Isometric Dilations of Representations of Product Systems via Commutants

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Abstract

We construct a weak dilation of a not necessarily unital CP-semigroup to an \( E \)-semigroup acting on the adjointable operators of a Hilbert module with a unit vector. We construct the dilation in such a way that the dilating \( E \)-semigroup has a pre-assigned product system. Then, making use of the commutant of von Neumann correspondences, we apply the dilation theorem to proof that covariant representations of product systems admit isometric dilations.

1 Introduction

Let \( S = \mathbb{R}_+ \) or \( S = \mathbb{N}_0 \). Our scope is to proof the following theorem on existence of isometric dilations of covariant representations of product systems, a problem suggested by one of the authors of [MS02] on the occasion of a meeting in Bangalore in December 2005.

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1.1 Theorem. Let $F^0 = (F_t)_{t \in S}$ be a product system of correspondences over a $C^*$–algebra (a $W^*$–algebra) $M$.\[1] Let $\sigma^0 = (\sigma_t)_{t \in S}$ be a (normal) covariant representation of $F^0$ on a Hilbert space $G$.\[2] Then there exists an isometric dilation $\tau^0 = (\tau_t)_{t \in S}$ of $\sigma^0$.\[3]

However, the heart of our proof is the following theorem about existence of a (weak) dilation of a CP-semigroup with a pre-assigned product system that has an obvious extension to $W^*$–modules.

1.2 Theorem. Let $E^0 = (E_t)_{t \in S}$ be a product system of correspondences over a unital $C^*$–algebra $B$ and let $\xi^0 = (\xi_t)_{t \in S}$ be a contractive (that is, $\langle \xi_t, \xi_t \rangle \leq 1$) unit[4] for $E^0$. Then there exists a unique strict weak dilation $(E, \vartheta, \xi)$ of the CP-semigroup $T = (T_t)_{t \in S}$ ($T_t = \langle \xi_t, \cdot \xi_t \rangle$) fulfilling:

1. The product system associated with the strict $E$–semigroup $\vartheta = (\vartheta_t)_{t \in S}$ on $B^a(E)$ is $E^0$.

2. $E$ is generated by $E^0$ in the sense that $\bigcup_{t \in S} \vartheta_t(\xi_t^a)E$ is total in $E$.

We prove this theorem in Section 2 in a version for $C^*$–correspondences over a unital $C^*$–algebra, which extends easily to $W^*$–correspondences. (See also there for concise definitions.) After passing to the commutant system of $E^0$ Theorem 1.2 translates into the $W^*$–version of Theorem 1.1 in the special case when $\sigma_0$ is faithful an unital. To that goal in Section 3 we proof a couple of equivalences (Theorem 3.4) between structures present in a product system and structures present in its commutant system. In Section 4 we explain how the general $W^*$–case and the $C^*$–case boil down to Theorems 1.2 and 3.4.

We mention that Muhly and Solel (private communication) give a direct formulation of the construction of the isometric dilation in Theorem 1.1 without passing through commutants, that reduces the problem to the case of nondegenerate covariant representations. Then they apply their dilation result [MS02, Theorem 3.7] for nondegenerate representations to obtain the isometric dilation. We should like to say that, assuming the basic facts about commutants of von Neumann correspondences, the proof we give here is self-contained.

It is very well possible to read the three following sections in reverse order, that is, first reducing to faithful nondegenerate $\sigma_0$, then showing equivalence of the statement of Theorem 1.1 in this case with the statement of Theorem 1.2 and, finally, proving Theorem 1.2. Of course, reading in this order makes less clear that Theorem 1.2 is a result that is independent of the remainder.

[1] That is, apart from identification $u_{x,y}: E_t \otimes E_s \rightarrow E_{t+s}$ which compose associatively, we have $F_0 = M$ and the identifications $u_{0,0}(F_0 \otimes F_0) = F_0 = u_{0,0}(F_t \otimes F_0)$ are the canonical ones.

[2] That is, a family of linear ($\sigma$–weak) CC-maps $\sigma_t: F_t \rightarrow B(G)$ such that $\sigma_0$ is a (normal) representation of $B$ (not necessarily unital but of course $*$), all $\sigma_t$ are $B$–$B$–linear, and $\sigma_{t+s}(x_t \otimes y_s) = \sigma_t(x_t)\sigma_s(y_s)$.

[3] That is, a (normal) covariant representation $\tau^0$ on a Hilbert space $H \supseteq G$ such that $\tau_t(x_t^a)\tau_s(y_s) = \tau_0((x_t^a, y_s))$ and $p_G^a\tau_t(\bullet)p_G = \sigma_t$.

[4] That is, $u_{x,y}((\xi_t \otimes \xi_t)) = \xi_{t+s}$ and $\xi_0 = 1$. 

2
2 Proof of Theorem 1.2

Let us first clarify some notions used in Theorem 1.2. An \( E\text{-semigroup} \) is a semigroup of endomorphisms of a \(*\)-algebra. If the endomorphisms are assumed unital, then we say \( E_0\text{-semigroup} \). The product system associated with a strict[^1] \( E\text{-semigroup} \( \vartheta \) acting on \( \mathcal{B}^a(E) \) where \( E \) is a Hilbert \( \mathcal{B} \)-module with a unit vector \( \xi \in E \) (that is, \( \langle \xi, \xi \rangle = 1 \)) is obtained in exactly the same way as in Skeide [Ske02]: Put \( E_t := \vartheta_t(\xi \xi^*)E \). Then \( E_t \) becomes a correspondence over \( \mathcal{B} \) by defining the left action \( bx_t := \vartheta_t(\xi b \xi^*)x_t \) of \( b \in \mathcal{B} \). By \( x_t \circ y_t \mapsto \vartheta_t(x_t^*)y_t \), we define a unitary \( E_t \circ E_t \subseteq E \) giving back \( \vartheta_t \) as \( \vartheta_t(a) = a \circ \text{id}_E \in \mathcal{B}^{a}(\vartheta_t(1)E) \subseteq \mathcal{B}^{a}(E) \). Moreover, the restriction of this unitary to \( E_t \circ E_t \subseteq E \circ E_t \) defines a bilinear unitary \( u_{t,s} : E_t \circ E_t \to E_{s+t} \) and the family of all these bilinear unitaries defines a product system structure. If there is no danger of confusion we will suppress the \( t \) in the notation and simply write \( E_t \circ E_t = E_{s+t} \). Note that \( E_t \subseteq E \), so that it makes sense to speak about \( E \) being generated by \( E^0 \) in the sense of Theorem 1.2(2). The generalization of [Ske02] to \( E\)-semigroups has been discussed by Bhat and Lindsay [BL05]. We may also obtain a version for nonunital \( \mathcal{B} \) (that is, in particular, without unit vector) following the methods from Muhly, Skeide and Solel [MSS06] as explained in the case of \( E_0\)-semigroups in [Ske04]. However, note that large parts of the proof of Theorem 1.2 will not work.

