (see Appendix B). Therefore, α^m gives a factor $(\pm\lambda)^m$ and γ^n gives a factor q^n in the argument of the q–coherent state. Thus, denoting the normalization factor in the second case by $|c|_\odot$, we have to compute the expressions

$$\lim_{q \to 1} \lambda^m \frac{\langle e_{q^2}(\lambda) | e_{q^2}(q^n \lambda) \rangle}{e_{q^2}^{|\lambda|^2}} \quad = \quad \lim_{q \to 1} \lambda^m \frac{e_{q^2}^{q^n |\lambda|^2}}{e_{q^2}^{|\lambda|^2}}$$

in the first case, and

$$\lim_{q \to -1} \frac{1}{|c|_\odot^2 e_{q^2}^{|\lambda|^2}} \Big(|c_1|^2 \lambda^m \langle e_{q^2}(\lambda) | e_{q^2}(q^n \lambda) \rangle \; + \; \overline{c_1} c_2 (-\lambda)^m \langle e_{q^2}(\lambda) | e_{q^2}(-q^n \lambda) \rangle \; +$$

$$+ \; \overline{c_2} c_1 \lambda^m \langle e_{q^2}(-\lambda) | e_{q^2}(q^n \lambda) \rangle \; + \; |c_2|^2 (-\lambda)^m \langle e_{q^2}(-\lambda) | e_{q^2}(-q^n \lambda) \rangle \Big)$$

$$= \lim_{q \to 1} \frac{(|c_1|^2 \lambda^m + |c_2|^2 (-\lambda)^m) e_{q^2}^{q^n (-1)^n |\lambda|^2} + (\overline{c_1} c_2 (-\lambda)^m + \overline{c_2} c_1 \lambda^m) e_{q^2}^{q^n (-1)^{n+1} |\lambda|^2}}{|c|_\odot^2 e_{q^2}^{|\lambda|^2}}$$

in the second case where we transformed the limit $q \to -1$ into a limit $q \to 1$. The recursion formula for the q–exponential reads

$$e_{q^2}^{\pm q^{n+2} |\lambda|^2} \quad = \quad (1 \mp q^n |\lambda|^2) e_{q^2}^{\pm q^n |\lambda|^2}.$$

In the first case we see that

$$\lim_{q \to 1} \frac{e_{q^2}^{q^{2n} |\lambda|^2}}{e_{q^2}^{|\lambda|^2}} \; = \; (1 - |\lambda|^2)^n \quad \text{and} \quad \lim_{q \to 1} \frac{e_{q^2}^{q^{2n+1} |\lambda|^2}}{e_{q^2}^{|\lambda|^2}} \; = \; (1 - |\lambda|^2)^n \lim_{q \to 1} \frac{e_{q^2}^{q |\lambda|^2}}{e_{q^2}^{|\lambda|^2}},$$

if the latter limit exists. For the second case we consider

$$e_{q^2}^{-q^n|\lambda|^2} \quad < \quad e_{q^2}^{-q^{n+1}|\lambda|^2} \tag{5.2}$$

for $q \in (0,1)$. Since $|\lambda| \neq 0$, Proposition 5.1 yields

$$\frac{e_{q^2}^{-|\lambda|^2}}{e_{q^2}^{|\lambda|^2}} \quad \to \quad 0$$

as $q \to 1$. Therefore, $|c|_{\odot}$ converges to 1. It follows by (5.2) and the recursion formula that

$$\frac{e_{q^2}^{\pm(-1)^n q^n|\lambda|^2}}{e_{q^2}^{|\lambda|^2}} \quad \to \quad 0 \quad \text{for} \quad \pm(-1)^n = -1.$$

For $\pm(-1)^n = 1$ we obtain the same numbers as in the first case. Thus, it remains to calculate

$$\lim_{q\to 1} \frac{e_{q^2}^{q|\lambda|^2}}{e_{q^2}^{|\lambda|^2}}.$$

We define

$$F_q(x) \quad = \quad \frac{e_{q^2}^{qx}}{e_{q^2}^{x}}.$$

By looking at the factors $\frac{1-q^{2k}x}{1-q^{2k+1}x}$ of the infinite product we easily find that F_q is a strictly decreasing function of $x \in (0,1)$, i.e. $F_q(x)F_q(qx) < F_q(qx)F_q(qx) < F_q(qx)F_q(q^2x)$. Since $F_q(x)F_q(qx) = 1-x$, we have

$$1 - x \; < \; F_q(qx)^2 \; < \; 1 - qx. \tag{5.3}$$

Therefore,

$$\lim_{q\to 1} F_q(x) \quad = \quad \lim_{q\to 1} \frac{1-x}{F_q(qx)} \quad = \quad \sqrt{1-x},$$

and, henceforth,

$$\lim_{q\to 1} \frac{e_{q^2}^{q^n|\lambda|^2}}{e_{q^2}^{|\lambda|^2}} \quad = \quad \sqrt{1-|\lambda|^2}^{\,n}.$$

We insert this and obtain

$$\lim_{q\to 1} \langle \hat{e}_{q^2}(\lambda)|\rho_0(\gamma^n\alpha^m)|\hat{e}_{q^2}(\lambda)\rangle \quad = \quad \lambda^m \sqrt{1-|\lambda|^2}^{\,n}$$

and

$$\lim_{q\to -1} \langle c \odot \hat{e}_{q^2}(\lambda)|\rho_0(\gamma^n\alpha^m)|c \odot \hat{e}_{q^2}(\lambda)\rangle \quad = \quad \lambda^m \sqrt{1-|\lambda|^2}^{\,n} \begin{cases} |c_1|^2 + |c_2|^2(-1)^m & \text{for } n \text{ even} \\ \overline{c_1}c_2(-1)^m + \overline{c_2}c_1 & \text{otherwise} \end{cases}$$

which is the claimed result for $\chi = 0$.

The general case can be obtained by multiplying with the factor $e^{i\chi}$ for each γ and $e^{-i\chi}$ for each γ^*. ■

The excluded cases $|\lambda_0| = 0, 1$ can be obtained as the limit $\lambda \to \lambda_0$ of the above expressions. In our next step we include these cases by replacing λ with a function $\lambda(q)$ which converges to λ_0 as q tends to ± 1.

Notice that the approximating expressions, in (i) and (ii) of the foregoing proposition, are analytic functions in the variable $|\lambda|^2$ and can be continued analytically to $|\lambda|^2 < \frac{1}{1-q}$. For a given $\lambda_0 = e^{i\varphi}|\lambda_0| \in \overline{U_1(0)}$ we introduce the function

$$\lambda(q) \quad = \quad \begin{cases} \lambda_0 & \text{for } \left(1 - \frac{1}{2}|\lambda_0|^2\right)^{-\frac{\ln 2}{\ln q^2}} < 1 - q^2 \\ e^{i\varphi}(|\lambda_0| + \delta\lambda) & \text{otherwise} \end{cases}$$

on $\frac{1}{2} \leq q^2 < 1$, where $\delta\lambda$ is a non-negative real number such that

$$\left(1 - \frac{1}{2}(|\lambda_0| + \delta\lambda)^2\right)^{-\frac{\ln 2}{\ln q^2}} \quad = \quad 1 - q^2.$$

We explain why this is well-defined. Notice that $\kappa = -\frac{\ln 2}{\ln q^2}$ is a positive real number which tends to infinity as q tends to ± 1. Therefore, if the first case is not true, it is always possible to find a unique $\delta\lambda$ such that the second case is fulfilled. The lower boundary for q^2 guarantees that $|\lambda(q)| \leq 1$. (If $q^2 = \frac{1}{2}$, we have $\kappa = 1$ and $1 - q^2 = \frac{1}{2}$, i.e. $|\lambda_0| + \delta\lambda = 1$.) Obviously, $\delta\lambda$ converges to 0 as $q \to \pm 1$. The worst case for this convergence is $\lambda_0 = 0$, i.e. the convergence is uniformly in λ_0. We collect the properties of $\lambda(q)$.

