Interacting Fock Spaces and Subproduct Systems

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Abstract

Interacting Fock spaces are the most general $\mathbb{N}_0$–graded (pre-)Hilbert spaces with creation operators that have degree 1 and generate everything out of a single vacuum vector Ω. It is the creators alone that generate the space out of the vacuum; so the same is true for the non-selfadjoint operator algebra generated be the creators. A formal definition has been given by Accardi, Lu, and Volovich (1997). Forthcoming work by Accardi and Skeide (2008), gave a different but equivalent definition, and also several desirable properties (embeddability, and what we are going to call here regularity, but also embeddability in Cuntz-Pimsner -Toeplitz type algebras) have been pointed out there.

In these notes we show that every interacting Fock space is embeddable, provided we ask the question if it is the right way. (This requires a new more flexible definition. The definition does not allow for more interacting Fock spaces, but for more freedom how to obtain them and capture their structure, eliminating a piece of construction that is present in the preceding definitions. The theorem is, that this missing piece can always be chosen so that the interacting Fock space becomes embeddable. The same statement for regularity fails.) Embeddability allows to recover an interacting Fock space as so-called $\varkappa$–interacting Fock space. ($\varkappa$ is an operator on a usual full Fock space and the creators $a^*(x)$ of the former are recovered as $\varkappa\ell^*(x)$ on the latter, where $\ell^*(x)$ are the usual creators.) We show that interacting Fock spaces are classified by the $\varkappa$. We give criteria for when the creators of an interacting Fock spaces are bounded in general and under regularity. If all creators are bounded, then the Banach algebra and the C^*–algebra generated by them, embed into the tensor algebras and the Cuntz-Pimsner-Toeplitz algebras, respectively, associated with several suitably chosen C^*–correspondences.

We illustrate all this in the case of interacting Fock spaces coming from so-called subproduct systems, and determine for which $\varkappa$ the $\varkappa$–interacting Fock space comes from a subproduct system.

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1 Introduction

The gist of the definition of *interacting Fock space* by Accardi, Lu, and Volovich [ALV97, Definition 18.1], is:

1.1 Definition. Let H be a (complex) vector space and form the ***tensor algebra*** $\mathcal{F}(H) := \Omega\mathbb{C} \oplus \bigoplus_{n\in\mathbb{N}} H^{\otimes n}$ over H,where Ω is some nonzero reference vector, the ***vacuum***. For each $x \in H$, define the ***creation operator*** $\ell^*(x)$ on $\mathcal{F}(H)$ by

$$\ell^*(x)X_n \;:=\; x \otimes X_n \quad (X_n \in H^{\otimes n}, n \geq 1), \qquad\qquad \ell^*(x)\Omega \;:=\; x$$

(that is $\ell^*(x)X = xX$, the product in the tensor algebra with unit Ω, for all $X \in \mathcal{F}(H)$). Put $H^{\otimes 0} := \Omega\mathbb{C}$. Suppose on each $H^{\otimes n}$ ($n \in \mathbb{N}$) we have a semiinner product $(\bullet,\bullet)_n$ with kernel $\mathcal{N}_n$ and put $(\Omega,\Omega)_0 := 1$, so that $(\bullet,\bullet) := \bigoplus_{n\in\mathbb{N}_0}(\bullet,\bullet)_n$ is a semiinner product on $\mathcal{F}(H)$ with kernel $\mathcal{N} = \bigoplus_{n\in\mathbb{N}_0} \mathcal{N}_n$. Put $H_n := H^{\otimes n}/\mathcal{N}_n$ and $\mathcal{I} := \mathcal{F}(H)/\mathcal{N}$. Then

$$\mathcal{I} \;=\; \bigoplus_{n\in\mathbb{N}_0} H_n$$

(we omit the simple proof; essentially $H_n \ni X_n + \mathcal{N}_n = X_n + \mathcal{N} \in \mathcal{I}$ for $X_n \in H^{\otimes n}$). We say the pre-Hilbert space $\mathcal{I}$ is an ***ALV-interacting Fock space*** (denoting this situation by $\mathcal{I} = (H, ((\bullet,\bullet)_n)_{n\in\mathbb{N}_0})$), if

$$H \otimes \mathcal{N}_n \;\subset\; \mathcal{N}_{n+1} \tag{$*$}$$

(that is, $H \otimes \mathcal{N} \subset \mathcal{N}$), so that $a^*(x)\colon X + \mathcal{N} \mapsto \ell^*(x)X + \mathcal{N}$ well-defines the ***creation operators*** $a^*(x)$ on $\mathcal{I}$.

1.2 Remark. We collect some notes that should be mentioned but, otherwise (like all our remarks), should not interrupt the flow of reading.

1. The notion of *interacting Fock space* was motivated by an example due to Accardi and Lu [AL92, AL96], emerging from QED. In this example, actually, the semiinner product is on the tensor algebra over a $\mathcal{B}$–bimodule (all tensor products over $\mathcal{B}$), turning the quotient into a pre-Hilbert module. In fact, one might study also these more general ***interacting Fock modules***. We emphasize that, here, we are concerned only with the scalar case. The full Fock module does occur, however, in its "unperturbed" form, when we discuss that the algebras generated by the creators on an interacting Fock space embed into Cuntz-Pimsner-Toeplitz algebras. Still, it might be noteworthy that the papers [AL92, AL96] are likely to host the first occurrence of full Fock modules even before Pimsner [Pim97] and Speicher [Spe98], and that the three contexts are entirely different.

2. The only true difference between Definition 1.1 and [ALV97, Definition 18.1] is (apart from some unnecessary requirements in the latter that are fulfilled automatically) that, here, we do not require that the creators possess a (formal) adjoint (in which case they are well-defined, automatically), but that we produce well-definedness by the kernel condition. In fact, as a minor side effect, in these notes we also free a number of results from Accardi and Skeide [AS08] from the requirement that creators have adjoints.

3. As the reader will have noticed, by the construction in Definition 1.1, we are concerned with pre-Hilbert spaces, and both tensor products and direct sums are algebraic. Despite the fact that in the end we are interested basically in the case when the creators are bounded and that, therefore, we may and will complete the pre-Hilbert spaces in this case, for several of the strongest results it is indispensable to wait with this step until the last moment. In fact, even when all creators are bounded, certain operators that parametrize interacting Fock spaces, will remain unbounded.

4. The scope of the notion of *interacting Fock space* is to capture, in some sense, the most general situation of a Fock type pre-Hilbert space. What *in some sense* means, becomes clearer in a moment when we discuss the definition from [AS08]. We do not claim that all spaces that are somehow related to Fock spaces are captured. (The GNS-spaces of *temperature states* on the *CCR-algebras* are not. Also Fock spaces from *species* discussed by Guta and Maassen [GM02] are not. Actually, the latter would fit quite nicely into a description by Fock modules.) But we would not like to dispense with the properties that interacting Fock spaces possess.

The, in a sense, simplest class of interacting Fock spaces possible is captured in the following example by Accardi and Bozejko [AB98]. Despite its striking simplicity, it captures to a surprisingly large extent rudimentary forms of the most important structure results on interacting Fock spaces; for this reason we repeat it here once more.

1.3 Example. We consider the case $H = \mathbb{C}$, a so-called ***one-mode interacting Fock space***. So, $\mathcal{F}(\mathbb{C}) = \bigoplus_{n\in\mathbb{N}_0} \mathbb{C}^{\otimes n}$ and we denote $e_0 := \Omega$ and $e_n := 1^{\otimes n}$. A family of semiinner products is determined by the numbers $\ell_n = \langle e_n, e_n \rangle \geq 0$. For that the $\langle \bullet, \bullet \rangle_n$ determine an interacting Fock space, we must have $\ell_0 = 1$ and $\ell_n = 0 \Rightarrow \ell_{n+1} = 0$. These conditions are also sufficient. It follows that there are (unique, if $k_n = 0 \Rightarrow k_{n+1} = 0$) numbers k_n such that $\ell_n = k_n \ldots k_1$.

Suppose μ is a (nonzero) symmetric measure on the real line with finite moments of all orders. Then the orthogonal polynomials P_n of μ satisfy and are determined by the following recursion

$$P_0(t) = 1, \qquad P_1(t) = t, \qquad tP_n(t) = P_{n+1}(t) + k_n P_{n-1}(t) \quad (n \geq 1),$$

for unique (positive) numbers k_n. (If μ is not symmetric, then on the right-hand side of the recursion there is also term proportional to P_n. [AB98] take into account also this case; here, we ignore it.)

Since $\int P_m(t)P_n(t)\mu(dt) = \delta_{m,n}\ell_n$ and since the P_n are real, it follows that $e_n + \mathcal{N}_n \mapsto P_n$ defines an isometry from $\mathcal{I}$ onto $\operatorname{span} P_{\mathbb{N}_0} \subset L^2(\mu)$. The creation operator $a^* := a^*(1)\colon e_n+\mathcal{N}_n \mapsto e_{n+1} + \mathcal{N}_{n+1}$ has an adjoint $(a^*)^* =: a\colon e_n + \mathcal{N}_n \mapsto (e_{n-1} + \mathcal{N}_{n-1})k_n$ (with $e_{-1} := 0$), and the crucial observation in [AB98] is that (for symmetric μ) he isomorphism $\mathcal{I} \to \operatorname{span} P_{\mathbb{N}_0}$ acts as

$$(a^* + a)(e_n + \mathcal{N}_n) \longmapsto tP_n,$$

that is, $a^* + a$ on $\operatorname{span} P_{\mathbb{N}_0}$ is nothing but multiplication with the function t. In the context of these notes, we are more interested in the following fact. Suppose we equip $\mathcal{F}(\mathbb{C})$ with the canonical inner product where the e_n are orthonormal. Then we may embed $\mathcal{I}$ into $\mathcal{F}(\mathbb{C})$ via the (adjointable) isometry $\xi\colon e_n + \mathcal{N}_n \mapsto e_n\sqrt{\ell_n}$ and

$$\xi a^*(x)\xi^* = \varkappa\ell^*(x), \tag{$**$}$$

where $\varkappa$ is some square root of the operator $k\colon e_n \mapsto e_nk_n$ on $\mathcal{F}(\mathbb{C})$. In fact, one of the main results of these notes is that every interacting Fock space $\mathcal{I}$ can be recovered as $\xi\mathcal{I} \subset \overline{\mathcal{F}(H)}$ for a suitable pre-Hilbert space H in such a way that the creators have the same form as above. Moreover, the $\varkappa$ suitably parametrize interacting Fock spaces.

After this example, we return (really only for a moment) to the situation in Definition 1.1 where H is just a vector space. An ALV-interacting Fock space $\mathcal{I} = (H, (\langle\bullet,\bullet\rangle_n)_{n\in\mathbb{N}_0})$ comes shipped with the ***creator map*** $a^*\colon H \to \mathcal{L}(\mathcal{I})$ from H into the linear operators on $\mathcal{I}$, which is linear and satisfies

$$\operatorname{span} a^*(H)H_n = H_{n+1}. \tag{$***$}$$

This means, in particular, that everything in $\mathcal{I}$ is created out of the vacuum Ω by successive application of creation operators $a^*(x)$. In the definition by Accardi and Skeide [AS08], emphasis is put on the family of pre-Hilbert spaces H_n and the creator map a^*. A formulation of [AS08, Definition 2.2] that matches the situation of Definition 1.1 is:

1.4 Definition. Let $(H_n)_{n\in\mathbb{N}_0}$ be a family of pre-Hilbert spaces where $H_0 = \Omega\mathbb{C}$ for a fixed unit vector Ω, the ***vacuum***, and put $\mathcal{I} := \bigoplus_{n\in\mathbb{N}_0} H_n$. Let H be a vector space and suppose $a^*\colon H \to \mathcal{L}(\mathcal{I})$ is a linear map satisfying $(***)$. Then $\mathcal{I}$ is an ***interacting Fock space*** based on H (denoted as $\mathcal{I} = ((H_n)_{n\in\mathbb{N}_0}, a^*)$).

Let us convince ourselves that Definitions 1.1 and 1.4 speak about "the same" thing.

- We know already that every ALV-interacting Fock space $\mathcal{I} = (H, (\langle\bullet,\bullet\rangle_n)_{n\in\mathbb{N}_0})$ is an interacting Fock space based on H via $H_n = H^{\otimes n}/\mathcal{N}_n$ and $a^*\colon x \mapsto a^*(x)$ (obviously, by definition, having the same creators $a^*(x)$).

- Every interacting Fock space $\mathcal{I} = ((H_n)_{n\in\mathbb{N}_0}, a^*)$ based on H, comes along with a linear surjective map $\Lambda := \bigoplus_{n\in\mathbb{N}_0} \Lambda_n \colon \mathcal{F}(H) \to \mathcal{I}$ where $\Lambda_n \in \mathcal{L}(H^{\otimes n}, H_n)$ is defined by

$$\Lambda_n \colon x_n \otimes \ldots \otimes x_1 \;\longmapsto\; a^*(x_n)\ldots a^*(x_1)\Omega \tag{$****$}$$

  and $\Lambda_0 \colon \Omega \mapsto \Omega$. Then for the semiinner products $(\bullet,\bullet)_n := \langle \Lambda_n\bullet, \Lambda_n\bullet\rangle$ on $H^{\otimes n}$, the map $\Lambda_n(x_n \otimes \ldots \otimes x_1) \mapsto x_n \otimes \ldots \otimes x_1 + \mathcal{N}_n$ establishes a unitary $H_n \to H^{\otimes n} + \mathcal{N}_n$. Moreover, from

$$\Lambda_{n+1}(\ell^*(x)X_n) \;=\; a^*(x)\Lambda_n X_n,$$

  it follows that the semiinner products fulfill $(*)$ (and $(\Omega,\Omega)_0 = 1$) and that, under the stated isomorphism, the ALV-interacting Fock space $(H, ((\bullet,\bullet)_n)_{n\in\mathbb{N}_0})$ has the same creators as $\mathcal{I} = ((H_n)_{n\in\mathbb{N}_0}, a^*)$.

(This has not been clarified that explicitly in [AS08].) Note that an ALV-interacting Fock space, with the structures defined in the first part of the preceding discussion, **is** an interacting Fock space based on H, while the isomorphism that identifies in the second part an interacting Fock space as an ALV-interacting Fock space cannot be discussed away. We, therefore, as a ***convention***, will always consider ALV-interacting Fock space as interacting Fock spaces based on H. With this convention, being ALV for an interacting Fock space is an extra information telling how the interacting Fock space has been obtained.