Further, \( (E, \vartheta, \xi) \) is a weak dilation of \( T \), if \( \langle \xi, \vartheta_t(\xi b \xi^*) \xi \rangle = T_t(b) \); see Bhat and Skeide [BS00]. In general, if \( \xi_t := \vartheta_t(\xi \xi^*) \xi \) defines a unit for the product system associated with \( \vartheta_t \), then the mappings \( b \mapsto \langle \xi, \vartheta_t(\xi b \xi^*) \xi \rangle \) define a CP-semigroup. (In fact, we verify easily that \( \langle \xi_t, \xi_s \rangle = \langle \xi, \vartheta_t(\xi \bullet \xi_s^*) \xi \rangle \), and whatever a unit \( \xi^0 \) we choose the maps \( \langle \xi_t, \xi_s \rangle \) always define a CP-semigroup.) The dilation we construct has a product system \( E^0 \) which we pre-assign and a unit \( \xi^0 \) that gives back \( T \). Therefore, we speak of a dilation of \( T \) with pre-assigned product system \( (E^0, \xi^0) \).

**Proof of Theorem 1.2.** The idea is to mimic the construction in [BS00, Section 8] which treats the case when \( E^0 \) is the GNS-system of the CP-semigroup \( T \) (that is, the unit \( \xi^0 \) that gives back \( T \) as \( \langle \xi_t, \xi_s \rangle \) generates the whole product system). The construction in [BS00] is done by passing to the unitalized CP-semigroup \( \widetilde{T} \) on the unitalization \( \widetilde{\mathcal{B}} = \mathcal{C} \mathcal{I} \oplus \mathcal{B} \). Then the so-called minimal dilation of \( \widetilde{T} \) is constructed. This construction involves an inductive limit of correspondences (called the first inductive limit or two-sided inductive limit) giving the product system and an inductive limit of right Hilbert modules (called the second inductive limit or one-sided inductive limit) giving the right module on which the dilating \( E \)-semigroup lives. At the moment of the proof in [BS00] these constructions are already known and simply used. The major part of the work in [BS00, Section 8] consists in identifying how the dilation of \( T \) itself sits inside the dilation of \( \widetilde{T} \).

[^1]: This means the endomorphisms \( \vartheta_t \) in the semigroup are strictly continuous on bounded subsets of \( \mathcal{B}^a(E) \). Equivalently, the action of \( \mathcal{K}(E) \) alone is already nondegenerate on \( \vartheta_t(\text{id}_E)E \).
Here we have to redo the first inductive limit (or better to replace it with something similar) to obtain a product system of correspondences over $\tilde{B}$ with a unital unit that allows, then, to do the second inductive limit, also yielding a dilation (just not the minimal one) of $\tilde{T}$. The work to show how this dilation of $\tilde{T}$ contains a dilation of $T$ is very similar to [BS00]. We apologize for that with respect to [BS00] we find it convenient to change the order of certain components in column vectors. (For instance, $\tilde{B}$ is isomorphic to the $C^*$-algebraic direct sum $C \oplus \tilde{B}$. Here we shall write the component $C$ as upper component, while in [BS00] we wrote it as lower component.) We also mention a typo in [BS00, Theorem 8.4] where we wrote accidently that $\vartheta$ is an $E_0$–semigroup. (Of course, if $T$ is nonunital, then $\vartheta$ constructed in [BS00] is definitely not an $E_0$–semigroup but only an $E$–semigroup.)

A crucial point in [BS00] was to identify the GNS-module of $\tilde{T}_i$ and its cyclic vector $\tilde{\xi}_i$ in terms of the GNS-module $\mathcal{E}_i$ and cyclic vector $\xi_i$ of $T_i$.[6] Here we do the same for $E_i$ and $\xi_i$, just that now $E_i$ may be bigger than $\mathcal{E}_i$.[7] So put $\tilde{\xi}_i := \sqrt{1 - \langle \xi_i, \xi_i \rangle} \in \tilde{B}$ and denote by $\tilde{E}_i := \tilde{\xi}_i \tilde{B}$ the closed right ideal in $\tilde{B}$ generated by $\tilde{\xi}_i$. Turn the Hilbert $\tilde{B}$–module $\tilde{E}_i$ into a correspondence over $\tilde{B}$ by putting $b \tilde{E}_i = 0$ for every $b \in \tilde{B}$ (and, of course, $\mathbf{1}_{\tilde{E}_i} = \tilde{\xi}_i$). Moreover, $E_i$ is a correspondence over $\tilde{B}$ in the unique way.[8] Define the $\tilde{B}$–correspondence $\tilde{\mathcal{E}}_i := \tilde{E}_i \otimes E_i$. Observe that $\tilde{\xi}_i := \tilde{\xi}_i \otimes \xi_i$ is a unit vector in $\tilde{E}_i$ and that $\langle \tilde{\xi}_i, \otimes \xi_i \rangle$ is nothing but the unitalization of $T_i$. As $E_s \otimes \tilde{E}_i = \{0\}$, we find

$$\tilde{\mathcal{E}}_s \otimes \tilde{\mathcal{E}}_i = (\tilde{E}_s \otimes E_i) \oplus (\tilde{E}_s \otimes E_i) \oplus (E_s \otimes E_i),$$

and, similarly,

$$\tilde{\mathcal{E}}_{t_0} \otimes \cdots \otimes \tilde{\mathcal{E}}_{t_1} = \bigoplus_{k=0}^n \tilde{E}_{t_0} \otimes \cdots \otimes \tilde{E}_{t_{k+1}} \otimes E_{t_k} \otimes \cdots \otimes E_{t_1}.$$  (2.1)