Proposition 5.3 *For the function $\lambda(q)$ on $\frac{1}{2} \leq q^2 < 1$ which is assigned to any $\lambda_0 =\in \overline{U_1(0)}$ by the above definition the following holds.*

(i) $$0 < |\lambda(q)| \leq 1.$$

(ii) $\lambda(q)$ *is continuous.*

(iii) $$\lim_{q \to \pm 1} \lambda(q) = \lambda_0 \text{ uniformly in } \lambda_0.$$

(iv) *For any $\lambda_0 \neq 0$ we even have $\lambda(q) = \lambda_0$ for q sufficiently close to ± 1.*

(v) $$\left(1 - \frac{1}{2}|\lambda(q)|^2\right)^{-\frac{\ln 2}{\ln q^2}} \leq 1 - q^2 \text{ for all } \lambda_0.$$

After these preparations, we prove the following

Theorem 5.4 *Let (φ, x, y) be a point in $SU(2)$ and set $\lambda_0 = \sqrt{1 - x^2 - y^2}e^{i\varphi}$, $\chi = \arg(x + iy)$. Then we have*

(i)

$$\lim_{q \to 1} \langle \hat{e}_{q^2}(\lambda(q)) | \rho_\chi | \hat{e}_{q^2}(\lambda(q)) \rangle \quad = \quad \delta_{\varphi x y}.$$

(ii)

$$\lim_{q \to -1} \langle c \odot \hat{e}_{q^2}(\lambda(q)) | \rho_\chi | c \odot \hat{e}_{q^2}(\lambda(q)) \rangle \quad = \quad \langle c | \hat{\delta}_{\varphi x y} | c \rangle.$$

uniformly *in λ_0 and in the unit vector $c = (c_1, c_2) \in \mathbf{C}^2$.*

Proof Consider the proof of Proposition 5.2. The expressions to be calculated are the same except that the fixed number λ is replaced everywhere by $\lambda(q)$. The expressions are linear combinations of the functions $\frac{1}{|c|^2_\odot}$, $\frac{F_q(|\lambda|^2)}{|c|^2_\odot}$, and $\frac{e_{q^2}^{-|\lambda|^2}}{|c|^2_\odot e_{q^2}^{|\lambda|^2}}$ where the coefficients are polynomials $P(q, \lambda, \overline{\lambda})$. If we insert $\lambda(q)$, these coefficients assume their limits $P(\pm 1, \lambda_0, \overline{\lambda_0})$ uniformly in λ_0 because λ_0 does so. Of course, $|c|^2_\odot$

assumes its limit 1 uniformly in λ_0, if $\frac{e_{q^2}^{-|\lambda(q)|^2}}{e_{q^2}^{|\lambda(q)|^2}}$ does so. Thus, our proof is complete if we show that $\lim_{q\to 1} F_q(|\lambda(q)|^2) = \sqrt{1-|\lambda_0|^2}$ and $\lim_{q\to 1} \frac{e_{q^2}^{-|\lambda(q)|^2}}{e_{q^2}^{|\lambda(q)|^2}} = 0$ uniformly in λ_0.

Consider (5.3) which holds for $0 < x < 1$. We insert $F_q(qx) = \frac{1-x}{F_q(x)}$ and obtain

$$\frac{1-x}{\sqrt{1-qx}} \;<\; F_q(x) \;<\; \frac{1-x}{\sqrt{1-x}}.$$

This inequality also holds for $x = 1$, if we change the $<$ signs to $\leq$. Therefore,

$$\begin{aligned} |\sqrt{1-x} - F_q(x)| &\leq (1-x)\left(\frac{1}{\sqrt{1-x}} - \frac{1}{\sqrt{1-qx}}\right) = \frac{1-x}{\sqrt{1-x}\sqrt{1-qx}}(\sqrt{1-qx} - \sqrt{1-x}) \\ &= \sqrt{\frac{1-x}{1-qx}}\frac{x(1-q)}{\sqrt{1-qx}+\sqrt{1-x}}. \end{aligned}$$

We easily check that the function $\frac{x(1-x)}{1-qx}$ of x has a unique maximum on $(0,1)$ at $x_0 = \frac{1-\sqrt{1-q}}{q}$. Thus, $1 - x_0 = \sqrt{1-q}\frac{1-\sqrt{1-q}}{q}$ and $1 - qx_0 = \sqrt{1-q}$. Therefore,

$$\sqrt{\frac{x(1-x)}{1-qx}} \;\leq\; \frac{1-\sqrt{1-q}}{q} \;=\; x_0.$$

We obtain

$$|\sqrt{1-x} - F_q(x)| \;\leq\; \frac{1-\sqrt{1-q}}{q}\frac{\sqrt{x}(1-q)}{\sqrt{1-qx}+\sqrt{1-x}} \;\leq\; \frac{\sqrt{1-q}}{q}.$$

This is the uniform convergence of $F_q(x) \to \sqrt{1-x}$.

κ was given by $\kappa = -\frac{\ln 2}{\ln q^2}$. Obviously we have $q^{2\kappa} = \frac{1}{2}$. By $[\kappa]$ we denote the greatest integer less than or equal to κ. We have

$$q^{2k} \;\geq\; \frac{1}{2} \text{ for } k \leq [\kappa].$$

From (5.1) we obtain

$$\begin{aligned} \left(\frac{e_{q^2}^{-|\lambda|^2}}{e_{q^2}^{|\lambda|^2}}\right)^2 &= \prod_{k=0}^{\infty}\left(1 - \frac{4q^{2k}|\lambda|^2}{(1+q^{2k}|\lambda|^2)^2}\right) \leq \prod_{k=0}^{\infty}(1-q^{2k}|\lambda|^2) \\ &\leq \prod_{k=0}^{[\kappa]}(1-q^{2k}|\lambda|^2) \leq \left(1-\frac{1}{2}|\lambda|^2\right)^{[\kappa]+1} \leq \left(1-\frac{1}{2}|\lambda|^2\right)^{\kappa} \end{aligned}$$

for all λ. We insert $\lambda(q)$ and obtain

$$\frac{e_{q^2}^{-|\lambda(q)|^2}}{e_{q^2}^{|\lambda(q)|^2}} \;\leq\; \sqrt{\left(1-\frac{1}{2}|\lambda(q)|^2\right)^{\kappa}} \;\leq\; \sqrt{1-q^2} \;=\; \sqrt{1+q}\sqrt{1-q}$$

for $q^2 > \frac{1}{2}$. This is the uniform convergence of $\frac{e_{q^2}^{-|\lambda(q)|^2}}{e_{q^2}^{|\lambda(q)|^2}} \to 0$. ■