We stated Definition 1.4 for H just a vector space in order to be compatible with Definition 1.1. As compared with [AS08, Definition 2.2] (where H is required to be a pre-Hilbert space), in Definition 1.4 (like in Definition 1.1; see Remark 1.2(2)), we also have removed the condition that the $a^*(x)$ be adjointable. In either case, we will speak of an ***adjointable*** interacting Fock space if all creators have an adjoint. However, in applications H is (almost) always a pre-Hilbert space; so, from now on, as a ***convention***, we shall always assume (adding to Definitions 1.1 and 1.4) that H is a pre-Hilbert space. This means, $\mathcal{F}(H)$ does already possess an inner product $\langle\bullet,\bullet\rangle$ arising from tensor product and direct sum of pre-Hilbert spaces. In other words, $\mathcal{F}(H)$ is not just the tensor algebra over vector space H, but the (***algebraic***) ***full Fock space*** over the pre-Hilbert space H.

As already illustrated in Example 1.3, the interplay between the semiinner product $(\bullet,\bullet)$ on $\mathcal{F}(H)$ and the inner product $\langle\bullet,\bullet\rangle$ on $\mathcal{F}(H)$ plays a very important role. In fact, when interacting Fock spaces are obtained by introducing a semiinner product on $\mathcal{F}(H)$, then in practically all examples of this type the semiinner product is obtained from the original inner product of the pre-Hilbert space $\mathcal{F}(H)$ with the help of a positive operator $L = \bigoplus_{n\in\mathbb{N}_0} L_n$ (with $L_0\Omega = \Omega$) as $(\bullet,\bullet) := \langle\bullet, L\bullet\rangle$. We discuss such *positive operator induced* or *POI-interacting Fock spaces* in Section 2. In particular, we push forward to the more general Definitions 1.1 and 1.4 the result from [AS08] that POI-interacting Fock spaces are precisely those ALV-interacting Fock spaces that are *regular* interacting Fock spaces based on a pre-Hilbert space H in the sense that the canonical surjection $\Lambda\colon \mathcal{F}(H) \to \mathcal{I}$ has an adjoint $\mathcal{I} \to \overline{\mathcal{F}(H)}$. In Theorem 3.5, we show that a large class of interacting Fock spaces is non-regular.

As we just mentioned, almost all examples that have been recognized as interacting Fock spaces, are explicitly given as POI-interacting Fock spaces. A large class of examples arising as Fock spaces of so-called (discrete, one-parameter) *subproduct systems* (crucial for us, as the title of these notes tells), despite in the end being regular, too, comes along from the beginning as an interacting Fock space based on H. (The QED-example from [AL92, AL96] would be an example for an in interacting Fock *module* defined by analogy with Definition 1.1; we do not know if it could be obtained as the analogue of POI or not.) We discuss this in Section 7 and characterize all interacting Fock spaces that come from subproduct systems.

Already [AS08] pointed out *embeddability*, that is, existence of an *even*, *vacuum-preserving* isometry $\xi\colon \mathcal{I} \to \overline{\mathcal{F}(H)}$, as a crucial property, which an interacting Fock space based on H may possess or not. Under embeddability, creators can be written as in ($**$) for a suitable (even, vacuum-preserving) operator $\varkappa$ (that goes from the dense subspace $\xi\mathcal{I} \oplus (\xi\mathcal{I})^\perp$ of $\overline{\mathcal{F}(H)}$ to the dense subspace $(H \otimes (\xi\mathcal{I} \oplus (\xi\mathcal{I})^\perp)) \oplus \Omega\mathbb{C})$. Apart from the (minor) effort to push this forward to the situation of Definition 1.4 (dropping adjointability), we show two major results. Firstly, the operator $\varkappa$ is uniquely determined by ($**$) (and ξ) and satisfies two extra conditions; moreover, varying $\varkappa$ and the subspaces corresponding to $\xi\mathcal{I} \cong \mathcal{I}$, we get a parametrization (up to suitable isomorphism) of all interacting Fock spaces based on H in terms of $\varkappa$ (Theorem 4.5), so-called *$\varkappa$–interacting Fock spaces* (Definition 4.6). Secondly, we show that missing embeddability is a consequence of a badly chosen basing $a^*\colon H \to \mathcal{L}(\mathcal{I})$; in fact, in Theorem 3.4 we show that every interacting Fock space **can** be based embeddably. Putting these two results together, in Theorem 4.8 we get that every interacting Fock space is (isomorphic to) a $\varkappa$–interacting Fock space.

Both definitions, ALV-interacting Fock spaces (with its subclass of POI-interacting Fock spaces) and interacting Fock spaces based on a pre-Hilbert space (with its subclass of regular interacting Fock spaces), are relative to a chosen pre-Hilbert space H. For several reasons it is indispensable to come up with yet another (new) definition of (*abstract*) *interacting Fock space* (see Definition 3.1) that abandons this dependence on H. The pre-Hilbert space H parametrizes the set $A^* := a^*(H)$ of all creation operators by means of the creator function a^*. But what interests in quantum probability and operator algebras is the algebra $\mathcal{A}$ (or $*$–algebra $\mathcal{A}^*$) generated by these operators (plus, in some contexts, the ***vaccum state*** $\langle\Omega, \bullet\Omega\rangle$). If the parametrization is not well-done (for instance, if an interacting Fock space based on H is not embeddable), maybe it can be done better (changing the pre-Hilbert space H and the creator map a^*, but maintaining the set of creators A^*); Theorem 3.4 tells that precisely this is always possible. And regarding regularity, Theorem 3.5 tells precisely that there is no possibility to turn a so-called *non-nilpotent full* interacting Fock space into a regular interacting Fock space based on some pre-Hilbert space H. Moreover, Davidson, Ramsey, and Shalit [DRS11] showed that a subclass of subproduct systems (closely related to question from algebraic geometry and multivariate operator theory) is classified up to isomorphism by the isomorphism classes of the (non-selfadjoint) operator algebras $\overline{\mathcal{A}}$ generated by its creators. (This fails for the selfadjoint operator algebras $\overline{\mathcal{A}^*}$.)

Motivated by all this, Definition 3.1 substitutes in Definition 1.4 the pre-Hilbert space H and the creator map a^* by the vector subspace A^* of $\mathcal{L}(\mathcal{I})$ and the cyclicity condition in $(***)$ with $\operatorname{span} A^* H_n = H_{n+1}$. We will say, $\mathcal{I}$ is an (abstract) ***interacting Fock space***, and denote this situation by $\mathcal{I} = ((H_n)_{n\in\mathbb{N}_0}, A^*)$. Of course, choosing a pre-Hilbert space H and a linear surjection $a^*\colon H \to A^*$ we turn $\mathcal{I}$ into an interacting Fock space based on H.

1.5 Remark. Maintaining, at least in principle, the cyclicity condition in $(***)$ and the one-dimensional vacuum sector $\Omega\mathbb{C}$, what could possibly be further generalizations of the definition of interacting Fock space? The spaces H_n characterize a common domain of the creation operators; requiring (as we do) that the common domain is invariant under creation operators, makes sure that we can consider the algebra generated by the creators. But, suppose we weaken that condition requiring only that the creators map H_n into $\overline{H_{n+1}}$. Let us define D_n to be the span of the right-hand side of $(****)$, that is, what n creators can create out of the vacuum Ω. Then we would have to require that D_n is a subspace of H_n (in order to be able to apply the $(n+1)$st creator) and we would have to require the D_n is dense in H_n (substituting $(***)$). The direct sum over the D_n (with $D_0 = H_0$) is, then, an interacting Fock space in its own right, sitting densely in the direct sum over the H_n. One would, then, expect that what the creators do on the bigger domain H_n has to be determined by what they do on the smaller domain D_n. This amounts to requiring that the former sits in the closure of the latter, that is, a mild regularity condition without which probably nobody wants to work. In particular, bounded creators fulfill this, and the procedure is very similar to what we do in Section 7 when extracting from a subproduct system of Hilbert spaces the pre-Hilbert spaces that form an interacting Fock space. So, this does not add anything to the existing definitions.

A last possibility to get something more general, would be to allow that each creator has its own domain. But also here, any attempt to formulate $(***)$ also in this context, would lead to same result that nothing new is added. So, we really think that, in this sense, probably the definition of interacting Fock space has reached its maximum generality.

In Section 6 we address the question when the creators $a^*(x)$ are bounded. This question is interesting only when $\mathcal{I}$ is embeddably based on H, because in the case of an interacting Fock space based on H and an abstract interacting Fock space, it depends just on how a^* and A^*, respectively, are chosen. Consequently, our results are expressed in terms of the parameter $\varkappa$ in the general case and of the positive operator L in the case of POI-interacting Fock spaces. Among the results there are: Boundedness of all $a^*(x)$ (even boundedness of the creator map a^*) does not imply regularity; boundedness of $\varkappa$ is sufficient, but not necessary; boundedness of L is neither sufficient, nor necessary. The necessary and sufficient criterion that all $a^*(x)$ are bounded, given by the following (unbounded operator) inequality

$$\ell(x)L\ell^*(x) \ \leq \ M_xL$$

for all $x \in H$ (together with an analogue criterion for boundedness of the creator map), answers the long standing question when POI-interacting Fock spaces have bounded creators (creator maps).

It is known since [AS08] how, in the case of adjointable interacting Fock spaces, to embed the ($*$–)algebra $\mathcal{A}^{(*)}$ generated by the creators into the ($*$–)algebra generated by usual creators in a (usual) full (not interacting!) Fock module. (In [AS08], this was done in the general case, and in the case of unbounded creators it was a tough problem to control positivity of the inner products on this algebraic full Fock module over a bimodule over a $*$–algebra of unbounded operators.) Once we assured (for instance, by the criteria in Section 6) that all creators are bounded, we are back in the framework of C^*–correspondences and the ($*$–)algebra $\mathcal{A}^{(*)}$ embeds into a usual Pimsner-Toeplitz algebra on a (completed) Fock module; this is subject of Section 5. (A recent example is due to Kakariadis and Shalit [KS15].) This embedding into a *tensor algebra* (the non-selfadjoint version of Pimsner-Toeplitz algebra; see Muhly and Solel [MS98]), the containing Pimsner-Toeplitz algebra, and in the end the universal Cuntz-Pimsner algebras [Pim97] opens a whole range of new questions for future work.

Notation. For the discussion of interacting Fock spaces we need to work with pre-Hilbert spaces. Direct sums and tensor products are understood algebraically.[1] Consequently, we need the following spaces of operators. The space $\mathcal{L}(H, H')$ of linear maps from the pre-Hilbert space H to the pre-Hilbert space H'. Of course, here and in a similar way for all other spaces

[1] For at least two reasons, this is not exaggerated generality, but necessary and unavoidable flexibility. Firstly, it actually quite a bit lightens notation when we discuss spaces where, like the Boson Fock space (this is Example 2.3(1) for $q = 1$), the creation operators are unbounded; and we, surely, would not be happy to exclude classical examples like the Boson Fock space from the discussion. Secondly, yes, in the end we are interested in the case of bounded creation operators as they occur, for instance, from subproduct systems, and will complete the interacting Fock spaces; but, as our Theorems 6.4 and 6.6 show, it is not possible to characterize efficiently interacting Fock spaces with bounded creators by just bounded parameters $\varkappa$ or L. (This resembles a bit the characterization of morphisms of time ordered product systems from Barreto, Bhat, Liebscher, and Skeide [BBLS04, Secion 5.2], which is quite a bit easier than the characterization of *bounded* morphisms; see Bhat [Bha01, Section 6].) And we do not wish to loose these cases.

of operators, in the case $H' = H$ we will write $\mathcal{L}(H)$. The space $\mathcal{L}^a(H, H')$ contains those elements of $\mathcal{L}(H, H')$ that have an adjoint in $\mathcal{L}(H', H)$. We do **not** assume that an adjoint has maximal domain. (For $a \in \mathcal{L}^a(H, H')$, the domain of a^* is H' and H' is mapped by a^* into its codomain H.) We use the letter $\mathcal{B}$ to indicate the bounded parts of these spaces. For fixed H, H', an operator $a \in \mathcal{L}(H, \overline{H'})$ is called ***weakly adjointable*** if there exists $a^* \in \mathcal{L}^a(H', \overline{H})$ such that $\langle ah, h' \rangle = \langle h, a^*h' \rangle$ for all $h \in H, h' \in H'$. (Note: *Weakly adjointable* is a notion relative to two chosen pre-Hilbert spaces H and H'. So if $G = \overline{H'}$, being weakly adjointable as element of $\mathcal{L}(H, G)$ is a different thing from being weakly adjointable as element of $\mathcal{L}(H, \overline{H'})$.) If a is weakly adjointable, then $(a^*)^* = a$ considered as an element in $\mathcal{L}(H, \overline{H'})$ is a weak adjoint of a^*. If a is weakly adjointable, then it is a closeable densely defined operator $\overline{H} \to \overline{H'}$ in the usual sense, with core H. In particular, if H is a Hilbert space, then a weakly adjointable a is bounded. A (weakly) adjointable operator a is (***weakly***) ***self-adjoint*** if $a^* = a$. (Note: This means two different notions of what usually is called a *symmetric* operator. They differ by the implicit assumptions on domain and codomain. Weakly self-adjoint corresponds to the more frequent definition in functional analysis.)

Pre-Fock notation. All Fock-type spaces – in these notes and elsewhere – are in the first place *graded* vector spaces. This makes available the notion of linear maps with a *degree* in $\mathbb{Z}$. It is useful to do this once for all, and introduce a unified way to address these structures. However, they are more than just graded vector spaces, but have an important specialty about them: The vacuum; that is, a grade-zero space of a particular form. A ***pre-Fock space*** $\mathcal{I}$ is a(n $\mathbb{N}_0$–)graded vector space, that is, $\mathcal{I} = \bigoplus_{n\in\mathbb{N}_0} H_n$ for vector spaces H_n, with a distinguished non-zero vector $0 \neq \Omega \in \mathcal{I}$, the ***vacuum***, such that $H_0 = \Omega\mathbb{C}$. We sometimes write $\mathcal{I} = ((H_n)_{n\in\mathbb{N}_0}, \Omega)$.