Accordingly,

$$\tilde{\xi}_{t_0} \otimes \cdots \otimes \tilde{\xi}_{t_1} = \bigoplus_{k=0}^n \tilde{\xi}_{t_0} \otimes \cdots \otimes \tilde{\xi}_{t_{k+1}} \otimes \xi_{t_k} \otimes \cdots \otimes \xi_{t_1}.$$  

Of course, this is the tensor product of unit vectors and, therefore, itself a unit vector. (A direct verification of this trivial fact alone from the right-hand side would be quite tedious."

Rather than the lattices of interval partitions of $(0, t]$, we use the lattices

$$\mathcal{J}_t := \{ t = (t_n, \ldots, t_1) : n \in \mathbb{N}, t_1 > 0, t_n + \ldots + t_1 = t \}$$

[6] The GNS-construction for a CP-map $T$ on (or between) unital (pre-)$C^*$–algebra(s) is due to Paschke [Pas73, Theorem 5.2]. The result is a correspondence $E$ and vector $\xi \in E$ such that $T$ is recovered as $T = \langle \xi, \bullet \xi \rangle$. If $\xi$ is cyclic for $E$, that is if $\xi$ generates $E$ as a correspondence, then we speak of the GNS-module, as in this case everything is determined up to suitable unitary equivalence.

[7] We also should like to say that the version in [BS00] is formulated for pre-Hilbert modules and extends easily to Hilbert modules, while here we write immediately for Hilbert modules. This is mere convenience, and whatever we write down in this section would work also in the algebraic context without any completion.

[8] Recall that left actions of correspondences are nondegenerate by definition.
(t > 0) as in [BS00]. We define a partial order on $\mathcal{J}_s$ by $s \leq t$, if there are $s_j \in \mathcal{J}_{s_j}$, $s = (s_m, \ldots, s_1) \in \mathcal{J}_s$ such that $t = s_m \sim \cdots \sim s_1$, where the join $\sim$ of two tuples $s = (s_m, \ldots, s_1) \in \mathcal{J}_s$, $t = (t_n, \ldots, t_1) \in \mathcal{J}_t$ is defined as $s \sim t = (s_m, \ldots, s_1, t_n, \ldots, t_1)$. See [BS00, Observation 4.2] for why we find this lattice more useful than the lattice of interval partition (to which it is isomorphic). For $t = (t_n, \ldots, t_1) \in \mathcal{J}_t$, it follows that defines a bilinear isometric embedding $E_t \to E_{t_1} \circ \cdots \circ E_{t_1}$ sending $\tilde{\xi}_t$ to $\tilde{\xi}_{t_1} \circ \cdots \circ \tilde{\xi}_{t_1}$. (To check that this mapping is isometric on $\tilde{\xi}_t$ simply observe that the missing term $k = n$ in the sum has “square length” $\langle \xi_j, \xi_i \rangle$ and that the left action of $b \in B$ gives 0 as it should.) For $t \geq s$ (so that $t = s_m \sim \cdots \sim s_1, s_j \in \mathcal{J}_{s_j}, s = (s_m, \ldots, s_1) \in \mathcal{J}_s$) we put $\beta_{ts} = \beta_{s_m(s_m)} \cdots \beta_{s_1(s_1)}$. As in [BS00, Section 4] the $\beta_{ts}$ form an inductive system, the inductive limits $\tilde{\mathcal{E}}_t := \lim \text{ind}_{E_{t_j}} \tilde{E}_{t_j}$ form a product system $\tilde{\mathcal{E}}^\circ$ and the $\tilde{\xi}_t \in \tilde{\mathcal{E}}_t$ form a unital unit $\tilde{\xi}^\circ$ for $\tilde{\mathcal{E}}^\circ$. (Of course, $\tilde{\mathcal{E}}_s \circ \tilde{\mathcal{E}}_t$ imbeds into $\tilde{\mathcal{E}}_{s+t}$, and surjectivity follows from surjectivity of $\beta_{s(t),s+t}$, which can be used to insert a time point if one should be missing. In addition to [BS00, Section 4], see also Barreto, Bhat, Liebscher and Skeide [BBL04, Section 4.3] and Skeide [Ske03a, Ske01] for similar two-sided limits. See [BS00, Appendix A] for details about inductive limits of Hilbert modules and correspondences.)

Clearly, $\text{span}\ B\tilde{\xi}_tB$ contains $\tilde{\mathcal{E}}_t = (1 - 1)\tilde{\xi}_t$. This shows that $\tilde{\mathcal{E}}^\circ$ is generated (as a product system) by $\tilde{\mathcal{E}}^\circ$ and $\tilde{\xi}^\circ$. Also, since $\tilde{\xi}_t = (1 - 1)\tilde{\xi}_t$, we find that so that everything in $\tilde{\mathcal{E}}_t$ that lies in the complement of $E_t$ lies in the span of elements of the form $\tilde{\xi}_{s+t} \circ x_s$ ($0 \leq s < t, x_s \in E_s$). Note also that by (2.1), $1\tilde{E}_t$ is just $E_t$.