Notice that $\langle c|\hat{\delta}_{\varphi xy}|c\rangle$ can be written in the form

$$\langle c|\hat{\delta}_{\varphi xy}|c\rangle \;=\; \text{Tr } |c\rangle\langle c|\hat{\delta}_{\varphi xy} \;=\; \text{Tr } \hat{c}\hat{\delta}_{\varphi xy},$$

where we introduced the matrix $\hat{c} = \begin{pmatrix} |c_1|^2 & c_1\overline{c_2} \\ c_2\overline{c_1} & |c_2|^2 \end{pmatrix}$. If we assign to any matrix $\widehat{m}$ of unit trace the operator

$$\mathcal{M}_\lambda(\widehat{m}) = \frac{1}{1+(m_{12}+m_{21})\frac{e_{q^2}^{-|\lambda|^2}}{e_{q^2}^{|\lambda|^2}}} \Big(|\hat{e}_{q^2}(\lambda)\rangle\, m_{11}\, \langle \hat{e}_{q^2}(\lambda)| + |\hat{e}_{q^2}(\lambda)\rangle\, m_{12}\, \langle \hat{e}_{q^2}(-\lambda)| + \\ + |\hat{e}_{q^2}(-\lambda)\rangle\, m_{21}\, \langle \hat{e}_{q^2}(\lambda)| + |\hat{e}_{q^2}(-\lambda)\rangle\, m_{22}\, \langle \hat{e}_{q^2}(-\lambda)| \Big)$$

in $\mathcal{B}(h_0)$ which also has trace 1, then we obtain

$$\mathrm{Tr}\ \mathcal{M}_\lambda(\hat{c})\rho_\chi = \langle c \odot \hat{e}_{q^2}(\lambda)|\rho_\chi|c \odot \hat{e}_{q^2}(\lambda)\rangle.$$

Since any positive 2×2–matrix can be decomposed into the sum of at most two dyadic products $|c\rangle\langle c|$ and the normalization factors converge to 1 (uniformly in λ_0 if λ is replaced by $\lambda(q)$), we see that for any positive matrix $\widehat{m}$ of unit trace we obtain a family $\mathrm{Tr}\ \mathcal{M}_{\lambda(q)}(\widehat{m})\rho_\chi$ of states on $\mathcal{A}_q$ such that

$$\lim_{q\to -1} \mathrm{Tr}\ \mathcal{M}_{\lambda(q)}(\widehat{m})\rho_\chi = \mathrm{Tr}\ \widehat{m}\,\hat{\delta}_{\varphi xy}$$

uniformly in (φ, x, y) and $\widehat{m}$.

Now let $\varphi_\pm$ be arbitrary states on $\mathcal{A}_{\pm 1}$. By Sections 2 and 3 we know that there is a measure $d\nu^+$ on $SU(2)$ in the first case, and a matrix measure $d\hat{\nu}^- = \hat{n}\,d\nu^-$ on $SU(2)$, where $\hat{n}$ is a ν^-–integrable function with values in the positive matrices of unit trace, in the second case, such that

$$\varphi_+ = \int_{SU(2)} \delta_{\varphi xy}\, d\nu^+_{\varphi xy},$$

and

$$\varphi_- = \int_{SU(2)} \mathrm{Tr}\ \hat{n}(\varphi xy)\hat{\delta}_{\varphi xy}\, d\nu^-_{\varphi xy},$$

respectively. Obviously, $\langle \hat{e}_{q^2}(\lambda(q))|\rho_\chi|\hat{e}_{q^2}(\lambda(q))\rangle$ and $\mathrm{Tr}\ \mathcal{M}_{\lambda(q)}(\hat{n}(\varphi xy))\rho_\chi$ are $\nu^\pm$–integrable functions which converge uniformly in (φ, x, y) if q tends to ± 1. Therefore, we obtain by an application of the *theorem of majorized convergence* that the order of integration over $SU(2)$ and the limit $q \to \pm 1$ can be exchanged. The following is a simple corollary.

Theorem 5.5 *Arbitrary states $\varphi_\pm$ on $\mathcal{A}_{\pm 1}$ can be approximated by states on $\mathcal{A}_q$. We have*

$$\varphi_+ = \lim_{q\to 1} \int_{SU(2)} \langle \hat{e}_{q^2}(\lambda(q))|\rho_\chi|\hat{e}_{q^2}(\lambda(q))\rangle\, d\nu^+_{\varphi xy}$$

and

$$\varphi_- = \lim_{q\to -1} \int_{SU(2)} \mathrm{Tr}\ \mathcal{M}_{\lambda(q)}(\hat{n}(\varphi xy))\rho_\chi\, d\nu^-_{\varphi xy}.$$

The approximation is uniformly in $\varphi_\pm$.

So far, we know how to approximate arbitrary states (or more generally arbitrary positive functionals on $\mathcal{A}_{\pm 1}$ by states (positive functionals normalized to the same constant at $\mathbf{1}$) on $\mathcal{A}_q$.

6 Conditionally positive functionals

First, let us agree on some notation. If ψ_q is any conditionally positive functional on $\mathcal{A}_q$, we denote by $\Psi_q = \psi_q \circ \mathcal{E}_q$ the corresponding raised functional on $\mathcal{F}$. By $\mathcal{K}_i^q \subset \mathcal{F}, i = 1, 2$, we denote the sets consisting of all $a \in \mathcal{F}$ such that $\mathcal{E}_q(a) \in K_i$ for the corresponding $q \in [-1, 1]$. Notice that $\delta_\varphi \circ \mathcal{E}_q$ does not depend on q. Consequently we can define the mappings $\delta_{\mathcal{F}} = \delta \circ \mathcal{E}_q$ and $\delta'_{\mathcal{F}} = \delta' \circ \mathcal{E}_q$. Since no confusion can arise, we will omit the subscript $\mathcal{F}$. The same will be done for the projections $(Id-\delta_{\mathcal{F}}\mathbf{1}) = (Id-\delta\mathbf{1})\circ\mathcal{E}_q$ and $\mathcal{P}_{\mathcal{F}} = \mathcal{P}\circ\mathcal{E}_q$. Notice that $\mathcal{P}_{-1}\circ\mathcal{E}_{-1} = \mathcal{P}\circ\mathcal{E}_q$, but $\mathcal{P}_1\circ\mathcal{E}_1 < \mathcal{P}\circ\mathcal{E}_q$ and, of course, $\mathcal{P}_1\circ\mathcal{E}_1$ does not vanish on $\mathcal{I}_q$ unless $q=1$. Thus, we can omit the superscript q in $\mathcal{K}_1^q = \mathcal{K}_1$ for all q and in $\mathcal{K}_2^q = \mathcal{K}_2$ for $q \neq 1$.

Notice that

$$\begin{aligned} \mathcal{F} &= \mathcal{K}_1 \oplus \mathbf{C}\mathbf{1} \\ \mathcal{K}_1 &= \mathcal{K}_2 \oplus \mathbf{C}\frac{\alpha-\alpha^*}{2i} \\ \mathcal{K}_2 &= \mathcal{K}_2^1 \oplus \mathbf{C}\frac{\gamma+\gamma^*}{2} \oplus \mathbf{C}\frac{\gamma-\gamma^*}{2i} \\ \mathcal{K}_2^1 &= \mathcal{K}^2 \oplus \mathbf{C}\frac{\beta+\beta^*}{2} \end{aligned}$$

where we defined $\mathcal{K}^2 = \mathrm{lin}(\mathcal{K}_1\cdot\mathcal{K}_1)$.