For every pair $\mathcal{I} = \bigoplus_{n\in\mathbb{N}_0} H_n$ and $\mathcal{J} = \bigoplus_{n\in\mathbb{N}_0} G_n$ of graded vector spaces, a linear map $a \in \mathcal{L}(\mathcal{I}, \mathcal{J})$ has ***degree*** $n \in \mathbb{Z}$ if $aH_m \subset G_{m+n}$ for all $m \in \mathbb{N}_0$ (where we use the conventions that $H_k = \{0\}$ for $k \leq -1$). We denote the set of all maps from $\mathcal{I}$ to $\mathcal{J}$ that have degree n by $\mathcal{L}_{(n)}(\mathcal{I}, \mathcal{J})$. The elements of $\mathcal{L}_{(0)}(\mathcal{I}, \mathcal{J})$ are called ***even***.

We say, an even map a from a pre-Fock space $\mathcal{I} = ((H_n)_{n\in\mathbb{N}_0}, \Omega)$ to a pre-Fock space $\mathcal{J} = ((G_n)_{n\in\mathbb{N}_0}, \Omega')$ is a ***Fock map*** if it is ***vacuum-preserving***, that is, if $a\Omega = \Omega'$.

If $\mathcal{I}$ and $\mathcal{J}$ are direct sums of pre-Hilbert spaces, then Ω and Ω' will be required to be unit vectors. Moreover, a vacuum-preserving map $a \colon \mathcal{I} \to \overline{\mathcal{J}}$ will also be called a ***Fock map*** if $aH_n \subset \overline{G_n}$, that is, if a is a Fock map into the algebraic direct sum over the completions $\overline{G_n}$.

We use analogue terminology in the category of right modules (bimodules) over a fixed unital algebra $\mathcal{B}$ with the variation that $H_0 = \Omega\mathcal{B}$ is required to be isomorphic as right module (as bimodule) to $\mathcal{B}$ via $\Omega \mapsto \mathbf{1}$.

2 POI-Interacting Fock spaces

2.1 Definition. Let H be a pre-Hilbert space. An operator $L \in \mathcal{L}(H, \overline{H})$ is ***positive*** (writing $L \geq 0$) if $\langle x, Lx\rangle \geq 0$ for all $x \in H$.

- We might have called this *weakly positive*, reserving *positive* for operators in $\mathcal{L}(H)$. We opted not to do so, and will refer to the latter situation as a ***positive operator on*** H.
- A positive operator is weakly selfadjoint. A positive operator on H is selfadjoint.
- Positivity induces a partial order among operators in $\mathcal{L}(H, \overline{H})$ by defining $L \geq L'$ if $L - L' \geq 0$.

2.2 Definition. An ALV-interacting Fock space $\mathcal{I} = (H, ((\bullet, \bullet)_n)_{n\in\mathbb{N}_0})$ (according to Definition 1.1 and, by our convention, with H being a pre-Hilbert space) is a ***positive operator induced*** or ***POI-interacting Fock space*** if there is a Fock map $L \in \mathcal{L}(\mathcal{F}(H), \overline{\mathcal{F}(H)})$ such that $(\bullet, \bullet) = \langle \bullet, L\bullet\rangle$.

- Recall the pre-Fock notations from the end of Section 1: L being a Fock map means that L goes into the pre-Fock space $\bigoplus_{n\in\mathbb{N}_0} \overline{H^{\otimes n}} \subset \overline{\mathcal{F}(H)}$ and as such is is even and vacuum-preserving. That is, $L = \bigoplus_{n\in\mathbb{N}_0} L_n$ and $L_0 = \mathsf{id}_{H_0}$.
- A Fock map L induces a POI-interacting Fock space via $(\bullet, \bullet) := \langle \bullet, L\bullet\rangle$ if and only if $L \geq 0$ (that is, $L_n \geq 0$ for all n) and $H \otimes \ker L \subset \ker L$ (that is, $H \otimes \ker L_n \subset \ker L_{n+1}$). The latter follows from $\mathcal{N} = \ker L$. (Indeed, if $X \in \ker L$, then $(X, X) = \langle X, LX\rangle = 0$, so $X \in \mathcal{N}$. If $X \in \mathcal{N}$, so that $(X, X) = 0$, then, by Cauchy-Schwartz inequality, $0 = (Y, X) = \langle Y, LX\rangle$ for all $Y \in \mathcal{F}(H)$. Since, $\mathcal{F}(H)$ is dense, $\overline{\mathcal{F}(H)} \ni LX = 0$, so $X \in \ker L$.)

Typical classes of examples are:

2.3 Example. 1. By setting

$$L_n \colon x_n \otimes \ldots \otimes x_1 \longmapsto \sum_{\sigma\in S_n} x_{\sigma(n)} \otimes \ldots \otimes x_{\sigma(1)} q^{\mathrm{inv}(\sigma)},$$

for $q \in [-1, 1]$ ($\mathrm{inv}(\sigma)$ being the number of *inversions* of the permutation $\sigma \in S_n$), we get Bozejko's and Speicher's [BS91] q–Fock space, whose creators and their adjoints satisfy the q–commutation relations

$$a(x)a^*(y) - q(a^*(y)a(x) = \langle x, y\rangle.$$

$q = 0$ (hence, $L = \mathsf{id}_{\mathcal{F}(H)}$) is just the full Fock space. The cases $q = 1$ and $q = -1$ give the *Boson* (or *symmetric*) and the *Fermion Fock space*, respectively. While in these extreme cases the $\frac{L_n}{n}$ are projections and, therefore, easily established to be positive, in the general case $0 < |q| < 1$ showing positivity is a tough problem.

2. A large class of examples, so-called *standard interacting Fock spaces* [ALV97], arises from $H \subset L^2(M,\mu)$ (usually, referred to as *test function space*) for some (σ–finite) measure space (M,μ) and L_n given by multiplication of the elements in $H^{\otimes n} \subset L^2(M^n,\mu^{\otimes n})$ with (measurable) positive functions on M^n. Standard interacting Fock spaces have been examined in particular by Lu and his coworkers. For instance, multiplying with the indicator function of the set $\{\alpha_n \geq \ldots \geq \alpha_1 \geq 0\}$ on $\mathbb{R}^n$, gives rise to the *time-ordered* or *chronological* or *monotone Fock space* examined first as interacting Fock space by Lu and Ruggieri [LR98].

Note that the first class has operators L_n that map into $H^{\otimes n}$, while in the second class (unless in very special cases) we will need completion.

2.4 Remark. Note that Example 1.3 is a standard interacting Fock space. In fact, $\mathbb{C}$ is the L^2 of a probability measure concentrated in a single point. What we did in that example, can be generalized to standard interacting Fock spaces. So, let L_n be positive measurable functions on M^n acting as multiplication operators on $L^2(M^n,\mu^{\otimes n})$ in such a way that for the dense subspace H of $L^2(M,\mu)$ all $H^{\otimes n}$ are in the natural domain of L_n. By the *Radon-Nikodym theorem*, the kernel condition on the L_n is satisfied (if and) only if there are positive ("L–almost surely" unique) measurable functions K_n on M^n such that $L_{n+1}(t_{n+1},t_n,\ldots,t_1) = K_{n+1}(t_{n+1},t_n,\ldots,t_1)L_n(t_n,\ldots,t_1)$, almost surely. In terms of operators, this reads

$$L_{n+1} = K_{n+1}(\mathsf{id}_H \otimes L_n),$$

so that

$$L_n = K_n(\mathsf{id}_H \otimes K_{n-1})\ldots(\mathsf{id}_{H^{\otimes(n-1)}} \otimes K_1)$$

for all $n \in \mathbb{N}$ (with initial condition $L_0 = \mathsf{id}_{\Omega\mathbb{C}}$). Also here, $\xi := \sqrt{L}$, considered as operator $\mathcal{I} \to \overline{\mathcal{F}(L^2(M,\mu))}$, defines an isometry fulfilling $(**)$, where (modulo adjusting domain and codomain appropriately) $\varkappa = \sqrt{K}$.

Applying brute force linear algebra to the kernel condition $\ker L_{n+1} \subset H \otimes \ker L_n$ (see [AS08, Lemma 5.4]), also for a general POI-interacting Fock space there exist $K_{n+1} \in \mathcal{L}(H \otimes \overline{H^{\otimes n}}, \overline{H^{\otimes(n+1)}})$ such that L_n is given by the preceding recursion. The recursion, yes, does capture entirely the kernel condition, by expressing the L_n in terms of the K_n. However if L_{n+1} and $\mathsf{id}_H \otimes L_n$ do not commute, it leaves completely out of control the question for which K_n the preceding sequence would consist of positive operators. We come back to this problem (and resolve it) in Section 4.

It is natural to ask, if all ALV-interacting Fock spaces are POI (answer no), and (if not) how they are distinguished. We, first, answer the second question.

2.5 Lemma. *Let H be a pre-Hilbert space. For another semiinner product $(\bullet,\bullet)$ on H, put $H_{\mathcal{N}} := H/\mathcal{N}$ where $\mathcal{N} := \ker(\bullet,\bullet)$. Define the quotient map $\Lambda\colon x \mapsto x+\mathcal{N}$. Then $(\bullet,\bullet) = \langle\bullet, L\bullet\rangle$ for some positive operator $L \in \mathcal{L}(H,\overline{H})$ if and only if Λ has an adjoint in $\mathcal{L}(H_{\mathcal{N}},\overline{H})$.*

Proof. If Λ has an adjoint, then $L := \Lambda^*\Lambda$ is the positive operators we seek. Conversely, if $\langle \Lambda x, \Lambda y \rangle = (x, y) = \langle x, Ly \rangle$, then for each $z = \Lambda y \in H_{\mathcal{N}}$ (Λ is surjective!), the linear functional $x \mapsto \overline{\langle \Lambda x, z \rangle}$ on H is bounded by $\|Ly\|$, so that there is a unique element in $\overline{H}$, denoted by $\Lambda^* z$, such that $\langle x, \Lambda^* z \rangle = \langle \Lambda x, z \rangle$. The map $\Lambda^* : z \mapsto \Lambda^* z$ is an adjoint of Λ. ■

2.6 Corollary. *For an interacting Fock space $\mathfrak{I}$ based on H the following properties are equivalent:*

1. *The operator Λ defined by $(****)$ has an adjoint $\Lambda^* \in \mathcal{L}(\mathfrak{I}, \overline{\mathfrak{F}(H)})$.*
2. *The corresponding ALV-interacting Fock space is POI.*

We say, an interacting Fock space based on H is ***regular*** or ***regularly based*** on H if Λ has an adjoint. The corollary says, then, that the POI-interacting Fock spaces obtainable from $\mathfrak{F}(H)$ via positive Fock maps L, are precisely the interacting Fock spaces regularly based on H.

Based on the following lemma, POI-interacting Fock spaces share an important property.

2.7 Lemma. *Let H be a pre-Hilbert space with a positive operator $L \in \mathcal{L}(H, \overline{H})$. Define the inner product $(x, y) := \langle x, Ly \rangle$ and put $H_L := H/\mathcal{N}_L$. Then there exists an isometry $H_L \to \overline{H}$. Equivalently: $\dim \overline{H_L} \leq \dim \overline{H}$.*

Proof. By Friedrich's theorem, L has a positive extension $\overline{L} : \mathcal{D}_{\overline{L}} \to \overline{H}$ which is self-adjoint in the usual sense (that is, $\mathcal{D}_{\overline{L}} = \mathcal{D}_{\overline{L}^*}$ is the maximal domain in $\overline{H}$ for an adjoint of $\overline{L}$). By spectral calculus, $\overline{L}$ has a unique positive square root $\overline{\lambda} : \mathcal{D}_{\overline{\lambda}} \to \overline{H}$, where $\mathcal{D}_{\overline{\lambda}} \supset \mathcal{D}_{\overline{L}} \supset H$ and $\langle \overline{\lambda} x, \overline{\lambda} x \rangle = \langle x, Lx \rangle$ for all $x \in \mathcal{D}_{\overline{L}}$. By $x + \mathcal{N}_L \mapsto \overline{\lambda} x$ we define an isometry $H_L \to \overline{H}$, which extends as an isometry $\overline{H_L} \to \overline{H}$. ■

2.8 Corollary. *If the interacting Fock space $\mathfrak{I}$ based on H is a POI-interacting Fock space, then $\mathfrak{I}$ is **embeddable** in the sense that there exists an isometric Fock map $\xi : \mathcal{I} \to \overline{\mathfrak{F}(H)}$.*

Proof. Apply Lemma 2.7 to $\overline{\mathcal{I}} = \overline{\mathcal{F}(H)/\mathcal{N}}$, component-wise. ■

2.9 Example. Suppose H is a separable infinite-dimensional Hilbert space and choose a Hamel basis S of H. Let H_1 be a pre-Hilbert space with orthonormal Hamel basis $(e_s)_{s \in S}$. Put $H_n := \{0\}$ for $n > 1$. Then $\mathfrak{I} := \mathbb{C}\Omega \oplus H_1$ with $a^*(s)\Omega := e_s$ is an AS-interacting Fock space based on H. But, $\dim \overline{H_1} = 2^{\aleph_0} > \dim H = \aleph_0$, so that H_1 does not embed into H, so $\mathcal{I}$ is not embeddable.

A fortiori, by Corollary 2.6, this non-embeddable $\mathcal{I}$ is not regular, too. But while missing embeddability can be repaired (and after "repairing" the example is regular; see the discussion following Definition 3.2), there are examples of non-regularity that cannot be repaired. Both is subject of the next section.

3 (Abstract) interacting Fock spaces

The notion of embeddability of an interacting Fock based on H, as defined in Corollary 2.8, has been recognized in Accardi and Skeide [AS08] as a property of outstanding importance; we reconfirm this in these notes by, in particular, the results in Section 4.

When an interacting Fock space $\mathcal{I}$ based on H is embeddable, we also will say, $\mathcal{I}$ is ***embeddably based*** on H. This formulation already suggests what comes next, in that the space $\mathcal{I}$ may be embeddably based on H or it may not be embeddably based on H, depending on *how* we base it on H. This choice includes both different choices for the creator map $a^*\colon H \to \mathcal{I}$ (for fixed H) and different choices for H itself. The following new, more flexible, and hopefully final (see Remark 1.5) definition of (abstract) interacting Fock space makes this precise.

3.1 Definition. Let $(H_n)_{n\in\mathbb{N}_0}$ be a family of pre-Hilbert spaces where $H_0 = \Omega\mathbb{C}$ for a fixed unit vector Ω, the ***vacuum***, and put $\mathcal{I} := \bigoplus_{n\in\mathbb{N}_0} H_n$. Let A^* be a linear subspace of $\mathcal{L}(\mathcal{I})$ satisfying

$$\operatorname{span} A^* H_n \;=\; H_{n+1}$$

for all $n \in \mathbb{N}_0$ (a condition that, clearly, replaces $(***)$ in Definition 1.4). Then $\mathcal{I}$ is an (***abstract***) ***interacting Fock space*** (denoted as $\mathcal{I} = ((H_n)_{n\in\mathbb{N}_0}, A^*)$). Usually, we will omit 'abstract', and just say 'interacting Fock space'.