From the unital unit $\tilde{\xi}^\circ$ we construct an inductive limit $\tilde{\mathcal{E}} = \lim \text{ind}_E \tilde{\mathcal{E}}_t$ as in [BS00, Section 5] with the help of the isometric embeddings $\tilde{\mathcal{E}}_t \to \tilde{\xi}_t \circ \tilde{\mathcal{E}}_t \subset \tilde{\mathcal{E}}_{s+t}$. (In [BS00] we discussed the minimal case, but the construction works without changing a word also in the general case. See, for instance, [BBL04, Section 4.4] or Skeide [Ske03b].) We have the factorization $\tilde{\mathcal{E}} = \tilde{\mathcal{E}} \circ \tilde{\mathcal{E}}_t$ such that $\tilde{\theta}(a) = a \circ \text{id}_{\tilde{E}_t}$ defines an $E_0$–semigroup acting on $B^a(\tilde{\mathcal{E}})$, having the product system $\tilde{\mathcal{E}}^\circ$. Further, we have a unit vector $\tilde{\xi} = \lim \text{ind}_E \tilde{\xi}_t$ in $\tilde{\mathcal{E}}$, satisfying $\tilde{\xi} = \tilde{\xi} \circ \tilde{\xi}$, such that $(\tilde{\mathcal{E}}, \tilde{\theta}, \tilde{\xi})$ is a dilation of $\tilde{T}$.

Next we show how the dilation of $\tilde{T}$ restricts to a dilation of $T$. We proceed as in [BS00, Section 8]. We define $E := \tilde{E}1$ and observe that $E$ is a Hilbert $B$–module with a unit vector $\xi := \tilde{\xi}1$. As multiplication with 1 from the right defines a central projection $p$ in $B^a(\tilde{E})$ onto $E$, an operator $a \in B^a(\tilde{E})$ is in $B^a(E)$, if and only if $pa(= ap) = a$. So, for $a \in B^a(E)$ it follows that

$$\tilde{\theta}(a) = a \circ \text{id}_{\tilde{E}_t} = (pa) \circ \text{id}_{\tilde{E}_t} = a \circ \text{id}_{E_t}.$$
where, by slight abuse of notation, we denote the projection onto \( E_t \) in \( \mathcal{B}^a(\tilde{E}_t) \) (that is, left multiplication with \( 1 \in \mathcal{B} \)) by \( \text{id}_{E_t} \). This shows that \( \tilde{\partial}_t \) leaves \( \mathcal{B}^a(E) \) invariant. Moreover,

\[
\partial_t(\xi \xi^*)E = \tilde{\partial}_t(\xi \xi^*)E = \tilde{\partial}_t(\xi \xi^*)\tilde{\partial}_t(\xi \xi^*)E = 1\tilde{E}_t,1 = E_t1 = E_t,
\]

so that the product system of \( \partial \) is \( E^\otimes \). (It is an easy exercise to show that the identification \( \tilde{E}_s \odot \tilde{E}_t = \tilde{E}_{s+t} \) restricted to elementary tensors from \( E_s \odot E_t \) gives the correct identification \( E_s \odot E_t = E_{s+t} \).) Also \( \tilde{\partial}_t(\xi b \xi^*) = \tilde{\partial}_t(\xi b \xi^*) = \tilde{\partial}_t(\xi b \xi^*) \) and \( \xi = \tilde{\xi} \odot \tilde{\xi} = \tilde{\xi} = \tilde{\xi} = \tilde{\xi} = 1 \). Therefore, \( \tilde{\partial}_t(\xi \xi^*) = \tilde{\xi} \odot \tilde{\xi} = \tilde{\xi} = 1 \). Then, we obtain back the unit \( E_t \) and \((E, \partial, \xi)\) is a dilation of \( T \).

For showing Condition 2 in Theorem 1.2, we first show that a dilation fulfilling Condition 2 is determined uniquely. Let consider an inner product \((x, y)\) of elements \( x = \theta_t(\xi \xi^*)x \in E_t \) \((x \in E)\) and \( y \in E_t \) \((y \in E)\) and \( y \odot z = \theta_s(\xi \xi^*)y \odot \theta_t(\xi \xi^*)z \in E_{s+t} \) \((y, z \in E)\). We find

\[
\langle x, y \odot z \rangle = \langle \theta_t(\xi \xi^*)x, \theta_s(\xi \xi^*)y \odot \theta_t(\xi \xi^*)z \rangle = \langle \theta_s(\xi \xi^*)x, \theta_t(\xi \xi^*)y \odot \theta_t(\xi \xi^*)z \rangle = \langle \xi, x, y \odot z \rangle,
\]

so that these inner products (and, therefore, all inner products of \( E \)) can be calculated by using the product system structure of \( E^\otimes \) and the unit \( E_0 \).

It remains to show that the dilation we constructed fulfills Condition 2. But this follows from totality of the elements \( \tilde{\xi}_{t-s} \odot x_s \) \((0 \leq s \leq t, x_s \in E_s)\) in \( \tilde{E}_t \) and from \( \tilde{\xi}_1 = \tilde{\xi}_1 - 1 \tilde{\xi}_1 = \tilde{\xi}_1 - 1 \tilde{\xi}_1 \) (see the crucial [BS00, Observation 8.1]). Indeed, the elements \( \tilde{\xi} \odot \tilde{x}_t \) \((t \in \mathbb{S}, \tilde{x}_t \in \tilde{E}_t)\) are dense in \( \tilde{E} \) so that the elements \( \tilde{\xi} \odot \tilde{x}_1 \) are dense in \( E \). Now for all \( 0 \leq s \leq t \) we have

\[
\tilde{\xi} \odot \tilde{\xi}_{t-s} \odot x_s = \tilde{\xi} \odot \tilde{\xi}_{t-s} \odot x_s = \tilde{\xi} \odot (\tilde{\xi}_{t-s} \odot x_s) = \tilde{\xi} \odot x_s - \tilde{\xi} \odot \tilde{\xi}_{t-s} \odot x_s.
\]

Now \( \tilde{\xi} \odot x_s \) is in \( E_s = \theta_s(\xi \xi^*)E \) and \( \tilde{\xi} \odot \tilde{\xi}_{t-s} \odot x_s \) is in \( E_t = \theta_t(\xi \xi^*)E \). This shows that Condition 2 is fulfilled.