Recall Definition (2.5) of the set G. By $\mathcal{G}$ we denote the corresponding set of generators of $\mathcal{F}$. We denote by

$$\mathcal{G}_n = \bigcup_{k=2}^{n} \mathcal{G}^k$$

for $n \geq 2$ the set of all monomials having length between 2 and n. In the sequel, we will approximate general conditionally positive functionals on $\mathcal{A}_{\pm 1}$ by sequences ψ_n of conditionally positive functionals on $\mathcal{A}_{q_n}$ where $q_n \to \pm 1$. The approximaton will be such that the deviation of ψ_n to its limit is less than $C\frac{1}{n}$ for all $g \in \mathcal{G}_n$, where $C > 0$ is an appropiate constant. Then ψ_n converges for all $a \in \mathcal{K}^2$, because $a \in \mathrm{lin}(\mathcal{G}_{n_0})$ for some n_0. On the other hand, since the $\mathcal{G}_n$ are finite sets, a limit, which exists for any $a \in \mathcal{K}^2$, can be performed uniformly on $\mathcal{G}_n$ (for fixed n).

By $\psi^{\pm}_{\lambda_0\chi}$ we denote the conditonally positive functionals

$$\begin{aligned} \psi^{+}_{\lambda_0\chi} &= \langle \hat{e}_{q^2}(\lambda(q))|\rho_\chi \circ \mathcal{P}|\hat{e}_{q^2}(\lambda(q))\rangle \\ \psi^{-}_{\lambda_0\chi} &= \mathrm{Tr}\ \mathcal{M}_{\lambda(q)}(\widehat{m}(\lambda_0,\chi))\rho_\chi \circ \mathcal{P} \end{aligned}$$

on $\mathcal{A}_q$, where $\widehat{m}$ is a function on $SU(2)$ with values in the positive 2×2–matrices of unit trace. We have

$$\begin{aligned} \lim_{q\to 1} \psi^{+}_{\lambda_0\chi} &= \delta_{\varphi xy}\circ\mathcal{P} \ = \ \delta^{+}_{\varphi xy} \\ \lim_{q\to -1} \psi^{-}_{\lambda_0\chi} &= \mathrm{Tr}\ \widehat{m}(\lambda_0,\chi)\hat{\delta}_{\varphi xy}\circ\mathcal{P} \ = \ \delta^{-}_{\varphi xy} \end{aligned}$$

uniformly on $SU(2)$. Notice, however, that the first expression differs from $\delta_{\varphi xy}\circ\mathcal{P}_1$ by the functional $x\delta'^x + y\delta'^y$. We will be concerned with this problem later.

Denote by M_n the set

$$M_n = \left\{ (\varphi xy) \in SU(2) \middle| x^2 + y^2 \geq \frac{1}{n} \right\}.$$

Let $\psi_\pm$ be the conditionally positive functionals on $\mathcal{A}_{\pm 1}$ given by Hunt's formulae

$$\begin{aligned} \psi_+ &= \int_{SU(2)} \delta_{\varphi xy}\circ\mathcal{P}_1\, d\mu^{+}_{\varphi xy} \\ \psi_- &= \int_{SU(2)} \mathrm{Tr}\ \widehat{m}(\lambda_0,\chi)\hat{\delta}_{\varphi xy}\circ\mathcal{P}\, d\mu^{-}_{\varphi xy}, \end{aligned} \tag{6.4}$$

where $d\mu^{\pm}$ are measures having no atom at identity and fulfilling the necessary conditions

$$M^{\pm} = \int_{SU(2)} (x^2+y^2)\, d\mu^{\pm}_{\varphi xy} < \infty$$

and $\widehat{m}$ is μ^-–integrable.

Proposition 6.1 *There are monotone sequences* $\{q_n^\pm\}_{n\geq 2}$ *with* $-1 < q_n^\pm < 1$ *and* $\lim_{n\to\infty} q_n^\pm = \pm 1$*, such that*

$$\Psi_\pm(a) \;=\; \lim_{n\to\infty} \int_{M_n} \Psi^\pm_{\lambda_0\chi}(a)\, d\mu^\pm_{\varphi xy}$$

for all sequences $\{q_n\}_{n\geq 2}$ *with* $q_n^+ < q_n < 1$ *and* $-1 < q_n < q_n^-$*, respectively, and all* $a \in \mathcal{K}^2$*.*

Remark 6.1 *The dependence on* q_n *is hidden in the raised conditionally positive functionals* $\Psi^\pm_{\lambda_0\chi}$ *which vanish on* $\mathcal{I}_{q_n}$*.*

Proof Since on $\mathcal{K}^2$ both the projections $\mathcal{P}$ and $\mathcal{P}_1$ disappear, the expressions converge by (6.4) to the stated values, if we replace $\Psi^\pm_{\lambda_0\chi}$ by their limits $\delta^\pm_{\varphi xy}$.

On the other hand, the limit of the integrands can be performed uniformly on $\mathcal{G}_n$ and $SU(2)$. We choose $q_n^\pm$ such that

$$|\Psi^\pm_{\lambda_0\chi}(g) - \delta^\pm_{\varphi xy}(g)| \;<\; \frac{1}{n^2} \;\leq\; \frac{x^2+y^2}{n}$$

for all $(\varphi, x, y) \in M_n$, $g \in \mathcal{G}_n$, and q_n closer to ± 1 than $q_n^\pm$. We obtain

$$\int_{M_n} |\Psi^\pm_{\lambda_0\chi}(g) - \delta^\pm_{\varphi xy}(g)|\, d\mu^\pm_{\varphi xy} \;<\; \frac{M^\pm}{n}.$$

This is our claimed convergence. Of course, $q_n^\pm$ can be chosen monotone. ■

Obviously, both the left- and right-hand side vanish on $\mathbf{1}$ and $\frac{\alpha-\alpha^*}{2i}$. And by Relation (e) we see that also $\Psi_\pm\left(\frac{\beta+\beta^*}{2}\right)$ is approximated properly by the right-hand side. Thus, we immediately can extend the foregoing proposition to

$$\mathcal{K}_\mathcal{P} \;=\; \mathcal{K}^2 \oplus \mathbf{C}\frac{\beta+\beta^*}{2} \oplus \mathbf{C}\frac{\alpha-\alpha^*}{2i} \oplus \mathbf{C1}$$

which is precisely the set, on which $\mathcal{P}$ and $\mathcal{P}_1$ coincide.

In the case when $q \to -1$ we even obtain by Relations (ã) and (ã)* that $\Psi_-(\gamma^{(*)})$ is approximated by the right-hand side. Henceforth, the approximation is valid on the whole of the algebra $\mathcal{F}$.

In the case when $q \to 1$ we have to add something which converges in a sufficiently uniform way to the functional

$$\begin{aligned}\delta_{\varphi xy} \circ \mathcal{P}_1 - \delta_{\varphi xy} \circ \mathcal{P} \;&=\; -\left(\delta'^x \delta_{\varphi xy}\left(\frac{\gamma+\gamma^*}{2}\right) + \delta'^y \delta_{\varphi xy}\left(\frac{\gamma-\gamma^*}{2i}\right)\right) \\ &=\; -(x\delta'^x + y\delta'^y) \;=\; \delta'^r\end{aligned}$$

with $r = (0, -x, -y)$. This functional will also be needed in order to write down the general gaußian part.