Clearly, an interacting Fock space based on H (and, therefore, any other interacting Fock space in the preceding sections) is turned into an interacting Fock space by setting $A^* := a^*(H)$. Conversely, choosing a pre-Hilbert space and a linear surjection $a^*\colon H \to A^*$, we base an interacting Fock space on H. Of course, the latter is always possible by choosing an arbitrary inner product on A^*, turning it that way into a pre-Hilbert space denoted H, and choosing for a^* the identification of H and A^*. Note that the resulting interacting Fock space based on H is ***injective*** in the sense that the creator map a^* is injective. So, every interacting Fock space is trivially not only ***baseable*** but even ***injectively baseable***.

3.2 Definition. An interacting Fock space $\mathcal{I} = ((H_n)_{n\in\mathbb{N}_0}, A^*)$ is ***embeddable*** (***regular***) if we can base it on a pre-Hilbert space H ***embeddably*** (***regularly***), that is, the resulting interacting Fock space based on H is embeddable (regular).

We know from Example 2.9 that there are interacting Fock spaces based on H that are not embeddable, hence, not regular. In this Section we will show that all interacting Fock spaces can be embeddably based, hence, are embeddable (Theorem 3.4), while there exist (in abundance) interacting Fock spaces that cannot be regularly based, hence, are not regular (Theorem 3.5).

For instance, in Example 2.9 we just have chosen a bad ***basing*** $a^*\colon H \to A^* := a^*(H)$. If we replace H with H_1 (that is, if we change the inner product on the vector space $H = H_1$) and the same a^* now considered as map $H_1 \to A^*$, then $\mathcal{I}$ is perfectly embeddable. In fact, it sits already as a subspace $\Omega\mathbb{C} \oplus H_1$ in $\mathcal{F}(H_1)$ and for ξ we may choose the canonical embedding. Of course, $\mathcal{I}$ based in this way on H_1 is regular. (Indeed, $L = \mathsf{id}_{H_0} \oplus \mathsf{id}_{H_1} \oplus \bigoplus_{n\geq 2} 0$.) So, this is not an example of an interacting Fock space that is not regular.

Example 2.9 and the proof of Lemma 2.7 suggest that missing orthogonal dimension of $\overline{H}$ is an obstacle to embeddability. We show now that this is essentially the only obstacle. The following lemma, making a somehow quite obvious statement with a surprisingly difficult proof, is key.

3.3 Lemma. *Let S be a total subset of a Hilbert space H. Then $\dim H \leq \#S$.*

PROOF. Choose a well-order $\leq$ on S. For each $s \in S$ define

$$H_s \;:=\; \{s' \colon s' < s\}^{\perp\perp}$$

if s is non-minimal, and define $H_s := \{0\}$ if s is minimal. (H_s is the Hilbert subspace of H generated by all s' with $s' < s$. Note that s may be an element of H_s or not.) Define the function $f \colon S \to H$ by setting

$$f(s) \;:=\; (\mathsf{id}_H - p_s)s,$$

where p_s is the projection onto H_s. Note that $\mathbb{C}s + H_s = \mathbb{C}f(s) + H_s$, but $f(s)$ is perpendicular to H_s, while s need not be. In particular, both spaces are closed since $\mathbb{C}f(s) + H_s$ is closed. Note, too, that H_s can also be written as

$$H_s \;=\; \overline{\bigcup_{t<s}\{s' \colon s' \leq t\}^{\perp\perp}}.$$

We claim $\{s' \colon s' \leq s\}^{\perp\perp} = \{f(s') \colon s' \leq s\}^{\perp\perp}$ for all $s \in S$. Indeed, denote by Σ the set of all $s \in S$ for which the statement is true. For some $s \in S$ suppose that $t \in \Sigma$ for all $t < s$. This means in particular that

$$H_s \;=\; \overline{\bigcup_{t<s}\{s' \colon s' \leq t\}^{\perp\perp}} \;=\; \overline{\bigcup_{t<s}\{f(s') \colon s' \leq t\}^{\perp\perp}}.$$

Then

$$\{s' \colon s' \leq s\}^{\perp\perp} \;=\; \Big\{\{s\} \cup \bigcup_{t<s}\{s' \colon s' \leq t\}\Big\}^{\perp\perp} \;=\; \overline{\mathbb{C}s + \overline{\bigcup_{t<s}\{s' \colon s' \leq t\}^{\perp\perp}}} \;=\; \mathbb{C}s + H_s \;=\; \mathbb{C}f(s) + H_s$$
$$=\; \overline{\mathbb{C}f(s) + \overline{\bigcup_{t<s}\{f(s') \colon s' \leq t\}^{\perp\perp}}} \;=\; \Big\{\{f(s)\} \cup \bigcup_{t<s}\{f(s') \colon s' \leq t\}\Big\}^{\perp\perp} \;=\; \{f(s') \colon s' \leq s\}^{\perp\perp},$$

so that also $s \in \Sigma$. By transfinite induction, $\Sigma = S$.

Define $S_0 := \{s \in S : f(s) \neq 0\}$. For each $s \in S_0$ put $e_s := \frac{f(s)}{\|f(s)\|}$ so that all e_s ($s \in S_0$) are unit vectors. Put $E := (e_s)_{s \in S_0}$. Since $\overline{\mathsf{span}}\{e_s \colon s \in S_0\} = \overline{\mathsf{span}}\{f(s) \colon s \in S\}$, E is total. Since $e_s \perp H_s$ for all $s \in S$, the set E is orthonormal. So E is an ONB. Therefore, $\dim H = \#S_0 \leq \#S$. ■

3.4 Theorem. *Every (abstract) interacting Fock space is embeddable.*

Proof. Let $\mathcal{I} = ((H_n)_{n\in\mathbb{N}_0}, \Omega, A^*)$ be an interacting Fock space. Choose a vector space basis S of A^*. Equip A^* with the inner product that makes S orthonormal and denote the arising pre-Hilbert space by H. Let a^* denote the canonical identification. Then $\mathcal{J}$ is an interacting Fock space based on H.

For each $n \in \mathbb{N}$, the set $\Lambda_n(S \otimes \ldots \otimes S)$ spans H_n. In particular, it is total for $\overline{H_n}$. By Lemma 3.3, $\dim \overline{H_n} \leq \# S^n = \dim \overline{H^{\otimes n}}$. By Lemma 2.7, there exists an isometry $\xi_n \colon H_n \to \overline{H^{\otimes n}}$. Then the Fock map ξ with components ξ_n is the desired isometry. ■

In a sense, this shows that it is good to look at interacting Fock spaces as abstract ones. If they come along with a basing $a^* \colon H \to A^*$ and turn out to be embeddable, this is fine. However, if an interacting Fock space based on H turns out to be not embeddable, then it is better to change the basing. The results that follow from embeddability (see Section 4) are too important to allow their loss by insisting in an unfortunate choice of a basing.

We thank Roland Speicher who asked, when the second author was on sabbatical leave in Kingston, if the condition of embeddability might not be automatic. The answer, it is automatic provided we choose a reasonable basing, confirms his suspect *cum grano salis*. The proof turned out to be much more subtle than expected. It would not have been possible without the crucial Lemma 3.3. Despite making a sufficiently natural and intuitive statement, we felt that its proof was particularly difficult to find.

After having shown that every interacting Fock space is embeddable, of course, we wish to know if the same is true for regularity: Is every interacting Fock space regular, that is, does every interacting Fock space arise, by basing it appropriately on a suitable pre-Hilbert space, as a POI-interacting Fock space? The following theorem answers this question in the negative sense.

Clearly, for every interacting Fock space $\mathcal{I} = ((H_n)_{n\in\mathbb{N}_0}, A^*)$, necessarily $A^* \subset \mathcal{L}_{(1)}(\mathcal{I})$. On the other hand, for every sequence $(H_n)_{n\in\mathbb{N}_0}$ of pre-Hilbert spaces H_n with $H_0 = \Omega\mathbb{C}$ for some unit vector Ω, the pair $\mathcal{I} = ((H_n)_{n\in\mathbb{N}_0}, \mathcal{L}_{(1)}(\mathcal{I}))$ is an interacting Fock space, provided that $H_n = \{0\}$ implies $H_{n+1} = \{0\}$. (This is the only, necessary and sufficient, condition that assures that we get all of H_{n+1} by applying degree one maps to elements of H_n.) We call such $\mathcal{I}$ the ***full interacting Fock space*** over $(H_n)_{n\in\mathbb{N}_0}$. We say, an interacting Fock space is ***non-nilpotent*** if $H_n \neq \{0\}$ for all n.

3.5 Theorem. *Every non-nilpotent full interacting Fock space is non-regular.*

Proof. Let $\mathcal{I} = ((H_n)_{n\in\mathbb{N}_0}, \mathcal{L}_{(1)}(\mathcal{I}))$ be a non-nilpotent full interacting Fock space. Choose a (sufficiently big) pre-Hilbert space H and a surjective linear map $a^* \colon H \to \mathcal{L}_{(1)}(\mathcal{I})$.

Since $H_n \neq \{0\}$ for all n, we may fix a sequence of unit vectors $\Omega_n \in H_n$ (for simplicity, with $\Omega_0 = \Omega$). For every sequence $(c_n)_{n\in\mathbb{N}}$ of complex numbers define the operator $c := \sum_{n\in\mathbb{N}} \Omega_n c_n \Omega_{n-1}^*$ in $\mathcal{L}_{(1)}(\mathcal{I})$. Since a^* is surjective, there exists $x_c \in H$ such that $a^*(x_c) = c$. By definition,

$$\Lambda(x_c^{\otimes n}) \;=\; c^n\Omega \;=\; \Omega_n c_n \dots c_1,$$

so, $\langle \Omega_n, \Lambda(x_c^{\otimes n})\rangle = c_n \dots c_1$.

Now, if Λ had an adjoint $\Lambda^* \in \mathcal{L}(\mathcal{I}, \overline{\mathcal{F}(H)})$, then

$$|c_n \dots c_1| \;=\; \left|\langle \Lambda^*\Omega_n, x_c^{\otimes n}\rangle\right| \;\leq\; \|\Lambda^*\Omega_n\|\,\|x_c^{\otimes n}\| \;=\; \|\Lambda^*\Omega_n\|\,\|x_c\|^n$$

Since $\Lambda^*\Omega_n \neq 0$ (for instance, because Λ is surjective, or by inserting the special choice $c_k = 1$ for all k), we would get

$$\|x_c\| \;\geq\; \sqrt[n]{\frac{|c_n \dots c_1|}{\|\Lambda^*\Omega_n\|}}$$

for all c and n. Choosing $c_n > 0$ recursively by setting $c_1 := \|\Lambda^*\Omega_1\|$ and

$$c_{n+1} \;:=\; (n+1)^{n+1}\frac{\|\Lambda^*\Omega_{n+1}\|}{c_n \dots c_1},$$

we would get for this particular choice of c that

$$\|x_c\| \;\geq\; \sqrt[n+1]{c_{n+1}\frac{c_n \dots c_1}{\|\Lambda^*\Omega_{n+1}\|}} \;=\; n+1$$

for all n. As this is not possible, Λ cannot have an adjoint. ■

Note that, in particular, the interacting Fock space $\mathcal{I} = \oplus_{n\in\mathbb{N}_0}\Omega_n\mathbb{C}$ with $A^* = \mathcal{L}_{(1)}(\mathcal{I})$ is not regular. Of course this changes entirely if we take $\mathcal{I} = \mathcal{F}(\mathbb{C})$ with the usual creators $\ell^*(\mathbb{C}) = \ell^*(1)\mathbb{C}$ which are only a quite small subset of $\mathcal{L}_{(1)}(\mathcal{F}(\mathbb{C}))$. More generally, also the one-mode interacting Fock spaces (Example 1.3) are regular independently of the number of direct summands. This shows how very much the structure of an interacting Fock space depends on how many creators we allow on the pre-Hilbert space $\mathcal{I}$.

In Theorem 3.4, we completely settled the question of embeddability; we will not be able to do the same for regularity in these notes. Some (non-)possibilities open up several direction for future work, and will be discussed there. Example 6.3 presents another non-regular interacting Fock space.

Full interacting Fock spaces with their operator ($*$–)algebras generated by $A^* = \mathcal{L}_{(1)}(\mathcal{I})$ are not "bad guys". In fact, we will see in Section 5 that these (possibly unbounded) operator ($*$–)algebras are analogues of tensor algebras [MS98] (Pimsner-Toeplitz algebras [Pim97]). Theorem 3.5 just tells we might be better up, not looking at them as operator algebras of an interacting Fock space.

4 Squeezings: Embedded interacting Fock spaces

In this section, we examine the consequences of having a Fock embedding $\xi\colon \mathcal{I} \to \overline{\mathcal{F}(H)}$ of an interacting Fock space based on H into $\overline{\mathcal{F}(H)}$. The formula in $(**)$, which expresses the images $\xi a^*(x)\xi^*$ of the creators when acting on the subspace $\xi\mathcal{I}$ of $\overline{\mathcal{F}(H)}$ in terms of the usual Fock creators $\ell^*(x)$ *squeezed* by an operator $\varkappa$ as $\varkappa\ell^*(x)$, has been observed already in [AS08]. But in this section we go far beyond [AS08, Theorem 5.5 and Corollary 5.7], and obtain a classification of interacting Fock spaces (based on H or not) in terms of such *squeezings* $\varkappa$.

This is the moment to specify better when we consider two interacting Fock spaces to be "the same". Recall that we have the two fundamentally different notions of interacting Fock space and interacting Fock space based on H, the latter being "the same" as ALV-interacting Fock space, while POI-interacting acting Fock spaces are a subspecies of ALV-interacting Fock spaces corresponding to interacting Fock spaces that are based regularly.