2.1 Remark. The discussion in [BS00, Section 12] shows how to adapt the arguments to von Neumann modules using the appendices of [BS00]. Taking into account that every \( W^* \)-module may be considered as a von Neumann module (by choosing a concrete representation of the underlying \( W^* \)-algebra), the result holds also for \( W^* \)-modules.

2.2 Remark. We would like to note that, like in [BS00, Section 8], the \( \mathbb{C} \)-linear codimension of \( E \) in \( \tilde{E} \) is 1. More precisely, \( \Omega := \tilde{\xi} \odot (1 - 1) \) is a vector with “length” \( \langle \Omega, \Omega \rangle = \tilde{1} - 1 \) such that \( \tilde{E} = \mathbb{C} \Omega \oplus E \). This follows by looking at (2.1) and from the fact that \( \Omega := \tilde{\xi} \odot (1 - 1) \) \((t > 0)\) is a vector with “length” \( \langle \Omega, \Omega \rangle = \tilde{1} - 1 \) such that \( \tilde{E}_t = \mathbb{C} \Omega_t \oplus \tilde{E}_t \) and further \( \tilde{E}_t = \mathbb{C} \Omega_t \oplus \tilde{E}_t \). Finally, also \( \Omega^\otimes = (\Omega_t)_{t \in \mathbb{S}} \) (with \( \Omega_0 := 1 \)) is a unit for \( E^\otimes \).
3 Duality between dilations

The scope of this section is to establish a duality between dilations with pre-assigned product system \((E^\circ, \xi^\circ)\) of a normal CP-semigroup \(T\) determined by a unit \(\xi^\circ\) in a product system \(E^\circ\) of (concrete) von Neumann correspondences over a von Neumann algebra \(\mathcal{B} \subset \mathcal{B}(G)\) and isometric dilations of the covariant representation \(\sigma^\circ\) of the commutant system \(E^\circ\) on \(G\) with \(\sigma_0' = \text{id}_{\mathcal{B}'}\) that is associated with that CP-semigroup.

Recall that a von Neumann algebra is a strongly closed \(*\)-algebra \(\mathcal{B} \subset \mathcal{B}(G)\) of operators acting nondegenerately on a Hilbert space \(G\). As usual, by \(\mathcal{B}' \subset \mathcal{B}(G)\) we denote the commutant of \(\mathcal{B}\). Similarly, a concrete von Neumann \(\mathcal{B}\)-module is a subset \(E\) of \(\mathcal{B}(G, H)\), where \(H\) is another Hilbert space, such that

1. \(E\) is a right \(\mathcal{B}\)-submodule of \(\mathcal{B}(G, H)\), that is, \(E\mathcal{B} \subset E\),
2. \(E\) is a pre-Hilbert \(\mathcal{B}\)-module with inner product \(\langle x, y \rangle = x^*y\), that is, \(E^*E \subset \mathcal{B}\),
3. \(E\) acts nondegenerately on \(G\), that is, \(\text{span} \, EG = H\), and
4. \(E\) is strongly closed.

If we wish to underline the Hilbert space \(H\), we will also write the pair \((E, H)\) for the concrete von Neumann \(\mathcal{B}\)-module. One may show (see Skeide [Ske00, Ske05b]) that a subset \(E\) of \(\mathcal{B}(G, H)\) fulfilling 1–3 (that is, \(E\) is a concrete pre-Hilbert \(\mathcal{B}\)-module) is a concrete von Neumann \(\mathcal{B}\)-module, if and only if \(E\) is self-dual, that is, if and only if \(E\) is a \(W^*\)-module over the von Neumann algebra \(\mathcal{B} \subset \mathcal{B}(G)\) considered as a \(W^*\)-algebra. The definition of concrete von Neumann modules is due to Skeide [Ske05a].

Identifying \(xg \in H\) with \(x \circ g \in E \circ G\), we see from 3 that \(H\) and \(E \circ G\) are canonically isomorphic.

3.1 Remark. In fact, if \(E\) is a pre-Hilbert module over a pre-\(C^*\)-algebra \(\mathcal{B} \subset \mathcal{B}(G)\), then one may construct the Hilbert space \(E \circ G\) with an embedding \(x \mapsto L_x \in \mathcal{B}(G, E \circ G)\) where we put \(L_x g := x \circ g\), transforming \(E\) into a concrete pre-Hilbert \(\mathcal{B}\)-module. For a von Neumann algebra \(\mathcal{B} \subset \mathcal{B}(G)\) we defined in Skeide [Ske00] that \(E\) is a von Neumann \(\mathcal{B}\)-module, if and only if \(E\) is strongly closed. Of course, in that way also a \(W^*\)-module over a \(W^*\)-algebra \(M\) may be turned into a von Neumann module after choosing a faithful normal unital representation of \(M\) on a Hilbert space \(G\), thus, turning \(M\) into a von Neumann algebra.

Returning to the concrete von Neumann \(\mathcal{B}\)-module \((E, H)\) we note that the representation \(b' \mapsto \text{id}_E \circ b'\) of \(\mathcal{B}'\) on \(E \circ G\) induces a normal unital representation \(\rho'\) of \(\mathcal{B}'\) on \(H\) which acts as \(\rho'(b')xg = xb'g\). We call \(\rho'\) the commutant lifting associated with \(E\). From \(\rho'\) we obtain back \(E\) as

\[
E = C_{\mathcal{B}'}(\mathcal{B}(G, H)) := \{x \in \mathcal{B}(G, H) : \rho'(b')x = xb' (b' \in \mathcal{B}')\}, \tag{3.1}
\]
(See [Ske05b] for a proof by calculating the double commutant of the linking von Neumann algebra in \( B(G \oplus H) \). This proof also shows that the commutant \( \rho'(B')' \) of the range of \( \rho' \) in \( B(H) \) may be identified with the von Neumann algebra \( \mathcal{B}^a(E) \subset B(H) \).) Conversely, if \( (\rho', H) \) is a normal unital representation of \( B' \) on the Hilbert space \( H \), then \( E := C_{\mathcal{B}}(\mathcal{B}(G, H)) \) as in (3.1) defines a concrete von Neumann \( B \)-module in \( \mathcal{B}(G, H) \), which gives back \( \rho' \) as commutant lifting. (The only critical task, nondegeneracy in Condition 3, follows from Muhly and Solel [MS02, Lemma 2.10].) We noted in Skeide [Ske03a] that the correspondence between von Neumann \( B \)-modules and normal unital representations of \( B' \) establishes an equivalence of categories (here with the adjointable maps, there with the bounded operators that intertwine the representations as morphisms). In [Ske05a] we gave a precise formulation for the category of concrete von Neumann \( B \)-modules, where the functor is, really, bijective and not only an equivalence.