Proposition 6.2 *There are positive numbers* ϵ_n*, and a monotone function* $q_0(t)$ *on* $(0,1)$ *with* $0 < q_0(t) < 1$ *and* $\lim_{t\to 0} q_0(t) = 1$*, such that*

$$\left| \frac{\Psi^+_{\left(e^{it\varphi}\sqrt{1-t^2(x^2+y^2)}\right)(\chi)}(g)}{t} - \delta'^r(g) \right| \;<\; \frac{1}{n^2}$$

for all $g \in \mathcal{G} \cup \mathcal{G}_n$*,* $r \in SU(2)$*,* $t < \epsilon_n$ *and all functions* $q(t)$ *such that* $q_0(t) < q(t) < 1$*.*

Remark 6.2 *Actually,* r *is a vector in* $\mathbf{R}^3$*. By* $r \in SU(2)$ *we mean that the components* (φ, x, y) *of* r *describe an element of* $SU(2)$ *where the parameter* φ *lies in* $[-\frac{\pi}{2}, \frac{3\pi}{2})$*.*

PROOF First choose ϵ_n such that

$$\left|\frac{\delta_{(t\varphi)(tx)(ty)}(g)}{t} - \delta'^r(g)\right| < \frac{1}{2n^2}$$

for all $g \in \mathcal{G} \cup \mathcal{G}_n$, $r \in SU(2)$, and $t < \epsilon_n$. This is possible, because $\mathcal{G} \cup \mathcal{G}_n$ is finite, $g \in \mathcal{K}_1$, i.e. $\delta(g) = 0$, and $r \in [-\pi, \pi]^3 \subset \mathbf{R}^3$.

Then choose $q_0(t)$, such that

$$\left|\frac{\Psi^+_{\left(e^{it\varphi}\sqrt{1-t^2(x^2+y^2)}\right)(\chi)}(g) - \delta_{(t\varphi)(tx)(ty)}(g)}{t}\right| < \frac{1}{2n^2}$$

for all $g \in \mathcal{G} \cup \mathcal{G}_n$, $t \in (0, 1)$, and $q \in (q_0(t), 1)$. This is possible, because the approximation of $\delta_{\varphi xy}$ is uniform on $SU(2)$. Of course, $q_0(t)$ can be chosen monotone. ■

Now we use Proposition 6.2 in order to approximate $\Psi_+(\gamma^{(*)})$.

Corollary 6.3 *We have*

$$\Psi_+ = \lim_{n\to\infty} \int_{M_n} \left(\Psi^+_{\lambda_0\chi} + \frac{\Psi^+_{\left(\sqrt{1-t_n^2(x^2+y^2)}\right)(-\chi)}}{t_n} \right) d\mu^+_{\varphi xy}$$

for all sequences $\{t_n\}_{n\geq 2}$*, and* $\{q_n\}_{n\geq 2}$ *with* $0 < t_n < \epsilon_n$*, and* $\max(q_n^+, q_0(t_n)) < q_n < 1$.

PROOF On $\gamma^{(*)}$ the difference between the first summand in the integrand and its limit $x \pm iy$ can be estimated from above by $\frac{1}{n^2}$. The difference between the second summand and its limit $-(x \pm iy)$ can also be estimated by $\frac{1}{n^2}$. Therefore, the integrals over these differenes converge to 0.

On $\mathcal{K}_\mathcal{P}$ the second term converges to

$$\int_{SU(2)} \delta'^r \, d\mu^+_{\varphi xy}$$

with $r = (0, -x - y)$ which can be seen by the same estimates as for the first term. Since $\delta'^r = 0$ on $\mathcal{K}_\mathcal{P}$, this limit is 0. ■

We collect the results obtained so far.

Theorem 6.4 *There are universal sequences* $\{t_n\}_{n\geq 2}$*, and* $\{q_n^\pm\}_{n\geq 2}$*, such that any pair of conditionally positive functionals* $\psi_\pm$ *on* $\mathcal{A}_{\pm 1}$ *with integral representation (6.4) can be approximated as limits of conditionally positive functionals on* $\mathcal{A}_{q_n^\pm}$ *in the form*

$$\psi_+ = \lim_{n\to\infty} \int_{M_n} \left(\psi^+_{\lambda_0\chi} + \frac{\psi^+_{\left(\sqrt{1-t_n^2(x^2+y^2)}\right)(-\chi)}}{t_n} \right) d\mu^+_{\varphi xy}$$

$$\psi_- = \lim_{n\to\infty} \int_{M_n} \psi^-_{\lambda_0\chi} \, d\mu^-_{\varphi xy}.$$

Since δ, δ', and δ'' do not depent on q, the problem of approximating a gaußian part on $\mathcal{A}_{-1}$ is already solved. By Proposition 6.2 we also solved the problem of approximating the functional δ'^r on $\mathcal{A}_1$. (The general case $r \in \mathbf{R}^3$ can be reduced to the case when $r \in SU(2)$.) Thus, up to this moment we are able to express conditionally positive functionals ψ_+, and ψ_- on $\mathcal{A}_1$, and $\mathcal{A}_{-1}$ having no gaußian and anti-gaußian parts, respectively, by limits of conditionally positive functionals on $\mathcal{A}_q$.

Now we come to the last yet missing building blocks δ''^r and $\underline{\psi}_r$ for $r \in \mathbf{R}^3$ and $r \in \mathbf{C} \times \mathbf{R}^2$ which are needed to express the gaußian and anti-gaußian part, respectively. On $\mathcal{K}^2$ we have

$$\underline{\psi}_r = \lim_{t\to 0} \left\langle \frac{\hat{e}_1 + t\varphi\hat{e}_2}{t} \middle| \hat{\delta}_{0(tx)(ty)} \middle| \frac{\hat{e}_1 + t\varphi\hat{e}_2}{t} \right\rangle$$

and, of course,

$$\frac{\delta''^r}{2} \quad = \quad \lim_{t\to 0} \frac{\delta_{(t\varphi)(tx)(ty)}}{t^2}.$$

If we set $\widehat{m} = \frac{1}{1+|\varphi|^2t^2}\left(\begin{smallmatrix} 1 & \overline{\varphi}t \\ \varphi t & |\varphi|^2t^2 \end{smallmatrix}\right)$ in order to express the state

$$\frac{\left\langle \frac{\hat{e}_1+t\varphi\hat{e}_2}{t} \,|\bullet|\, \frac{\hat{e}_1+t\varphi\hat{e}_2}{t} \right\rangle}{1+|\varphi|^2t^2}$$

in the form Tr $\bullet\,\widehat{m}$, we obtain by a proof completely analogous to that of Proposition 6.2

Proposition 6.5 *There are positive numbers ϵ_n, and monotone functions $q_\pm(t)$ on $(0,1)$ with $q_\pm(t)$ between 0 and ±1 and $\lim_{t\to 0} q_\pm(t) = \pm 1$, such that*

$$\left| \frac{\Psi^+_{\left(e^{it\varphi}\sqrt{1-t^2(x^2+y^2)}\right)(\chi)}(g)}{t^2} - \frac{\delta''^r(g)}{2} \right| \quad < \quad \frac{1}{n},$$

and

$$\left| (1+|\varphi|^2t^2) \frac{\Psi^-_{\left(1\,\sqrt{1-t^2(x^2+y^2)}\right)(\chi)}(g)}{t^2} - \underline{\psi}_r(g) \right| \quad < \quad \frac{1}{n},$$

for all $g \in \mathcal{G}_n$, $r \in SU(2)$, $t < \epsilon_n$ and all functions $q(t)$, such that $q(t)$ between $q_\pm(t)$ and ±1.