4.1 Definition. 1. The interacting Fock spaces $\mathcal{I} = ((H_n)_{n\in\mathbb{N}_0}, A^*)$ and $\mathcal{I}' = ((H'_n)_{n\in\mathbb{N}_0}, A^{*\prime})$ are ***isomorphic*** if there exists a ***Fock unitary*** $u = \bigoplus_{n\in\mathbb{N}_0} u_n$ (that is, the u_n are unitaries $H^{\otimes n} \to H'^{\otimes n}$ and $u_0 = \mathsf{id}_{\Omega\mathbb{C}}$, where $H_0 = \Omega\mathbb{C} = H'_0$) such that

$$uA^*u^* \;=\; A^{*\prime}.$$

2. The interacting Fock spaces $\mathcal{I} = ((H_n)_{n\in\mathbb{N}_0}, a^*)$ and $\mathcal{I}' = ((H'_n)_{n\in\mathbb{N}_0}, a^{*\prime})$ based on (the same) H are ***isomorphic*** if

$$\Lambda X \;\longmapsto\; \Lambda' X$$

($X \in \mathcal{F}(H)$) defines a unitary $u\colon \mathcal{I} \to \mathcal{I}'$.

There are other reasonable notions of isomorphism, which we postpone to future work. We collect some more or less obvious properties.

4.2 Observation. Recall that for an interacting Fock space based on H not only Λ is defined in terms of a^* by $(****)$, but that also Λ determines a^* via $a^*(x)\Lambda X = \Lambda(x \otimes X)$.

1. Clearly, the unitary u for isomorphic interacting Fock spaces based on H, is a a Fock unitary. Moreover, by the preceding reminder, $ua^*(x)u^* = a^{*\prime}(x)$ for all $x \in H$. Therefore, isomorphic interacting Fock spaces based on H are also isomorphic as interacting Fock spaces.

   Conversely, suppose we have two interacting Fock spaces that are isomorphic via u. If we base the first one on H via $a^*\colon H \to A^*$, then by $a'^*\colon x \mapsto ua^*(x)u^*$ we turn the second one into an isomorphic interacting Fock space based on H. Moreover, by $\Lambda X \mapsto \Lambda' X$ we recover the u we started with.

2. By the discussion following Definition 1.4 we know: Every ALV-interacting Fock space $\mathcal{I} = (H, ((\bullet,\bullet)_n)_{n\in\mathbb{N}_0})$ **is** (understood as) an interacting Fock space based on H with the canonical basing $a^*\colon x \mapsto a^*(x)$ (where $a^*(x)$ are the creators with which an ALV-interacting Fock space comes along); every interacting Fock space $\mathcal{I} = ((H_n)_{n\in\mathbb{N}_0}, a^*)$ based on H is (canonically) **isomorphic** to the ALV-interacting Fock space coming from the semiinner product $(\bullet,\bullet) := \langle\Lambda\bullet,\Lambda\bullet\rangle$ on $\mathcal{F}(H)$. Moreover, if $\mathcal{I}' = (H, ((\bullet,\bullet)'_n)_{n\in\mathbb{N}_0})$ is another ALV-interacting Fock space isomorphic to the interacting Fock space $\mathcal{I}$ based on H, then $(\bullet,\bullet)' = \langle\Lambda'\bullet,\Lambda'\bullet\rangle = \langle\Lambda\bullet,\Lambda\bullet\rangle = (\bullet,\bullet)$, that is, as ALV-Fock spaces it is identical to the ALV-interacting Fock space arising from $\mathcal{I}$. *A fortiori* there is one and only one POI-interacting Fock space isomorphic to a given interacting Fock space regularly based on H.

We now fix an interacting Fock space $\mathcal{I}$ based on H and assume it is embedded via a fixed $\xi\colon \mathfrak{I} \to \overline{\mathcal{F}(H)}$. In this situation, we say $\mathcal{I}$ is an ***embedded*** interacting Fock space, and it is understood that an interacting Fock space to be embedded has to be based.

By assuming that $\mathcal{I}$ is embeddably based (always possible by Theorem 3.4), we do not loose any interacting Fock space. (After all, choosing a basing does not change the interacting Fock space.) Clearly, the Fock isometry ξ may be viewed as a Fock unitary u_ξ onto $\xi\mathcal{I} \subset \overline{\mathcal{F}(H)}$. Clearly, defining $a^*_\xi\colon x \mapsto u_\xi a^*(x) u^*_\xi$ turns $\xi\mathcal{I}$ into an interacting Fock space based on H isomorphic to $\mathcal{I}$. (And if we started with another embedding ξ', then the interacting Fock spaces $\xi\mathcal{I}$ and $\xi'\mathcal{I}$ based on H are isomorphic via $u_{\xi'}u^*_\xi$.) So, starting with an interacting Fock space embeddably based on H, actually embedding it, we stay in the same isomorphism class of interacting Fock spaces based on H.

We distinguished carefully between the unitary u_ξ onto $\xi\mathcal{I}$ and the isometry ξ into $\overline{\mathcal{F}(H)}$. We, tacitly, used already that a unitary u between pre-Hilbert spaces always has an adjoint, namely, $u^* = u^{-1}$. This is not so, for an isometry. (One may show that an isometry has an adjoint if and only if its range is complemented in its codomain; see, for instance, Skeide [Ske01, Proposition 1.5.13].) Fortunately, our isometry ξ goes into a Hilbert(!) space and, like every isometry from a pre-Hilbert space into a Hilbert space, it has a densely defined, surjective adjoint $\xi^*\colon \mathcal{D}_{\xi^*} := \xi\mathfrak{I} \oplus (\xi\mathfrak{I})^\perp \to \mathfrak{I}$, determined by $\xi^*(\xi x) = x$ and $\xi^* y = 0$ for $y \in (\xi\mathcal{I})^\perp$. (The complement $(\xi\mathcal{I})^\perp$ is taken in the Hilbert space $\overline{\mathcal{F}(H)}$, and since $\mathcal{D}_{\xi^*}$ has zero-complement in this Hilbert space, it is dense; the last conclusion may fail for subspaces of pre-Hilbert spaces.) It follows that

$$a \longmapsto \xi a \xi^*$$

defines an algebra monomorphism from the algebra $\mathcal{L}(\mathcal{I})$ onto the corner $\mathcal{L}(\xi\mathcal{I}) \subset \mathcal{L}(\mathcal{D}_{\xi^*}) = \begin{pmatrix} \mathcal{L}(\xi\mathcal{I}) & \mathcal{L}((\xi\mathcal{I})^\perp,\xi\mathcal{I}) \\ \mathcal{L}(\xi\mathcal{I},(\xi\mathcal{I})^\perp) & \mathcal{L}((\xi\mathcal{I})^\perp) \end{pmatrix}$. If a has an adjoint, a^*, then $\xi a^* \xi^*$ is, clearly, an adjoint of $\xi a \xi^*$. So, $\xi \bullet \xi^*$, when restricted to $\mathcal{L}^a(\mathcal{I})$ is actually a $*$–monomorphism. Moreover, since ξ respects the vacuum, $\xi\bullet\xi^*$

also respects the ***vacuum expectation*** $\langle\Omega, \bullet\Omega\rangle$. Since ξ is even, also the degree of a is preserved, that is, the monomorphism itself is an even map.

So, via ξ, we have identified $\mathcal{I}$ as a subspace $\xi\mathcal{I}$ of $\mathcal{D}_{\xi^*} \subset \overline{\mathcal{F}(H)}$ and we have identified A^* (and the algebras generated by it) as a subspace $\xi A^*\xi^*$ (and the algebras generated by it) of $\mathcal{L}(\mathcal{D}_{\xi^*})$. The map corresponding to Λ for this interacting Fock space $\xi\mathcal{I}$ is $\Lambda_\xi = u_\xi\Lambda$, when considered as map onto $\xi\mathcal{I}$, as it has to be by definition. However, we prefer to consider it as map $\lambda := \xi\Lambda\colon \mathcal{F}(H) \to \overline{\mathcal{F}(H)}$, taking also into account that its range is actually $\xi\mathcal{I} \subset \mathcal{D}_{\xi^*} \subset \overline{\mathcal{F}(H)}$. Recall that λ depends on ξ. But for reasons of readability in formulae with indices, we dispense with the idea, calling it λ_ξ. (We will rather write λ' to indicate when it is originating in another ξ'. It is clear that $\lambda' = \xi'\xi^*\lambda$.)

We are now almost ready to formulate and prove $(**)$ in this general context. The only question that remains to be made precise in order to make sense out of $\varkappa\ell^*(x)$, is the question of the appropriate domain and codomain of $\varkappa$. As we wish that $\varkappa\ell^*(x) = \xi a^*(x)\xi^*$, the codomain should coincide with domain $\mathcal{D}_{\xi^*}$ of ξ^*. The domain should contain what $\ell^*(x)$ generates out of $\mathcal{D}_{\xi^*}$. We just mention that $\mathcal{F}(H) = (H \otimes \mathcal{F}(H)) \oplus \Omega\mathbb{C}$ in the obvious way; consequently, for every subspace $\mathcal{D}$ of $\overline{\mathcal{F}(H)}$ we get the subspace $(H \otimes \mathcal{D}) \oplus \Omega\mathbb{C}$ of $\overline{\mathcal{F}(H)}$, and the latter is dense if (and only if) the former is dense.

4.3 Theorem. *Let $\mathcal{I}$ be an embedded (via ξ) interacting Fock space (based on H). There exists a unique vacuum-preserving map (necessarily also a Fock map) $\varkappa \in \mathcal{L}((H \otimes \mathcal{D}_{\xi^*}) \oplus \mathbb{C}\Omega, \mathcal{D}_{\xi^*})$ such that*

$$\varkappa\ell^*(x) \;=\; \xi a^*(x)\xi^*$$

(that is, Equation $(**)$).Therefore, the algebra monomorphism $a \mapsto \xi a\xi^*$ sends $a^*(x)$ to $\varkappa\ell^*(x)$.

If $a^*(x)$ has an adjoint $a(x) \in \mathcal{L}(\mathfrak{I})$, then $\xi a(x)\xi^* = (\varkappa\ell^*(x))^*$ (though, $\varkappa$ need not be adjointable).

In either case, the $(*$–$)$monomorphism respects the vacuum state $\langle\Omega, \bullet\Omega\rangle$.

Moreover, λ can be recovered as the unique Fock map satisfying the equation

$$\lambda \;=\; \varkappa((\mathrm{id}_H \otimes \lambda) + \mathrm{id}_{\Omega\mathbb{C}}), \tag{4.1}$$

that is, as the unique λ whose components satisfy the recursion

$$\lambda_{n+1} \;=\; \varkappa_{n+1}(\mathrm{id}_H \otimes \lambda_n) \quad \text{and} \quad \lambda_0 \;=\; \mathrm{id}_{\mathbb{C}\Omega},$$

that is,

$$\lambda_n \;=\; \varkappa_n(\mathrm{id}_H \otimes \varkappa_{n-1}) \ldots (\mathrm{id}_H^{\otimes(n-1)} \otimes \varkappa_1) \quad (n \geq 1).$$

Notes on the proof. Why 'notes on the proof'? Well, for adjointable interacting Fock spaces and without the uniqueness statement, this theorem is [AS08, Theorem 5.5 and Corollary 5.7].

The proof in [AS08] does not depend on adjointability, and once we have $\varkappa$ satisfying $(**)$, it was just an omission in [AS08] not to have noticed uniqueness. However, the proof in [AS08] went by first proving (by brute-force linear algebra [AS08, Lemma 5.4]) existence of $\varkappa$ satisfying the recursion with λ. And while $(**)$ fixes $\varkappa$, the recursion alone does not. (It may be considered a sort of "lucky punch" that the freedom in choosing $\varkappa$ for satisfying the recursion has been used "wisely" to also satisfy $(**)$.) Starting from $(**)$ and uniqueness, straightens up and simplifies the proof considerably, so we sketch this briefly.

$\varkappa$ is determined uniquely by $\varkappa\ell^*(f) = \xi a^*(f)\xi^*$ on the span of the ranges of all $\ell^*(x)$, that is, on $H \otimes \mathcal{D}_{\xi^*}$. The remaining uncertainty is taken away by the requiring $\varkappa$ as vacuum-preserving.

For existence of $\varkappa$, we simply put $\varkappa\Omega := \Omega$ and define it on $H \otimes \mathcal{D}_{\xi^*} = H \otimes (\xi\mathcal{I} \oplus (\xi\mathcal{I})^\perp)$ as $(**)$ suggests: Necessarily, $\varkappa(x \otimes Y) = \xi a^*(x)\xi^* Y = 0$ for $Y \in (\xi\mathcal{I})^\perp$. And for $\xi X \in \xi\mathcal{I}$ we obtain $\varkappa(x \otimes \xi X) = \xi a^*(x)\xi^*\xi X = \xi a^*(x)X$. Since ξ is an isometry, this map $\varkappa$ is well defined.

By definition, this $\varkappa$ satisfies $(**)$. And it is routine (using how Λ and a^* determine each other as explained in the beginning of Observation 4.2 and the interplay between Λ and λ via ξ) to verify (4.1). ■

Let us sum up again what we achieved. From an interacting Fock space $\mathcal{I}$ based on H and embedded via ξ, we extracted the pre-Fock subspace $\xi\mathcal{I}$ of $\overline{\mathcal{F}(H)}$ and the operator $\varkappa$ from the (dense, pre-Fock) subspace $(H \otimes \mathcal{D}_\xi) \oplus \Omega\mathbb{C}$ to the (dense, pre-Fock) subspace $\mathcal{D}_{\xi^*} := \xi\mathcal{I} \oplus (\xi\mathcal{I})^\perp$. From $\varkappa$ we reconstruct λ via the recursion encoded in (4.1), and from λ we reconstruct $\xi a^*(x)\xi^*$ (or, better, from Λ_ξ, the surjective corestriction of λ, we reconstruct $u_\xi a^*(x)u_\xi^* \in \mathcal{L}(\xi)$ as explained in the beginning of Observation 4.2), which, when embedded into $\mathcal{L}(\mathcal{D}_{\xi^*})$, becomes $\xi a^*(x)\xi^*$). That is, we have encoded the entire information about the embedded interacting Fock space $\mathcal{I}$, and up to isomorphism about the interacting Fock space $\mathcal{I}$ based on H, in the operator $\varkappa$ (including, of course, how its domain and codomain are made up out of $\xi\mathcal{I}$), and $\varkappa$, on the other hand, is uniquely determined by $\mathcal{I}$ and ξ, that is, by the embedded interacting Fock space $\mathcal{I}$. Moreover, if we started from another embedding, ξ', then the everything is under control via the partial isometry $\xi'\xi^*$ in the sense that $\varkappa' = \xi'\xi^*\varkappa((\mathsf{id}_H \otimes \xi\xi'^*) \oplus \mathsf{id}_{\Omega\mathbb{C}})$ and the corresponding $u_{\xi'}u_\xi^*$ is an isomorphism between the interacting spaces $\xi\mathcal{I}$ and $\xi'\mathcal{I}$ based on H.