A **concrete von Neumann correspondence** over a von Neumann algebra \( B \) is a concrete von Neumann \( B \)-module \( (E, H) \) with a left action of \( B \) such that \( \rho: B \to \mathcal{B}^a(E) \to B(H) \) defines a normal (unital) representation of \( B \) on \( H \). We call \( \rho \) the **Stinespring representation** associated with \( E \).

**3.2 Remark.** If \( E \) is the GNS-module of a (normal) CP-map, then \( \rho \) is, indeed, the Stinespring representation, while \( \rho' \) is (a restriction of) the representation constructed by Arveson [Arv69] in the section “lifting commutants”.

We observe that \( \rho(B) \subset \mathcal{B}^a(E) = \rho'(B')' \), that is, \( \rho' \) and \( \rho \) have mutually commuting ranges. Extending the correspondence between von Neumann \( B \)-modules and representations of \( B' \), we find a correspondence between von Neumann \( B \)-modules and representations of \( B' \), we find a correspondence between von Neumann \( B \)-module \( (E, H) \) and pairs \( (\rho', \rho, H) \) of normal unital representations with mutually commuting ranges. As an equivalence this has been observed in [Ske03a], while in [Ske05a] it is shown that for the category of concrete von Neumann \( B \)-modules (with bilinear adjointable maps as morphisms) we obtain a bijective functor (with the bounded maps that intertwine both the representations of \( B' \) and the representations of \( B \)).

A last almost trivial observation (once again in [Ske03a] up to equivalence and in [Ske05a], really, bijective) consists in noting that in the representation picture the roles of the representations \( \rho' \) and \( \rho \) are absolutely symmetric. Therefore, if we switch \( B \) and \( B' \), that is, if we interprete \( \rho \) as commutant lifting of \( B \) and \( \rho' \) as Stinespring representation of \( B' \), by

\[
E' := C_{\mathcal{B}}(\mathcal{B}(G, H)) := \{ x' \in \mathcal{B}(G, H) : \rho(b)x' = x'b \ (b \in B) \} \tag{3.2}
\]

we obtain a von Neumann \( B' \)-module which is turned into a von Neumann \( B' \)-correspondence by defining a left action via \( \rho' \). We call \( E' \) the **commutant** of \( E \). The commutant is a bijective functor from the category of concrete von Neumann \( B \)-modules onto the category of
concrete von Neumann \( B' \)–correspondences (in each case with the bilinear adjointable maps as morphisms that are, really, the same algebra \( B^a(E) \cap B^a(E') = \rho'(B')' \cap \rho(B)' \) of operators in \( B(H) \)). Obviously, \( E'' := (E')' = E \).

3.3 Remark. Muhly and Solel [MS04] have discussed a version of the commutant for \( W^* \)–algebras, called \( \sigma \)–dual, where \( \sigma \) is a faithful representation of the underlying \( W^* \)–algebra, that must be chosen, and the \( \sigma \)–dual depends on \( \sigma \) (up to Morita equivalence of correspondences [MS05a]). They extend the \( \sigma \)–dual to correspondences from \( A \) to \( B \) in [MS05a]. In [Ske05a] we discuss also a version of this for von Neumann algebras and von Neumann correspondences.

We are now ready to formulate how product systems and dilations of CP-semigroups and of covariant representations behave under the commutant. We mention that in the presence of invariant vector states the list may be extended to a duality of CP-maps that includes a duality between tensor dilations of a CP-maps on \( B \) and extensions from \( B' \) to \( B(G) \) of the dual of that CP-map; see Gohm and Skeide [GS05].

3.4 Theorem. Let \( B \subset B(G) \) be a von Neumann algebra (acting nondegenerately on the Hilbert space \( G \)) and denote by \( B' \) its commutant.

1. The commutant establishes a one-to-one correspondence between product systems \( E^\circ \) of concrete von Neumann correspondences over \( B \) (in the sense of [Ske05a]) and product systems \( E'^\circ \) of concrete von Neumann correspondences over \( B' \). Of course, \( E'^\circ = E^\circ \). The product systems \( E^\circ \) and \( E'^\circ \) have the same morphisms \( w^\circ = (w_t)_{t \in S} \), \( w_t \in B^a_{bil}(E_t) = B^a(E_t) \). (In fact, the commutant is a bijective functor between categories of product systems of concrete von Neumann correspondences.)

2. Normal \( E \)–semigroups \( \vartheta \) on \( B^a(E) \) with \( (E, H) \) a concrete von Neumann \( B \)–module that have \( E^\circ \) as associated product system correspond one-to-one to normal isometric covariant representations \( \tau^\circ \) of \( E'^\circ \) on \( H \). Moreover, \( \vartheta \) is an \( E_0 \)–semigroup, if and only if \( \tau^\circ \) is nondegenerate. \( E \) is strongly full, if and only if \( \tau^\circ \) is faithful (in which case necessarily every \( E_t \) is strongly full, respectively, the left action of \( B' \) on every \( E'_t \) is faithful).

3. Contractive units \( \xi^\circ \) for \( E^\circ \) correspond one-to-one to normal covariant representations \( \sigma^{\tau^\circ c} \) of \( E^{\sigma^\circ} \) on \( G \) with \( \sigma^{\tau^\circ c}_0 = \text{id}_{B^\circ} \).