This means that the gaußian and anti-gaußian part can be approximated by $\Psi^\pm$ at least on $\mathcal{K}^2$. Again the statement remains true on $\mathcal{K}_\mathcal{P}$ and in the case when $q \to -1$ it is even true on the whole algebra $\mathcal{F}$. For $q \to 1$ consider the sequence of conditionally positive functionals

$$\psi^0_n \quad = \quad \frac{\psi^+_{\left(e^{it_n\varphi}\sqrt{1-t_n^2(x^2+y^2)}\right)(\chi)}}{t_n^2} + \frac{\psi^+_{\left(e^{is_n\varphi}\sqrt{1-s_n^2(x^2+y^2)}\right)(-\chi)}}{t_ns_n}$$

on $\mathcal{A}_{q_n}$. Choose $t_n \geq t_{n+1} \to 0$ such that

$$\left| \frac{\delta_{(t_n\varphi)(t_nx)(t_ny)}(g)}{t_n^2} - \frac{\delta''^r(g)}{2} \right| \quad < \quad \frac{1}{n},$$

and then $s_n \geq s_{n+1} \to 0$ such that

$$\left| \frac{\delta_{(s_n\varphi)(-s_nx)(-s_ny)}(g)}{t_ns_n} \right| \quad < \quad \frac{1}{n},$$

for all $g \in \mathcal{G}_n$. We have

$$\frac{\delta_{(t_n\varphi)(t_nx)(t_ny)}}{t_n^2} + \frac{\delta_{(s_n\varphi)(-s_nx)(-s_ny)}}{t_ns_n} \quad \longrightarrow \quad \frac{\delta''^r}{2}$$

on $\mathcal{K}^1_2$. On the other hand,

$$\frac{\delta_{(t_n\varphi)(t_nx)(t_ny)}(\gamma^{(*)})}{t_n^2} + \frac{\delta_{(s_n\varphi)(-s_nx)(-s_ny)}(\gamma^{(*)})}{t_ns_n} \quad = \quad 0 \quad = \quad \delta''^r(\gamma^{(*)}).$$

Thus, the convergence is also on $\mathcal{K}_2$. Now we choose $q_n \leq q_{n+1} \to 1$ such that

$$\left| \Psi^+_{\left(e^{it_n\varphi}\sqrt{1-t_n^2(x^2+y^2)}\right)(\chi)}(g) - \delta_{(t_n\varphi)(t_nx)(t_ny)}(g) \right| \quad < \quad \frac{t_n^2}{n},$$

and

$$\left| \Psi^+_{\left(e^{is_n\varphi}\sqrt{1-s_n^2(x^2+y^2)}\right)(\chi)}(g) - \delta_{(s_n\varphi)(s_nx)(s_ny)}(g) \right| \quad < \quad \frac{t_ns_n}{n}$$

for all $g \in \{\beta+\beta^*, \gamma, \gamma^*\} \cup \mathcal{G}_n$. Since $\Psi^+_{\lambda_0\chi}$ and δ''^r are 0 on $\alpha - \alpha^*$ and $\mathbf{1}$, we obtain the following

Proposition 6.6 *We have*

$$\lim_{n\to\infty} \psi_n^0 \quad = \quad \frac{\delta''r}{2}.$$

Up to this point we are able to split up a given conditionally positive functional on $\mathcal{A}_1$ or $\mathcal{A}_{-1}$ into several parts, and to approximate any of these parts by sequences of conditionally positive functionals on $\mathcal{A}_{q_n}$, where q_n converges to ± 1. We emphasized that all approximations also work if the sequence q_n is replaced by a sequence q'_n where q'_n is closer to its limit ± 1 than q_n. Therefore, the sequences q_n belonging to different parts of the conditionally positive functional can be chosen to be the same. We summarize.

Theorem 6.7 *For any conditionally positive functional ψ on $\mathcal{A}_{\pm 1}$ there is a sequence $\{q_n\}$ with $\lim_{n\to\infty} q_n = \pm 1$ and a sequence $\{\psi_n\}$ of conditionally positive functionals on $\mathcal{A}_{q_n}$, such that*

$$\psi \quad = \quad \lim_{n\to\infty} \psi_n.$$

Due to the last remark it is also always possible to find a family ψ_q of conditionally positive functionals on $\mathcal{A}_q$, such that

$$\psi \quad = \quad \lim_{q\to\pm 1} \psi_q.$$

It can be shown that (for fixed q) the weak* closed cone generated by all functionals of the form $\varphi \circ \mathcal{P}$ (where now $\mathcal{P}$ runs over all projections onto K_2 respecting the $*$) consists of all conditionally positive functionals; see [6]. Similarly, the weak* closed cone generated by all functionals of the form $\varphi \circ (\mathrm{Id} - \mathbf{1}\delta)$ consists of all infinitesimal generators [7]. Together with Section 5 it is possible to use this in order to obtain the principal possibility for the approximation result in this section. However, notice that we found an explicit form of the approximation in terms of coherent states.

Appendix

A q–Analysis

We present the well-known results on q–analysis in a slightly modified form which is more convinient for our purposes. The proofs of formulae are omitted if they consist in simple computation.

A.1 q–Derivative and q–integral

Definition A.1 *Let $q \in (-1, 1)$ be a real number and $S_0 \subset \mathbf{C}$ a star shaped area having 0 as star point. By $C_\omega(S_0)$ we denote the space of analytic functions on S_0. We introduce the two linear mappings $d_q, \int_q : C_\omega(S_0) \to C_\omega(S_0)$ by*

(i) $$d_q(f) = \frac{df}{d_q z}, \quad \textit{where} \quad \frac{df}{d_q z}(w) \;=\; \frac{f(w) - f(qw)}{w}.$$

(ii) $$\int_q (f) = \int_q f, \quad \textit{where} \quad \left(\int_q f\right)(w) \;=\; \int_0^w f(z)\, d_q z \;=\; \sum_{k=0}^{\infty} q^k w f(q^k w).$$

By expanding f into a power series, we easily see that $d_q f$ and $\int_q f$ are indeed in $C_\omega(S_0)$.

Remark A.1 *In order to obtain the usual notions of q–derivative and q–integral we have to divide our derivative by $(1-q)$ and to multiply our integral by $(1-q)$. By looking at the corresponding expressions*

$$\frac{f(w) - f(qw)}{w - qw}, \quad \text{and} \quad \sum_{k=0}^{\infty} (q^k w - q^{k+1} w) f(q^k w)$$

we immediately see that they tend to the derivative and integral, respectively, of usual analysis as q tends to 1.

Theorem A.1 *(Main Theorem)*

$$(i) \qquad \int_q d_q f = f - f(0)$$

$$(ii) \qquad d_q \int_q f = f.$$

Proof By replacing the infinite sum in the definition of the integral by a finite sum and then performing the limit, we see that

$$\left(\int_q d_q f\right)(w) = f(w) - \lim_{n\to\infty} f(q^{n+1}w)$$

and

$$\left(d_q \int_q f\right)(w) = f(w) - \lim_{n\to\infty} q^{n+1} f(q^{n+1}w).$$

From this the statements follow. ■

By direct computation we obtain the following rules.