Additionally, let us observe that $\varkappa$ fulfills the following two properties: Firstly, $\varkappa$ is onto $\xi\mathcal{I}$ (simply because λ is onto $\xi\mathcal{I}$)). Secondly, $\varkappa$ is 0 on the subspace $H \otimes (\xi\mathcal{I})^\perp$ (as computed in the proof of Theorem 4.3).

We now show that these two conditions are the only conditions a Fock-map $\varkappa$ has to satisfy in order to be the $\varkappa$ of an embedded interacting Fock space. To that goal, we now free the discussion from the embedding ξ.

4.4 Definition. Let $\mathcal{I}$ be a pre-Fock subspace of $\overline{\mathcal{F}(H)}$, and define the dense, pre-Fock subspace $\mathcal{D}_\mathcal{I} := \mathcal{I} \oplus \mathcal{I}^\perp$ of $\overline{\mathcal{F}(H)}$. A Fock map $\varkappa\colon ((H \otimes \mathcal{D}_\mathcal{I}) \oplus \Omega\mathbb{C} \to \mathcal{D}_\mathcal{I}$ is called a ***squeezing*** (***relative***

to $\mathcal{I}$) if $\varkappa$ is onto $\mathcal{I}$ and vanishes on $H \otimes \mathcal{I}^\perp$.

Observe that the *squeezed creators* $\varkappa\ell^*(x)$ (co)restrict to maps $\mathcal{I} \to \mathcal{I}$, which we denote by $a^*_\varkappa(x)$. This gives rise to the linear map $a^*_\varkappa \colon H \to \mathcal{L}(\mathcal{I})$. Occasionally, we leave out the subscript $\varkappa$ when there is no danger of confusion.

4.5 Theorem. *If $\varkappa$ is a squeezing relative to $\mathcal{I} = \bigoplus_{n\in\mathbb{N}_0} H_n \subset \overline{\mathcal{F}(H)}$, then $\mathcal{I}_\varkappa := ((H_n)_{n\in\mathbb{N}_0}, a^*_\varkappa)$ is an interacting Fock space based on H. Moreover, the (unique) $\varkappa_{\xi_\varkappa}$ constructed by Theorem 4.3 from the canonical embedding $\xi_\varkappa \colon \mathcal{I} \to \mathcal{I} \subset \overline{\mathcal{F}(H)}$ is $\varkappa$.*

4.6 Definition. We call $\mathcal{I}_\varkappa$ a ***$\varkappa$–interacting Fock space***, and denote it by $\mathcal{I}_\varkappa = (H, \varkappa)$ (also here leaving occasionally out the subscript).

Proof of Theorem 4.5. There is not really much to prove. $\varkappa$ being a squeezing, by surjectivity of $\varkappa$ it follows that $\varkappa\ell^*(x)$ maps $\overline{H^{\otimes n}}$ onto $\mathcal{I} \cap \overline{H^{\otimes(n+1)}} = H_{n+1}$ and by $\varkappa$ being 0 on $H \otimes \mathcal{I}^\perp$ it follows that to exhaust the range it is sufficient to restrict to $\mathcal{I} \cap \overline{H^{\otimes n}} = H_n$. Therefore, $\operatorname{span} a_\varkappa(H)H_n = H_{n+1}$. Clearly, $\varkappa$ does satisfy $(**)$ for the canonical embedding $\xi_\varkappa$, so by the uniqueness statement in Theorem 4.3, $\varkappa$ coincides with $\varkappa_{\xi_\varkappa}$. ■

4.7 Corollary. *We, thus, established a one-to-one correspondence between embedded interacting Fock spaces and squeezings.*

The following theorem is a mere corollary of Theorems 3.4 and 4.3.

4.8 Theorem. *Every interacting Fock space is isomorphic to a $\varkappa$–interacting Fock space (suitably varying H, $\mathcal{I} \subset \overline{\mathcal{F}(H)}$, and $\varkappa$ relative to $\mathcal{I}$).*

Every interacting Fock spaces based embeddably on H is isomorphic to a $\varkappa$–interacting Fock space for a squeezing $\varkappa$ relative to a pre-Fock subspace $\mathcal{I}$ of $\overline{\mathcal{F}(H)}$.

We have already discussed the influence of different choices ξ how to embed into $\overline{\mathcal{F}(H)}$ a given interacting Fock space based on H. Maybe a bit surprisingly, the answer is the same if we vary also H, that is, if we vary also the basing. Without the obvious proof, we state the following:

4.9 Proposition. *Let $\varkappa$ and $\varkappa'$ be squeezings relative to pre-Fock subspaces $\mathcal{I} \subset \overline{\mathcal{F}(H)}$ and $\mathcal{I}' \subset \overline{\mathcal{F}(H')}$, respectively. Then the interacting Fock spaces $\mathcal{I}_\varkappa$ and $\mathcal{I}'_{\varkappa'}$ are isomorphic (as interacting Fock spaces) if and only if there is a partial Fock isometry $v \in \mathcal{B}(\mathcal{D}_\mathcal{I}, \mathcal{D}_{\mathcal{I}'})$ with $v^*v = \mathsf{id}_\mathcal{I}$ and $vv^* = \mathsf{id}_{\mathcal{I}'}$ such that*

$$\varkappa' = v\varkappa((\mathsf{id}_H \otimes v^*) \oplus \mathsf{id}_{\Omega\mathbb{C}}).$$

We see for getting an interacting Fock space as a $\varkappa$–interacting Fock space, it does not only not matter (via an extremely obvious relation among different $\varkappa$) how we embed it, but it does not even depend (up to the same obvious relation) on how we based it, as long as we based it embeddably.

Recall that $\varkappa$–interacting Fock spaces **are** embedded Fock spaces and, therefore, based. Some properties of an interacting Fock space (for instance, boundedness of the set A^*) are intrinsic; other properties (for instance, regularity of a basing) depend on the basing. This raises several question how these properties can be seen by looking only at $\varkappa$, or by guaranteeing existence of certain good choices for $\varkappa$. Regarding regularity – a property with reference to a given basing –, we close this section by stating the quite obvious result that regularity does not depend on the representative within the same isomorphism class of interacting Fock spaces based on the same pre-Hilbert space H.

4.10 Proposition. *If $\mathcal{I}$ and $\mathcal{I}'$ are isomorphic interacting Fock space based on H, then $\mathcal{I}$ is regular if and only if $\mathcal{I}'$ is regular.*

Proof. Let u be the isomorphism. Then if Λ^* exists, $\Lambda^* u^*$ is an adjoint of Λ', and *vice versa*. ■

4.11 Corollary. *Suppose ξ is a Fock embedding into $\overline{\mathcal{F}(H)}$. Then $\mathcal{I}$ is regular if and only if $\xi\mathcal{I}$ is regular, that is, if λ has an adjoint.*

In the following section we address questions of boundedness. More general questions require more refined notions of isomorphism and more reasonable choices for our basings. As with this we run into problems that do not allow for a single solution but split into subclasses, we postpone the discussion to future work.

5 Boundedness: Cuntz-Pimsner-Toeplitz algebras

As already noticed in Accardi and Skeide [AS08, Section 4], if $\mathcal{I}$ is an adjointable(**!**) interacting Fock space (in [AS08] based on H, but that is irrelevant), then we may define the ***full Fock module***

$$\mathcal{F}(\mathcal{L}^a_{(1)}(\mathcal{I})) \;:=\; \mathcal{L}^a_{(0)}(\mathcal{I}) \oplus \bigoplus_{n\in\mathbb{N}} \operatorname{span}(\underbrace{\mathcal{L}^a_{(1)}(\mathcal{I})\ldots\mathcal{L}^a_{(1)}(\mathcal{I})}_{n\text{ times}})$$

($\mathcal{L}^a_{(n)}$ denoting the adjointable part of $\mathcal{L}_{(n)}$) on which the adjointable operators on $\mathcal{I}$ (that is, in particular, the elements of A^*) act by operator multiplication. How is this a Fock module? Well, $\mathcal{L}^a_{(0)}(\mathcal{I})$ is a $*$–algebra of operators in $\mathcal{L}^a(\mathcal{I})$ and for each n ($n = 0$ included), $\mathcal{L}^a_{(n)}(\mathcal{I})$ is a bimodule over $\mathcal{L}^a_{(0)}(\mathcal{I})$ with an *inner product* $\langle X_n, Y_n\rangle := X_n^* Y_n$. Moreover, the tensor product $\mathcal{L}^a_{(n)}(\mathcal{I}) \odot \mathcal{L}^a_{(m)}(\mathcal{I})$ over $\mathcal{L}^a_{(0)}(\mathcal{I})$ sits naturally as $\operatorname{span}\mathcal{L}^a_{(n)}(\mathcal{I})\mathcal{L}^a_{(m)}(\mathcal{I})$ in $\mathcal{L}^a_{(n+m)}(\mathcal{I})$. We do not explain in detail how to make this more precise.[2] Here, we are interested in the case when A^* consists of **bounded** operators. In this case we really get a (completed) full Fock module and embed the operators and algebras into Cuntz-Pimsner-Toeplitz algebras. In the end, we free this from the unnecessary hypothesis that the elements of A^* are adjointable. Criteria that show how boundedness of A^* is reflected by other ways to describe interacting Fock spaces ($\varkappa$, λ, L, ...), are postponed to Section 6.

Since in this section we put emphasis on A^* and do not consider $\mathcal{I}$ to be based (Example 5.6 being the only exception), a^* stands for a typical element of A^*, and not for a basing.

Clearly, if $A^* \subset \mathcal{B}^a(\mathcal{I})$, then we restrict everything to the bounded portions, and define

$$\mathcal{F}(\mathcal{B}^a_{(1)}(\mathcal{I})) \;:=\; \mathcal{B}^a_{(0)}(\mathcal{I}) \oplus \bigoplus_{n\in\mathbb{N}} \operatorname{span}(\underbrace{\mathcal{B}^a_{(1)}(\mathcal{I})\ldots\mathcal{B}^a_{(1)}(\mathcal{I})}_{n\text{ times}}),$$

on which, again, the elements of A^* act by operator multiplication. Now, $\mathcal{B}^a_{(0)}(\mathcal{I})$ is a pre-C^*–algebra of operators in $\mathcal{B}^a(\mathcal{I})$ and $\mathcal{B}^a_{(n)}(\mathcal{I})$ is a pre-correspondence (that is like a correspondence but not necessarily complete and possibly over a pre–C^*–algebra with contractive left action) over $\mathcal{B}^a_{(0)}(\mathcal{I})$. (Even if all H_n are Hilbert spaces, $\mathcal{I}$, and with $\mathcal{I}$ also $\mathcal{B}^a_{(0)}(\mathcal{I})$ and $\mathcal{B}^a_{(n)}(\mathcal{I})$, will not be complete, unless $\mathcal{I}$ is nilpotent.) We may complete, and obtain

$$\mathcal{F}(\overline{\mathcal{B}^a_{(1)}(\mathcal{I})}) \;:=\; \overline{\mathcal{B}^a_{(0)}(\mathcal{I})} \oplus \overline{\bigoplus_{n\in\mathbb{N}}} \operatorname{span}(\underbrace{\mathcal{B}^a_{(1)}(\mathcal{I})\ldots\mathcal{B}^a_{(1)}(\mathcal{I})}_{n\text{ times}}) \;=\; \overline{\bigoplus_{n\in\mathbb{N}_0}} \mathcal{B}^a_{(1)}(\mathcal{I})^{\odot n}. \tag{5.1}$$

5.1 Remark. Still, while $\overline{\mathcal{B}^a_{(n)}(\mathcal{I}) \odot \mathcal{B}^a_{(m)}(\mathcal{I})}$ is contained in $\overline{\mathcal{B}^a_{(n+m)}(\mathcal{I})}$, it is usually only a proper subset. If we insist in equality, we have to pass to the von Neumann objects $\mathcal{B}_{(0)}(\overline{\mathcal{I}}) = \overline{\mathcal{B}^a_{(0)}(\mathcal{I})}^s$ and $\mathcal{B}_{(1)}(\overline{\mathcal{I}}) = \overline{\mathcal{B}^a_{(1)}(\mathcal{I})}^s$ (strong closure in $\mathcal{B}(\overline{\mathcal{I}})$). We ignore this ramification in these notes.

[2] It occupies the whole lengthy [AS08, Section 3] (see also Skeide [Ske01, Appendix C]) to develop a notion of positivity in general $*$–algebras that is sufficiently general for applications (for instance, the *square of white noise* Fock module in [AS00]) and still allows to control positivity in the tensor product, before the Fock module of an interacting Fock space $\mathcal{I}$ can be defined in [AS08, Section 4].

Let us briefly recall a couple of general facts about full Fock modules and inducing representations.

Firstly, if E is a correspondence over a C^*–algebra $\mathcal{B}$, then the ***full Fock module*** over E is the correspondence $\mathcal{F}(E) := \overline{\bigoplus}_{n\in\mathbb{N}_0} E^{\odot n}$. (Here $E^{\odot 0} := \mathcal{B}$. But if $\mathcal{B}$ is unital, then we will write it as $E^{\odot 0} := \omega\mathcal{B}$, with the central unit vector $\omega := \mathbf{1} \in \mathcal{B}$.) For each $x \in E$, the ***creator*** $\ell^*(x)\colon X \mapsto x \odot X$ is an adjointable operator on $\mathcal{F}(E)$, denoted $\ell^*(x) \in \mathcal{B}^a(\mathcal{F}(E))$. Since $\mathcal{F}(E)$ is a correspondence and since $\mathcal{B}$ acts faithfully from the left on the direct summand $E^{\odot 0} = \mathcal{B}$, also $\mathcal{B}$ sits as a C^*–subalgebra in $\mathcal{B}^a(\mathcal{F}(E))$. The ***tensor algebra*** over E is the Banach subalgebra of $\mathcal{B}^a(\mathcal{F}(E))$ generated by $\ell^*(E)$ and $\mathcal{B}$ (Muhly and Solel [MS98]). The ***Cuntz-Pimsner-Toeplitz algebra*** over E is the C^*–subalgebra of $\mathcal{B}^a(\mathcal{F}(E))$ generated by $\ell^*(E)$ and $\mathcal{B}$ (Pimsner [Pim97]).