4. Let \( T \) be the CP-semigroup determined by either of the ingredients of a pair \((\xi^\circ, \sigma^{\tau^\circ c})\) as in Number 3. Then weak dilations \((E, \vartheta, \xi)\) of \( T \) with \( E^\circ \) as associated product system correspond one-to-one to isometric dilations \( \tau^\circ \) of \( \sigma^{\tau^\circ c} \). In particular, existence of one implies existence of the other.
Proof. 1. This was indicated in the case of product systems of von Neumann correspondences in [Ske03a]. Here we have concrete von Neumann correspondences \( E_i = C_{\mathcal{B}}(\mathcal{B}(G, H_i)) \) and \( E'_i = C_{\mathcal{B}}(\mathcal{B}(G, H_i)) \) as in [Ske05a], so that a product system \( E^\otimes \) gives rise to a family \( (E'_i)_{i \in \mathbb{Z}} \) of von Neumann \( \mathcal{B}' \)–correspondences. What is still missing is the product system structure of this family. Computations of this type have been detailed also in Muhly and Solel [MS05b] (in the language of \( \sigma \)–duals) so that here we may content ourselves with a sketchy description. The identification \( E_s \otimes E_t \to E_{s+t} \) is given by an operator \( u_{s,t} \in \mathcal{B}(E_s \otimes H_t, H_{s+t}) \) that intertwines both \( \rho_{s+t} \) and the canonical action \( \text{id}_E \circ \rho'_t \) of \( \mathcal{B}' \) on a \( E_s \otimes H_t \) as well as \( \rho_{s+t} \) and the canonical action of \( \mathcal{B} \) on \( E_s \otimes H_t \). On the other hand, \( E_s \otimes H_t \) is canonically isomorphic to \( E'_t \otimes H_s \). Indeed, consider an element \( y'_{s'}g \) in the total subset \( E'_t G \) of \( H_t \). Then \( x_s \otimes y'_{s'}g \mapsto y'_{s'} \circ x_s \) defines a unitary \( E_s \otimes H_t \to E'_t \otimes H_s \) which intertwines the respective actions of \( \mathcal{B} \) and also the respective actions of \( \mathcal{B}' \). The image of \( u_{s,t} \) on \( E'_t \otimes H_s \) determines an identification \( E'_t \otimes E'_s \to E'_{s+t} \). We leave it as an exercise to check associativity and also the statement about the morphisms.

2. For \( E_0 \)–semigroups and nondegenerate covariant representations this is [Ske04, Theorem 7.4] in the discrete case and [Ske04, Theorem 8.2] in general with [Ske04, Remark 8.3] pointing out the extensions we need here. We explain this very briefly. The von Neumann \( \mathcal{B} \)–module \( E \) on the semigroup side and \( \tau'_0 \) on the covariant representation side are related by considering \( \tau'_0 \) as the commutant lifting associated with \( E \). (From this the statement about the relation between strongly full and faithful follows.) Next if \( \vartheta \) is a normal endomorphism of \( \mathcal{B}'(E) \), then \( \vartheta(1)E \) factors into \( E \otimes E \) or, equivalently, \( \vartheta(1)H \) factors into \( E \otimes H \subset H \). Therefore, \( \tau'_0(y'_s) : x \mapsto x \otimes y'_s \in H \) defines an isometric covariant representation \( (\tau'_0, \tau'_0) \) of \( E'_t \) on \( H \). (This representation is nondegenerate, if and only if \( \vartheta \) is unital.) Similarly, if \( (\tau'_0, \tau'_0) \) is an isometric covariant representation of \( E'_t \) on \( H \), then \( \text{span} \tau'_0(E'_t)H \) is isomorphic to \( E \otimes H \) via \( \tau'_0(y'_s)x \mapsto x \otimes y'_s \). By \( \vartheta(a) = a \otimes \text{id}_H \in \mathcal{B} \) we define a homomorphism \( \mathcal{B}'(E) \to \mathcal{B}(\text{span} \tau'_0(E'_t)H) \subset \mathcal{B}(H) \). Actually, the range is contained in \( \tau'(\mathcal{B}'(E)) = \mathcal{B}'(E) \) so that \( \vartheta \) is an endomorphism. It remains to check that the semigroup property of the family \( \vartheta \) corresponds exactly to the factorization property the family \( \tau' \) must fulfill taking into account how, according to the first part, the product system structures of \( E^\otimes \) and of \( E'^\otimes \) are related. (One sees that, nicely enough, the order of elementary tensors \( z'_t \otimes y'_s \) acting on \( xg \in H \) becomes reversed. This explains, roughly, why everything is compatible with the first part. We leave details as an exercise.)

3. For a single pair of correspondences \( E_i \leftrightarrow E'_i \) this is more or less [MS02, Lemma 2.16] and the remark that follows it. (In our notations, [MS02, Lemma 2.16] asserts that covariant representations \( (\sigma'_t, \text{id}_{\mathcal{S}'}) \) of \( E'_t \) on \( G \) correspond one-to-one to contractions \( \xi'_t \in \mathcal{B}(H_t, G) \). By the remark following [MS02, Lemma 2.16] normality of \( \text{id}_{\mathcal{S}'} \) alone implies normality of \((\sigma'_t, \text{id}_{\mathcal{S}'})\). However, we emphasize that the correspondence \( E_i \) is absent in [MS02], so it can be excluded that the statement can be found in [MS02].) We leave it as an exercise to check that the \( \xi_t \) form a unit for \( E^\otimes \), if and only if the \((\sigma'_t, \text{id}_{\mathcal{S}'})\) form a covariant representation of
(In the special case when $E^\circ$ is generated by $\xi^\circ$, that is if $E^\circ$ is the GNS-system of the CP-semigroup associated with $\xi^\circ$, an application of \cite[Proposition 3.1]{Ske02} shows that the statement is equivalent to \cite[Theorem 3.9]{MS02}. However, we emphasize that the product system $E^\circ$ is absent in \cite{MS02}, so it can be excluded that the statement can be found in \cite{MS02}.)