Theorem A.2 *For $f, g \in C_\omega(S_0)$ we have*

(i)
$$\frac{d(fg)}{d_q z}(z) = f(z)\frac{dg}{d_q z}(z) + \frac{df}{d_q z}(z)g(qz)$$

(ii)
$$\frac{d(\frac{1}{f})}{d_q z}(z) = -\frac{\frac{df}{d_q z}(z)}{f(z)f(qz)}$$

(iii)
$$\int_0^w f(z)\frac{dg}{d_q z}(z)\, d_q z + \int_0^w \frac{df}{d_q z}(z) g(qz)\, d_q z = f(w)g(w) - f(0)g(0)$$

Theorem A.3 *For the non-negative powers of z we obtain*

$$(i) \qquad d_q(z^k) = (1-q^k)z^{k-1}$$

$$(ii) \qquad \int_q (z^k) = \frac{z^{k+1}}{1-q^{k+1}}.$$

Remark A.2 *Of course, it is possible to extend the operation of q–derivation to functions which are analytic on an area S such that $qS \subset S$. In the next paragraph such functions, actually, will appear.*

A.2 q–Exponential function and q–Eulerian integral

Theorem A.4 *The q–exponential function.*

(i) By setting

$$e_q^z = \prod_{k=0}^{\infty} \frac{1}{1-q^k z}$$

we define a meromorphic function on $\mathbf{C}\backslash\{q^{-k}|k \in \mathbf{N}_0\}$.

(ii) On $U_1(0)$ we have

$$e_q^z = \sum_{k=0}^{\infty} \frac{z^k}{(1-q)\cdots(1-q^k)}$$

(iii) e_q^z *is different from* 0 *everywhere. By setting*

$$(e_q^z)^{-1} = \prod_{k=0}^{\infty}(1-q^k z)$$

we define a transcendent function.

PROOF Consider the power series in (ii). Clearly, its radius of convergence is 1. (We have $\frac{|z|}{1-q^k} \leq \frac{|z|}{1-q^K}$ for $k > K$ and for any $z \in U_1(0)$ we can find $K \in \mathbf{N}_0$, such that $\frac{|z|}{1-q^K} < 1$.) We find

$$e_q^z - e_q^{qz} = z e_q^z.$$

and consequently

$$e_q^z = \frac{1}{1-z} e_q^{qz} = \frac{1}{1-z} \cdots \frac{1}{1-q^k z} e_q^{q^{k+1}z}.$$

Since $\lim_{k\to\infty} e_q^{q^{k+1}z} = 1$, we have that $e_q^{q^{k+1}z}$ is different from zero for almost all k. Therefore, we find

$$e_q^z = \lim_{k\to\infty} \frac{e_q^z}{e_q^{q^{k+1}z}} = \prod_{k=0}^{\infty} \frac{1}{1-q^k z}.$$

Now let z be with $|z| \geq 1$ and $z \neq q^{-k}, k \in \mathbf{N}_0$. We can find $K \in \mathbf{N}$ such that $q^K|z| < 1$. Therefore, we see by using

$$e_q^z = e_q^{q^K z} \prod_{k=0}^{K-1} \frac{1}{1-q^k z}$$

that e_q^z is analytic on the given domain.

Now suppose that e_q^z is 0 for some z. By the recursion formula we see that 0 must be an accumulation point of zeros. Therefore, the function has to assume the value 0 at 0 in contradiction to $e_q^0 = 1$. Thus, we can define the reciprocal of e_q^z on the whole complex plain. Since this function assumes the value 0 for $z = q^{-k}$, this function cannot be a polynomial. It must be transcendent. ■

Using the recursion formula and our derivation rule (ii), we obtain

Corollary A.5 *The operation of q–derivation can be performed for e_q^z and its reciprocal at any point of their domains. We obtain*

$$(i) \qquad \frac{de_q^z}{d_q z}(z) = e_q^z$$

$$(ii) \qquad \frac{d(e_q^z)^{-1}}{d_q z}(z) = (e_q^{qz})^{-1}.$$

REMARK A.3 *Suppose that f and $\widetilde{f}$ are two (non-vanishing) solutions of the q–differential equation (i). It is not difficult to see that their quotient must be a constant. Therefore, e_q^z is the only solution of (i), fulfilling $e_q^0 = 1$. A similar statement is true for (ii).*

REMARK A.4 *Notice that the usual form of the q–exponential is given by*

$$\sum_{k=0}^{\infty} \frac{z^k}{\left(\frac{1-q}{1-q}\right)\cdots\left(\frac{1-q^k}{1-q}\right)} = e_q^{(1-q)z}.$$

This function converges pointwise to the exponential for $q \to 1$.

Now we can describe the q–factorial $[k]_q! = (1-q)\cdots(1-q^k)$ by a q–Eulerian integral. (To obtain the usual definition one has to divide by $(1-q)^k$.)

Theorem A.6 *We have*

$$\int_0^1 \frac{z^k}{e_q^{qz}}\, d_q z \;=\; (1-q)\cdots(1-q^k).$$

Proof We prove the statement by induction. Since

$$\int_0^1 \frac{d_q z}{e_q^{qz}} \;=\; -((e_q^1)^{-1} - (e_q^0)^{-1}) \;=\; 1,$$

the statement is true for $k = 0$.

Now suppose that it is true for $k \geq 0$. We obtain

$$\begin{aligned}
\int_0^1 \frac{z^{k+1}}{e_q^{qz}}\, d_q z &= -\int_0^1 z^{k+1}(d_q(e_q^z)^{-1})(z)\, d_q z \\
&= \left[-z^{k+1}(e_q^z)^{-1}\right]_0^1 + \int_0^1 (d_q z^{k+1})(e_q^{qz})^{-1}\, d_q z \\
&= 0 + (1-q^{k+1})\int_0^1 \frac{z^k}{e_q^{qz}}\, d_q z.
\end{aligned}$$

This proves the statement for $k+1$. ■

Now we show an estimate which will be useful in the next appendix.

Proposition A.7 *For all $w \in [0,1]$ and $k \in \mathbf{N}_0$ we have*

$$\frac{w^{k+1}}{e_q^q} \;\leq\; \frac{w^{k+1}}{(1-q^{k+1})e_q^{qw}} \;\leq\; \int_0^w \frac{z^k}{e_q^{qz}}\, d_q z \;\leq\; \frac{w^{k+1}}{(1-q^{k+1})} \;\leq\; \frac{w^{k+1}}{1-q}.$$

Proof By the power series representation we see that e_q^z is a strictly increasing function on $[0,1)$. Notice also that $\left(\int_q \bullet\right)(w)$ is a monotone functional for positive w. This yields immediately the two inner estimates. The outer estimates are obvious. ■

B The representation ρ_0 as a representation on a Hilbert space of analytic functions

In this appendix we generalize the representation of the relation

$$aa^* - qa^*a \;=\; \mathbf{1}$$

given in [1] to a representation of $\mathcal{A}_q$, unitarily equivalent to ρ_0 (cf. also [2]). We will show that the scalar product stated in [1] turns indeed out to be the scalar product of the underlying Hilbert space. (The authors of [1] only showed that their scalar product exisits on a special orthonormal basis and yields the correct values. The proof of existence on arbitrary vectors was left out.) However, notice that α and α^* fulfill the slightly modified commutation rule

$$\frac{\alpha\alpha^* - q^2\alpha^*\alpha}{1-q^2} \;=\; \mathbf{1}.$$

Now consider the representation ρ_0 on the Hilbert space h_0 with orthonormal basis $\{e_k\}_{k\in\mathbf{N}_0}$. We introduce the q–coherent states as eigenvectors of $\rho_0(\alpha)$. It is easy to check that they must be of the form

$$e_{q^2}(\lambda) \;=\; \sum_{k=0}^{\infty} \frac{\lambda^k}{\sqrt{1-q^2}\cdots\sqrt{1-q^{2k}}} e_k \quad \text{with} \quad |\lambda| < 1, \tag{B.1}$$

where λ is the eigenvalue. Under $\rho_0(\alpha^*)$ these vectors behave like

$$\begin{aligned}\rho_0(\alpha^*)e_{q^2}(\lambda) &= \sum_{k=0}^{\infty}\frac{\lambda^k}{\sqrt{1-q^2}\cdots\sqrt{1-q^{2k}}}\sqrt{1-q^{2(k+1)}}e_{k+1}\\ &= \frac{1}{\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{\sqrt{1-q^2}\cdots\sqrt{1-q^{2k}}}(1-q^{2k})e_k \;=\; \frac{de_{q^2}(\lambda)}{d_{q^2}\lambda}.\end{aligned}$$

The scalar product of two such vectors is given by

$$\langle e_{q^2}(\mu)|e_{q^2}(\lambda)\rangle = e_{q^2}^{\overline{\mu}\lambda}.$$

Now we ask how to build up the identity operator out of terms of the form $|e_{q^2}(\lambda)\rangle\langle e_{q^2}(\lambda)|$.