Secondly, if E is Hilbert $\mathcal{B}$–module and if G is a correspondence from $\mathcal{B}$ to $\mathbb{C}$ (that is, G is a Hilbert space with a nondegenerate representation of $\mathcal{B}$), then $\mathcal{B}^a(E)$ acts (nondegenerately) on the Hilbert space $E \odot G$ via $\mathcal{B}^a(E) \ni a \mapsto a \odot \mathsf{id}_G \in \mathcal{B}(E \odot G)$. If the correspondence G is ***faithful*** (that is, if the left action defines a faithful representation of $\mathcal{B}$), then also the action of $\mathcal{B}^a(E)$ on $\mathcal{B}(E \odot G)$ is faithful. (In our applications to the Fock module $\mathcal{F}(E)$, G will be "very non-faithful" and we have to work to show by hand that the action of $\mathcal{B}^a(E)$ for that G is, nevertheless, faithful.) If E is a ***correspondence*** from $\mathcal{A}$ to $\mathcal{B}$ (that is, the left action of $\mathcal{A}$ on the Hilbert $\mathcal{B}$–module E defines a nondegenerate homomorphism), then the canonical homomorphism $\mathcal{A} \to \mathcal{B}^a(E) \to \mathcal{B}(E \odot G)$ defines a nondegenerate representation of $\mathcal{A}$ on $E \odot G$ (turning $E \odot G$ into a correspondence from $\mathcal{A}$ to $\mathbb{C}$), the representation ***induced*** from (the representation on) G by E.[3]

After these reminders, we return to the beginning. The $\mathcal{F}(\overline{\mathcal{B}^a_{(1)}(\mathcal{I})})$ defined above is, indeed the full Fock module $\mathcal{F}(E)$ for the correspondence $E := \overline{\mathcal{B}^a_{(1)}(\mathcal{I})}$ over the (unital!) C^*–algebra $\mathcal{B} := \overline{\mathcal{B}^a_{(0)}(\mathcal{I})}$. We wish to identify the C^*–algebra $\mathcal{B}^a(\mathcal{F}(E))$ as a subalgebra of $\mathcal{B}(\overline{\mathcal{I}})$; and we wish to do it in such a way that the creators $\ell^*(a_1) \in \mathcal{B}^a(\mathcal{F}(E)) \subset \mathcal{B}(\overline{\mathcal{I}})$ act like the operators $a_1 \in E \subset \mathcal{B}^a(\mathcal{I}) \subset \mathcal{B}(\overline{\mathcal{I}})$ act on $\mathcal{I} \subset \overline{\mathcal{I}}$. For that goal, we tensor $\mathcal{F}(E)$ with the representation space $G := H_0 = \Omega\mathbb{C} \subset \mathcal{I}$ of $\mathcal{B}$, which is left invariant by $\mathcal{B}$ because all elements of $\mathcal{B}$ are even. (Tensoring with $\mathcal{I}$ would, yes, guarantee faithfulness of the representation on $\mathcal{F}(E) \odot \mathcal{I} \cong \overline{\bigoplus}_{n\in\mathbb{N}_0} H_n^{n+1}$, but this space would be much too big, and it also would be quite tedious to invent a good notation for how $\ell^*(a_1) \odot \mathsf{id}_{\mathcal{I}}$ acts between the several direct summands.) The following is obvious; it provides a correct proof for [AS08, Theorem 4.1].

[3] There are several definitions of *C^*–correspondence* around. Despite the possibility to construct (tensor products and) the full Fock module also over Hilbert $\mathcal{B}$–modules with a degenerate left action by $\mathcal{B}$, in several places in the theory to our taste degeneracy of the left action is not acceptable. (Just one instance: The algebra should act as "identity correspondence" under tensor product.) So, we insist that a correspondence, to merit the name, has nondegenerate left action, by definition. On the other hand, while many authors allow for degenerate left action, in the construction of the full Fock module they insist in that the correspondence should be full, which we do not.

5.2 Proposition. *The map*

$$(a_n^* \odot \ldots \odot a_1^*) \odot \Omega \longmapsto a_n^* \ldots a_1^* \Omega \qquad (a_i^* \in A^* \subset E)$$

defines a unitary $\mathcal{F}(E) \bar{\odot} H_0 \to \overline{I}$ *and, under this isomorphism,* $\ell^*(a_1) \odot \mathsf{id}_{H_0} = a_1$ *for all* $a_1 \in E \subset \mathcal{B}(\overline{I})$*. Therefore, the map* $a^* \mapsto \ell^*(a^*)$ $(a^* \in A^*)$ *extends to a completely isometric isomorphism from the (∗–)algebra* $\mathcal{A}^{(*)}$ *generated by* A^* *onto the (∗–)subalgebra of the tensor algebra (the Cunt-Pimsner-Toeplitz algebra) of* E *generated by* $\ell^*(A^*)$*.*

5.3 Remark. Note that also the representation of $\mathcal{B} \subset \mathcal{B}^a(\mathcal{F}(E))$ on $\mathcal{F}(E) \odot H_0$, under the isomorphism with $\overline{I}$, is just the identity representation. This is enough to show that the representation $\mathcal{B}^a(\mathcal{F}(E)) \to \mathcal{B}^a(\mathcal{F}(E)) \odot \mathsf{id}_{H_0} \subset \mathcal{B}(\overline{I})$ of $\mathcal{B}^a(\mathcal{F}(E))$ (containing the Cuntz-Pimsner-Toeplitz algebra of E, containing the tensor algebra of E) on $\overline{I}$ is faithful. (Indeed, first of all for $0 \neq a \in \mathcal{B}^a(\mathcal{F}(E))$ there exist k, m, n and $X_n \in E^{\bar{\odot} n}$, $Y_m \in E^{\bar{\odot} m}$, $Z_k, Z_k' \in E^{\bar{\odot} k}$ such that $\langle (Z_k \odot \Omega), (\langle X_n, aY_m \rangle \odot \mathsf{id}_{H_0})(Z_k' \odot \Omega) \rangle \neq 0$. (Recall that $\langle X_n, aY_m \rangle \in \mathcal{B}$ is even, and that if $\langle X_n, aY_m \rangle \neq 0$, then also $\langle X_n, aY_m \rangle \odot \mathsf{id}_{H_0} \neq 0$.) By

$$0 \neq \langle (Z_k \odot \Omega), (\langle X_n, aY_m \rangle \odot \mathsf{id}_{H_0})(Z_k' \odot \Omega) \rangle = \Big\langle ((X_n \odot Z_k) \odot \Omega), (a \odot \mathsf{id}_{H_0})((Y_n \odot Z_k') \odot \Omega) \Big\rangle,$$

we see $a \odot \mathsf{id}_{H_0} \neq 0$.) We do not really need that result. Nevertheless, it is surely worthwhile mentioning it.

$\mathcal{B}$ and E, as defined above, are rather big. (If we passed to the von Neumann case, that is, taking strong closures everywhere, we would end up with the type I von Neumann algebras $\overline{\mathcal{B}}^s$ and $\overline{\mathcal{B}^a(\mathcal{F}(E))}^s$ which have isomorphic atomic centers ℓ^∞.) In view of our interest in the Banach (∗–)algebra generated by A^*, we had better try and keep the tensor algebra (the Cuntz-Pimsner-Toeplitz) algebra into which we embed as small as reasonably possible. More precisely, instead of E and $\mathcal{B}$ we had better pass to a subspace $F \subset E$ and to a C^*–subalgebra $C \subset \mathcal{B}$ such that F still contains A^* and such that F is a correspondence over C with respect to the inner product and bimodule operations inherited from $\mathcal{B}(\overline{I}) \supset F, C$.

5.4 Corollary. *Under these conditions, Proposition 5.2 remains true. That is,* $\mathcal{F}(F) \bar{\odot} H_0 \cong \overline{I}$ *via the same isomorphism, and* $a^* \mapsto \ell^*(a^*)$ $(a^* \in A^*)$ *extends to (completely isometric) embeddings of the tenor algebra and the Cuntz-Pimsner-Toeplitz algebra of* F *into* $\mathcal{B}^a(\mathcal{F}(F))$*. (Also Remark 5.3 remains true.)*

Note that even for fixed F, the Fock module $\mathcal{F}(F)$ and $\mathcal{B}^a(\mathcal{F}(E))$ and its tensor and Cuntz-Pimsner-Toeplitz subalgebras still depend on the choice of C. The corollary is, of course, true for all possible choices.

The condition that F be a Hilbert module over some C^*–subalgebra C of $\mathcal{B}$, means that F is a closed subspace of E invariant under the ternary product $(x, y, z) \mapsto x\langle y, z \rangle$; the minimal choice for C is $C_F := \overline{\operatorname{span}}\langle F, F\rangle$ (in which case F is full) and every other choice must contain C_F as an ideal. (See, for instance, the lemma in Skeide [Ske18, Section 0].) It is easy to see that the smallest choice containing A^*, the closed ternary subspace generated by A^*, is

$$E_{A^*} := \overline{\operatorname{span}} \bigcup_{n\in\mathbb{N}_0} A^*((A^*)^*A^*)^n; \qquad \mathcal{B}_{A^*} := \overline{\operatorname{span}} \bigcup_{n\in\mathbb{N}} ((A^*)^*A^*)^n.$$

No smaller choice for F and C fitting the assumptions of Corollary 5.4 is possible. But is E_{A^*} a correspondence over $\mathcal{B}_{A^*}$? Or, more generally, if we have a closed ternary subspace F of E containing A^* and a C^*–subalgebra C of $\mathcal{B}$ containing C_F (so that F is a Hilbert C–module), is F a C–correspondence? This means actually two questions regarding the left action of C:

1. Is F invariant under C, that is, is $CF \subset F$?

2. Does C act nondegenerately on F, that is, is $\overline{\operatorname{span}}\, CF \supset F$?

Both questions together may be united in the single question whether $\overline{\operatorname{span}}\, CF = F$; but we prefer to keep the two questions separate.

As far as the second question is concerned, this problem can be resolved once for all by passing to the unitalization $\widetilde{C}$ of C, provided the answer to the first question is affirmative. (Recall that $\mathcal{B}$ is unital, so if $\mathbf{1}_{\mathcal{B}} \notin C$, then by identifying the new unit $\widetilde{\mathbf{1}}_C$ with $\mathbf{1}_{\mathcal{B}}$, $\widetilde{C}$ may be naturally identified as a subalgebra of $\mathcal{B}$. This is independent on whether C has its own unit $\mathbf{1}_C$ or not.) Note that if we do so, then even if F was a full Hilbert C–module, it is now a definitely non-full Hilbert $\widetilde{C}$–module. But, as explained in Footnote [3], for us this is not a problem. (This also explains as simply as possible how and why, as claimed in Footnote [3], the construction of $\mathcal{F}(F)$ for degenerate left actions of C works, too. Simply pass to $\widetilde{C}$ and construct $\mathcal{F}(F)$ for the $\widetilde{C}$–correspondence F. Then pass to $\mathcal{F}(F) \bar{\odot} C = \overline{\operatorname{span}}\, \mathcal{F}(F)C$, which removes from $\mathcal{F}(F)$ the only (one-dimensional subspace spanned by the) element that has inner products outside C. Corollary 5.4 remains true for $\mathcal{F}(F) \bar{\odot} C$.) A case where nondegeneracy is clear, is when $C \ni \mathbf{1}_{\mathcal{B}}$. It is easy to see that for interacting Fock spaces coming from subproduct systems (to be discussed in Section 7) $\mathcal{B}_{A^*}$ acts non-degenerately on E_{A^*} if and only if the subproduct system is actually a product system (in which case the interacting Fock space is actually a full Fock space $\mathcal{F}(H)$ and we really recover $\mathcal{B}_{A^*} = \mathbb{C}$ and $E_{A^*} = \overline{H}$). Also if $\mathcal{I} \neq H_0$ is nilpotent, then $\mathcal{B}_{A^*}$ necessarily acts degenerately on E_{A^*}. (Indeed, since $H_{N+1} = \{0\}$, A^* annihilates $H_N \neq \{0\}$, so none of the (even!) elements in $\mathcal{B}_{A^*}$ can reach $H_N \backslash \{0\}$.)

So, after we have resolved (in an uncomplicated, pragmatic way) the second question (nondegeneracy), we are left with the first question (invariance). For E_{A^*} and $\mathcal{B}_{A^*}$, the only answer we can give is "*rather no than yes*"; it depends highly on the interacting Fock space in question. Typical elements of E_{A^*} are products or ***words*** of elements or ***letters*** that come aternatingly from A^* and from $(A^*)^*$, starting and ending with a letter from A^*. The typical elements of $\mathcal{B}_{A^*}$ are similar alternating words, but the first letter is from $(A^*)^*$ instead of A^* (while the last one is still from A^*). If we multiply a word of E_{A^*} from the left with a word of $\mathcal{B}_{A^*}$, then the last letter of the latter (an element of A^*) meets the first letter of the former (also an element of A^*). So the resulting product word is no longer alternating. Whether or not it can be written as the limit of linear combinations of alternating words is totally unclear.

With the notation $\varepsilon_i = \pm 1$, putting $a_n^{\varepsilon_n} \dots a_1^{\varepsilon_1} = \mathbf{1}_{\mathcal{B}}$ for $n = 0$, and, for $a^* \in A^*$, putting $a^1 := a^*$, $a^{-1} := (a^*)^*$, one smaller choice is

$$\mathcal{B}_{\mathcal{I}} := \overline{\mathsf{span}}\Big\{a_n^{\varepsilon_n} \dots a_1^{\varepsilon_1} : n \in \mathbb{N}, a_i^* \in A^*, \textstyle\sum_{i=1}^n \varepsilon_i = 0\Big\} \tag{5.2a}$$

and

$$E_{\mathcal{I}} := \overline{\mathsf{span}}\Big\{a_n^{\varepsilon_n} \dots a_1^{\varepsilon_1} : n \in \mathbb{N}, a_i^* \in A^*, \textstyle\sum_{i=1}^n \varepsilon_i = 1\Big\}. \tag{5.2b}$$

Clearly, $E_{\mathcal{I}}$ is a full Hilbert $\mathcal{B}_{\mathcal{I}}$–module. It is unclear if $\mathcal{B}_{\mathcal{I}}$ acts nondegenerately, but, clearly, it leaves $E_{\mathcal{I}}$ invariant. If $\mathcal{B}_{\mathcal{I}}$ should act degenerately on $E_{\mathcal{I}}$, then we would pass to $\widetilde{\mathcal{B}}_{\mathcal{I}}$ by adding to the generating set in (5.2a) the term for $n = 0$. (Modulo completion, this is the choice that has been discussed in [AS08, Theorem 4.6].) Then $E_{\mathcal{I}}$ is considered a (definitely non-full) correspondence over $\widetilde{\mathcal{B}}_{\mathcal{I}}$.