4. This is simply 2. and 3. put together. Of course, a unit vector $\xi \in E \subset B(G,H)$ is an isometry and allows to identify $G$ as a subspace of $H$ and provides us with a projection $\xi\xi^*$ onto that subspace that compresses the isometric representation $\tau^\circ$ to $\sigma^\circ$. Conversely, if $G \subset H$ and the projection $p$ onto $G$ compresses $\tau^\circ$ to $\sigma^\circ$, then $p$ is in the intertwiner space $E$. The unit vector $\xi := p \uparrow G$ has all the desired properties, that is, $\xi$ turns the $E$–semigroup $\vartheta$ (dual to $\tau^\circ$) into a dilation of $T$.

4 Proof of Theorem 1.1

let us discuss first the von Neumann case. We appeal to Theorem 3.4 and the existence result Theorem 1.2, where $M$ plays the role of $B'$ and $F^\circ$ plays the role of $E^\circ$. For that goal we have to show that the general case boils down to the case when $\sigma_0$ is faithful and nondegenerate. But this is easy.

First of all, as $\sigma_0(1)\sigma(y_t)\sigma_0(1) = \sigma_t(y_t)$ we may simply pass to the subspace $\sigma_0(1)G$ of $G$, so that now $\sigma_0$ is nondegenerate.

Then, if $\sigma_0$ is not faithful, we simply “add” a faithful nondegenerate representation $\hat{\sigma}_0$ of $\ker \sigma_0$ on a Hilbert space $\hat{G}$. More precisely, we pass to the covariant representation $\hat{\sigma}^\circ$ on $\hat{G} := \hat{G} \oplus G$ that is defined by setting

$$\hat{\sigma}_0 := \hat{\sigma}_0 \oplus \sigma_0 \quad \quad \quad \hat{\sigma}_t := 0 \oplus \sigma_t \quad (t > 0).$$

Then every isometric dilation of $\hat{\sigma}^\circ$ compresses further to $G$, giving back $\sigma^\circ$.

Now we set $B := \hat{\sigma}_0(M)' \subset B(\hat{G})$, so that $M \cong B'$ as $W^*$–algebras. Put $E_i = C_{B'}(B(\hat{G}, H_i))$ where $H_i := F_i \odot \hat{G}$ with $\rho'_i$ and $\rho_i$ the canonical action of $B'$ and of $B$, repectively, so that $F^\circ \cong E^\circ$ as product system of $W^*$–correspondences. ($E^\circ$ is precisely what \cite{MS04, MS05b} would call the $\hat{\sigma}_0$–dual of $F^\circ$.) Under these isomorphisms the covariant representation $\sigma^\circ$ of $F^\circ$ induces a normal covariant representation $\sigma^\circ$ of $E^\circ$ on $\hat{G}$ with $\sigma'_0 = \text{id}_B$. So dilating as in Theorem 1.2 the CP-semigroup $T$ on $B$ associated to the unit $\xi^\circ$ corresponding to $\sigma^\circ$ by Theorem 3.4(3), by Theorem 3.4(4) we obtain an isometric dilation of $\sigma^\circ$ and, therefore, an isometric dilation of $\sigma^\circ$.

Now let us discuss the $C^*$–case, that is, $M$ is a $C^*$–algebra and $F^\circ$ a product system of $C^*$–correspondences over $M$ with a covariant representation $\sigma^\circ$ on a Hilbert space $G$. Our scope is simply to pass to the double commutant $F''^\circ$ of $F^\circ$ which is a product system of concrete von
Neumann modules, to show that \( \sigma \odot \) extends to a normal covariant representation \( \sigma'' \odot \) of that double commutant and, then, to apply the preceding discussion to \( F'' \odot \). To that goal we must choose a faithful representation of \( M \), and we must choose it carefully if we want that the left action of \( M \) is sufficiently normal.

So let \( K \) be the representation space of the universal enveloping von Neumann algebra \( M^{**} \) of \( M \). In this way we identify \( M \subset \mathcal{B}(K) \) and \( M^{**} = M'' \). Put \( K_t := F_t \odot K \). On \( K_t \) we have the normal commutant lifting \( \pi'_t \) of \( M' \). (The fact that \( \pi'_t \) is normal follows easily from the fact that \( \mathcal{B}(K, K_t) \) has enough intertwiners \( y_t \in F_t \) for the action of \( M' \) via \( \pi'_t \) on \( K_t \) and the defining action of \( M' \) on \( K_t \).) The left action of \( M \) on \( F_t \) gives rise to a representation \( \pi_t \) of \( M \) on \( K_t \). By the universal property of \( M'' \) this representation extends to a unique normal representation \( \pi''_t \) of \( M'' \) on \( K_t \). Put \( F''_t := C_M(\mathcal{B}(K, K_t)) \supset F_t \). Clearly, \( (F''_t, K_t) \) is a concrete von Neumann correspondence over \( M'' \) and the product system structure of \( F \odot \) extends uniquely to \( F'' \odot \).

We return to the covariant representation \( \sigma \odot \) of \( F \odot \). First, we observe that \( \sigma_0 \) extends uniquely to a normal representation \( \sigma''_0 \) of \( M'' \). Then, as in the proof Theorem 3.4(3), \( \sigma \odot \) gives rise to a unit \( \xi'' \odot \) for \( F'' := (F'') \odot \). Further, each \( \xi''_t \in F'_t \) gives rise to a normal representation \( (\sigma''_t, \sigma''_0) \) of \( F''_t \), which clearly extends \( (\sigma_t, \sigma_0) \). (This is exactly the statement of the remark following [MS02, Theorem 2.16].) Of course, the \( \sigma''_t \) form a normal representation \( \sigma'' \) of \( F'' \) so that now we are ready to apply the \( W^* \)–version of Theorem 1.1, obtaining an isometric dilation \( \tau'' \odot \) of \( \sigma'' \) that restricts to an isometric dilation \( \tau^\odot \) of \( \sigma^\odot \).

References