We write $\lambda = |\lambda|e^{i\varphi}$ in polar coordinates and intergate over φ. For fixed $|\lambda|$ the double sum is absolutely convergent and no problem can arise. We obtain

$$\int_0^{2\pi}|e_{q^2}(|\lambda|e^{i\varphi})\rangle\langle e_{q^2}(|\lambda|e^{i\varphi})|\,\frac{d\varphi}{2\pi} = \sum_{k=0}^{\infty}|e_k\rangle\frac{|\lambda|^{2k}}{(1-q^2)\cdots(1-q^{2k})}\langle e_k|.$$

If we now could perform the q^2–Eulerian integral (see Appendix A) with respect to the variable $|\lambda|^2$ in order to eliminate q^2–factorial in the denominator, we would obtaine a representation of the identity. However, in order to perform the q^2–integral the integrand has to be evaluated at $|\lambda|^2 = 1$ where the sum is no longer norm convergent. On the other hand, due to Proposition A.7 we have for $w \in [0,1)$

$$\int_0^w\int_0^{2\pi}|e_{q^2}(|\lambda|e^{i\varphi})\rangle\langle e_{q^2}(|\lambda|e^{i\varphi})|\,\frac{d\varphi}{2\pi}\,\frac{d_{q^2}|\lambda|^2}{e_{q^2}^{q^2|\lambda|^2}} = \sum_{k=0}^{\infty}|e_k\rangle\frac{\int_0^w\frac{|\lambda|^{2k}}{e_{q^2}^{q^2|\lambda|^2}}\,d_{q^2}|\lambda|^2}{(1-q^2)\cdots(1-q^{2k})}\langle e_k|.$$

It is easy to see that

$$\lim_{w\to1}\int_0^w\int_0^{2\pi}|e_{q^2}(|\lambda|e^{i\varphi})\rangle\langle e_{q^2}(|\lambda|e^{i\varphi})|\,\frac{d\varphi}{2\pi}\,\frac{d_{q^2}|\lambda|^2}{e_{q^2}^{q^2|\lambda|^2}} = \mathbf{1}$$

in the strong topology. Notice that the order of integrals does not matter. Using the notation

$$\int f(\lambda)\,d_{q^2}^2\lambda = \lim_{w\to1}\frac{1}{2\pi}\int_0^w\int_0^{2\pi}f(|\lambda|e^{i\varphi})\,d\varphi\,d_{q^2}|\lambda|^2,$$

we obtain

$$\int\frac{|e_{q^2}(\lambda)\rangle\langle e_{q^2}(\lambda)|}{e_{q^2}^{q^2|\lambda|^2}}\,d_{q^2}^2\lambda = \mathbf{1}.$$

Obviously, this remains true if in the integrand λ is replaced by $\overline{\lambda}$.

Consider the Hilbert space H_{f} of analytic functions which is defined by assuming that

$$\left\{\frac{z^k}{\sqrt{1-q^{2k}}}\right\}_{k\in\mathbf{N}_0}$$

forms an orthonormal basis. Notice that the scalar products for different $q \in (-1,1)$ induce the same topology. Therefore, H_{f} consists of all power series with square summable coefficients. The scalar product of two elements $f, g \in H_{\mathrm{f}}$ can be obtained as

$$\langle f|g\rangle = \int\frac{\overline{f(\lambda)}g(\lambda)}{e_{q^2}^{q^2|\lambda|^2}}\,d_{q^2}^2\lambda.$$

By

$$f \in h_0 \quad \longmapsto \quad f(z) \;=\; \langle e_{q^2}(\overline{z})|f\rangle \in H_{\mathrm{f}}$$

we obviously define a Hilbert space isomorphism. For the representation operators in this image we obtain

$$\begin{aligned}
\rho_0(\alpha^*)f(z) &= \langle e_{q^2}(\overline{z})|\rho_0(\alpha^*)f\rangle \;=\; \langle \rho_0(\alpha)e_{q^2}(\overline{z})|f\rangle \\
&= zf(z) \\
\rho_0(\alpha)f(z) &= \langle \rho_0(\alpha^*)e_{q^2}(\overline{z})|f\rangle \\
&= \frac{df}{d_{q^2}z}(z) \\
\rho_0(\gamma^{(*)})f(z) &= f(qz).
\end{aligned}$$

The q–coherent state belonging to the eigenvalue λ is given by the q–exponential

$$\langle e_{q^2}(\overline{z})|e_{q^2}(\lambda)\rangle \;=\; e_{q^2}^{z\lambda}.$$

Let $\varphi = \langle f|\rho_0|f\rangle$ be a state with GNS–representation ρ_0. (Notice that by irreducibilty any non-vanishing vector in h_0 is cyclic for ρ_0.) Since φ is hermitian and ρ_0 does not distinguish between γ and γ^*, it is sufficient to know its values on $\alpha^{*n}\gamma^m$ for $n, m \in \mathbf{N}_0$. We obtain

$$\varphi(\alpha^{*n}\gamma^m) \;=\; \int \overline{f(z)}z^n f(q^m z)\, d_{q^2}^2 z.$$

References

[1] Arik, M., Coon, D.D., *Hilbert spaces of analytic functions and generalized coherent states*, J. Math. Phys. 17, 524-527 (1976)

[2] Bozejko, M., Speicher, R., *An example of generalized Brownian motion*, Commun. Math. Phys. 137, 519-531 (1991)

[3] Hunt, G.A., *Semi-groups of measures on Lie groups*, Trans. Amer. Math. Soc. 81, 264-293 (1956)

[4] Schürmann, M., *White noise on bialgebras*, (Lect. Notes Math., vol.1544), Berlin Heidelberg New York, Springer, 1993

[5] Schürmann, M., Skeide, M., *Infinitesimal generators on the quantum group* $SU_q(2)$, to appear in Infinite Dimensional Analysis, Quantum Probability & Related Topics

[6] Skeide, M., *The Lévy-Khintchine formula for the quantum group* $SU_q(2)$, Thesis, Heidelberg, 1994

[7] Skeide, M., *Hunt's formula on* $SU_q(2)$*; a unified view*, to appear in Journal of Open Systems and Information Dynamics

[8] Skeide, M., *Infinitesimal generators on* $SU_q(2)$ *do not depend on* q, to appear in Quantum Probability & Related Topics X, Singapore, New Jersey, London, Hong Kong, World Scientific

[9] Vaksman, L.L., Soibelman, Y.S., *Algebra of functions on the quantum group* $SU(2)$, Functional Anal. Appl. 22, 170-181 (1988)

[10] Woronowicz, S.L., *Twisted* $SU(2)$ *group. An example of a non-commutative differential calculus*, Publ. RIMS 23, 117-181 (1987)