An even smaller choice, not discussed before, is

$$\mathcal{B}_{\mathcal{I}}^{NC} := \overline{\mathsf{span}}\Big\{a_n^{\varepsilon_n} \dots a_1^{\varepsilon_1} : n \in \mathbb{N}, a_i^* \in A^*, \textstyle\sum_{i=1}^k \varepsilon_i \geq 0 \forall k \leq n, \sum_{i=1}^n \varepsilon_i = 0\Big\} \tag{5.3a}$$

and

$$E_{\mathcal{I}}^{NC} := \overline{\mathsf{span}}\Big\{a_n^{\varepsilon_n} \dots a_1^{\varepsilon_1} : n \in \mathbb{N}, a_i^* \in A^*, \textstyle\sum_{i=1}^k \varepsilon_i \geq 0 \forall k \leq n, \sum_{i=1}^n \varepsilon_i = 1\Big\}. \tag{5.3b}$$

(***NC*** is referring to the fact that the difference of tuples occurring in (5.2a) and (5.3a) resembles the difference between *pair partitions* and *non-crossing pair partitions* of the set $\{1, \dots, n\}$ for even n.) Clearly, $\mathcal{B}_{\mathcal{I}}^{NC}$ is an algebra and $E_{\mathcal{I}}^{NC}$ is invariant under left and right multiplication by elements of $\mathcal{B}_{\mathcal{I}}^{NC}$.

5.5 Proposition. *$\mathcal{B}_{\mathcal{I}}^{NC}$ is a C^*–algebra and the restriction of the inner product of E turns $E_{\mathcal{I}}^{NC}$ into a full Hilbert $\mathcal{B}_{\mathcal{I}}^{NC}$–module.*

Proof. Suppose we have a word $a_n^{\varepsilon_n} \dots a_1^{\varepsilon_1}$ from the generating set in (5.3a), that is, $\sum_{i=1}^k \varepsilon_i \geq 0 \forall k \leq n$ and $\sum_{i=1}^n \varepsilon_i = 0$. Then

$$\textstyle\sum_{i=1}^k (-\varepsilon_{n+1-i}) = -\sum_{i=n-k+1}^n \varepsilon_i = -(0 - \sum_{i=1}^{n-k} \varepsilon_i) \geq 0$$

for all $k \leq n$. Therefore, also the word $(a_n^{\varepsilon_n} \dots a_1^{\varepsilon_1})^* = a_1^{-\varepsilon_1} \dots a_n^{-\varepsilon_n}$ is from the generating set. Therefore the Banach subalgebra $\mathcal{B}_{\mathcal{I}}^{NC}$ of $\mathcal{B}$ is a C^*–algebra.

In a similar way one shows that $x, y \in E_{\mathcal{I}}^{NC}$ implies $\langle x, y\rangle \in \mathcal{B}_{\mathcal{I}}^{NC}$. So, $E_{\mathcal{I}}^{NC}$ is a Hilbert $\mathcal{B}_{\mathcal{I}}^{NC}$–module.

Since every generating word $a_n^{\varepsilon_n} \ldots a_1^{\varepsilon_1}$ of $\mathcal{B}_{\mathcal{I}}^{NC}$ contains a factor of the form $a_{i+1}^{-}a_i^{+}$, the Hilbert $\mathcal{B}_{\mathcal{I}}^{NC}$–module $E_{\mathcal{I}}^{NC}$ is full. ■

Again, if $\mathcal{B}_{\mathcal{I}}^{NC}$ should act degenerately on $E_{\mathcal{I}}^{NC}$, we may pass to the unitalization $\widetilde{\mathcal{B}}_{\mathcal{I}}^{NC} \ni \mathbf{1}_{\mathcal{B}}$.

Summing up, we have presented three (usually) different ways to embed the Banach (C^*–)algebra $\mathcal{A}^{(*)}$ generated by A^* into a tensor (Cuntz-Pimsner-Toeplitz) algebra. It is noteworthy that the latter (two) have no choice but containing $\mathcal{B}_{A^*}$, which coincides with the Banach algebra generated by the set $(A^*)^*A^*$ and is a C^*–algebra. It is usually not contained in $\mathcal{A}$, so the containing tensor algebras will usually be bigger than $\mathcal{A}$.

5.6 Example. Let $\mathcal{I} = \Omega\mathbb{C} \oplus H \oplus \Omega_2\mathbb{C}$ for a pre-Hilbert space H and some unit vector Ω_2, and assume H has an anti-unitary involution $x \mapsto \bar{x}$. Turn $\mathcal{I}$ into an interacting Fock space based on H by defining $a^*(x)$ as

$$\Omega \longmapsto x, \qquad y \longmapsto \Omega_2\langle \overline{x}, y\rangle, \qquad \Omega_2 \longmapsto 0.$$

(The involution serves to assure that $x \mapsto a^*(x)$ is linear.) One easily checks that the adjoint $a(x)$ of $a^*(x)$ acts as

$$\Omega \longmapsto 0, \qquad y \longmapsto \Omega\langle x, y\rangle, \qquad \Omega_2 \longmapsto \overline{x}.$$

We prefer to write these as finite-rank operators, getting $a^*(x) = x\Omega^* + \Omega_2\overline{x}^*$ and, consequently, $a(x) = \Omega x^* + \overline{x}\Omega_2^*$. Clearly, $a(x)$ leaves $\mathcal{I}$ invariant, so $\mathcal{I}$ with $A^* := a^*(H)$ is an adjointable interacting Fock space with bounded creators.

For simplicity (in particular, notationally), we assume H is a Hilbert space. We find $\mathcal{B} = \mathbb{C} \oplus \mathcal{B}(H) \oplus \mathbb{C}$ and $E = \mathcal{B}_{(1)}(\mathcal{I}) = \begin{pmatrix} H\Omega^* & \\ & \Omega_2 H^* \end{pmatrix}$. From $a(x)a^*(y) = \Omega\langle x, y\rangle\Omega^* + \overline{x}\,\overline{y}^*$, we see that $\mathcal{B}_{A^*} \subset \mathbb{C} \oplus \mathcal{K}(H) \oplus 0$. Given x, x', by choosing a unit vector y perpendicular to both, we see that $a(x)a^*(y)a(y)a^*(x') = \bar{x}\bar{x}'^*$, so $\mathcal{B}_{A^*}$ contains all rank-one operators on H. Therefore, $\mathcal{B}_{A^*} = \mathbb{C} \oplus \mathcal{K}(H) \oplus 0$ and, consequently, $E_{A^*} = E$. From $E_{A^*} \subset F \subset E$ for any possible choice fulfilling the hypotheses of Corollary 5.4, we find $E_{\mathcal{I}}^{NC} = E_{\mathcal{I}} = E$. The only nonzero word in $\mathcal{B}_{\mathcal{I}}$ that does not evidently factor as a product of a word from $\mathcal{B}_{A^*}$ and smaller words, is $a(x')a(x)a^*(y)a^*(y') = \Omega\langle x', \bar{x}\rangle\langle \bar{y}, y'\rangle\Omega^*$ and, therefore, already in $\mathcal{B}_{A^*}$, so, $\mathcal{B}_{\mathcal{I}} = \mathcal{B}_{A^*}$. From $\mathcal{B}_{A^*} \subset \mathcal{B}_{\mathcal{I}}^{NC} \subset \mathcal{B}_{\mathcal{I}}$, we conclude that also $\mathcal{B}_{\mathcal{I}}^{NC} = \mathcal{B}_{A^*}$.

So, E_{A^*}, $E_{\mathcal{I}}^{NC}$, and $E_{\mathcal{I}}$ all coincide with $E = \mathcal{B}_{(1)}(\mathcal{I})$, and $\mathcal{B}_{\mathcal{I}}^{NC}$ and $\mathcal{B}_{\mathcal{I}}$ both coincide with $\mathcal{B}_{A^*}$ but are different from $\mathcal{B} = \mathcal{B}_{(0)}(\mathcal{I})$. Since E is invariant under $\mathcal{B}$, it is invariant under any subalgebra of $\mathcal{B}$. However, since $\mathcal{B}_{A^*}E = \begin{pmatrix} H\Omega^* & \\ & 0 \end{pmatrix} \neq E$, the action of $\mathcal{B}_{A^*}$ is degenerate. So, we have to pass to the unitalization $\widetilde{\mathcal{B}}_{A^*} = \mathcal{B}_{A^*} + \mathsf{id}_{\mathcal{I}}\mathbb{C}$. (Note that this does not coincide with

$\mathbb{C} \oplus \widetilde{\mathcal{K}(H)} \oplus \mathbb{C}$; indeed, the latter contains $\mathsf{id}_H \in \widetilde{\mathcal{K}(H)}$, while the former does not.) So, the tensor (Cuntz-Pimsner-Toeplitz) algebras into which we embed $\mathcal{A}^{(*)}$ differ only by how much $\mathcal{B}$ differs from $\widetilde{\mathcal{B}}_{A^*}$ (respectively, from $\mathcal{B}_{A^*}$ if we do not insist in nondegenerate left actions).

Going one step further to $\mathcal{I} = \Omega\mathbb{C} \oplus H_1 \oplus H_2 \oplus \Omega_3\mathbb{C}$ with various choices for A^*, allows to produce more distinctive examples.

So far, we assumed an interacting Fock space $\mathcal{I}$ with bounded A^* that is adjointable. We briefly show how to free the preceding discussion and results from the hypothesis of adjointability.

So, we now only assume that all elements of A^* are bounded, but not necessarily adjointable. (Of course, they are all weakly adjointable.) We may complete all pre-Hilbert spaces $\mathcal{I}$ and H_n and extend every element a^* of A^* to a (now adjointable) operator in $\mathcal{B}(\overline{\mathcal{I}})$, which we continue denoting by a^*. (We do not assume that $\mathcal{I}$ is based. In fact, completing H, wishing to extend also the map $H \to A^*$ involves unavoidably to change also A^*.) Clearly, $\overline{\mathsf{span}}\, A^*\overline{H}_n = \overline{H}_{n+1}$.

We also may immediately start with a family $(H_n)_{n\in\mathbb{N}_0}$ of Hilbert spaces where $H_0 = \Omega\mathbb{C}$, the Hilbert space $\mathcal{I} = \bigoplus_{n\in\mathbb{N}_0} H_n$, and with a subset $A^* \subset \mathcal{B}(\mathcal{I})$ such that

$$\overline{\mathsf{span}}\, A^* H_n \;=\; H_{n+1}. \tag{5.4}$$

In this case, we may define the pre-Hilbert subspaces

$$\underline{H}_n \;:=\; \mathsf{span}\, A^{*n}\Omega$$

of H_n. Since all elements of A^* are bounded, we may show by induction that $\underline{H}_n$ is dense in H_n for all n. Obviously, the elements of A^* send $\underline{H}_n$ into $\underline{H}_{n+1}$. It follows that the H_n and $\mathcal{I}$ may be obtained by the completion procedure described above, from the interacting Fock space $\underline{\mathcal{I}}$ obtained from the $\underline{H}_n$ with the set $\underline{A}^*$ of all (co)restrictions of the elements of A^* to operators on $\underline{\mathcal{I}}$.

So, it does not really matter if we complete an interacting Fock space with bounded (but not necessarily adjointable) creators, or if we start start with a Hilbert-space-version of interacting Fock space where the axiom corresponding to $(***)$ is replaced with the weaker condition in (5.4). But, once we have Hilbert spaces, the elements of A^* **are** adjointable. It is clear that everything about E, E_{A^*}, $E_{\mathcal{I}}^{NC}$, and $E_{\mathcal{I}}$ (with the corresponding versions of $\mathcal{B}$) goes through exactly, as before. We do not give details.

6 Boundedness: Criteria

In the preceding section we have seen the nice consequences when A^* has only bounded elements; in this section we wish to examine when the latter happens. Well, if we just have an (abstract) interacting Fock space, then we cannot do much more than just look at A^* and check if its elements are bounded. What we mean is that in this section we will assume that $\mathcal{I}$ is based on H via the creator map $a^* \colon H \to A^*$ (so that there is Λ) or even embeddably based (so that there is $\varkappa$) or that it is regularly based (so that there is L). Recall that the first two things can be done for every interacting Fock space, while the last is limited to regular ones. We wish to understand boundedness of the creators in A^* in terms of Λ, $\varkappa$, or L.

The question of boundedness has several layers. First of all, note that $a^*(x)$ is bounded if and only if all restrictions to the n–particle sectors H_n have finite norms $\|a^*(x)\|_n := \|a^*(x) \upharpoonright H_n\|$ and if $\sup_n \|a^*(x)\|_n$ $(= \|a^*(x)\|)$ is finite. (The same is true for Λ, $\varkappa$, L ...) For being unbounded it is sufficient to show that $\|a^*(x)\|_n = \infty$ for one n. On the other hand, if all $\|a^*(x)\|_n$ are finite and $a^*(x)$ is unbounded just because the supremum is not finite, then this unboundedness is of a much nicer type. (For instance the symmetric Fock space, that is, Example 2.3(1) for $q = 1$, has creators of that type.) Such operators, clearly, have adjoints on the same invariant domain; their unboundedness is technically not more complicated than that of a selfadjoint operator with discrete spectrum. $a^*(x)$ that are unbounded on an n–particle sector, may be arbitrarily irregular. All the criteria for boundedness in this section have (more or less obvious) versions for boundedness on each n–particle sector (but not necessarily global), but we dispense with formulating them.

On the other hand, apart from the question whether $a^*(x)$ is bounded for every x, we may ask whether the creator map a^* itself is bounded or not. This is a question we will address.

Let us start with an example illustrating that even for a POI-interacting Fock space boundedness of the operator L (or its *square root* Λ) does not guarantee boundedness of the creators $a^*(x)$.

6.1 Example. Let $H = L^2[0,1]$ (as functions of $t \in [0,1]$). For L_1 choose multiplication by t, for L_2 choose $\mathrm{id}_{H\otimes H}$, and put $L_n = 0$ for $n \geq 3$, so that L is bounded. Then for $y_n = I\!I_{[0,\frac{1}{n}]}$ we find $\|y_n\|_{\mathcal{I}} = \sqrt{\int_0^{\frac{1}{n}} t\,dt} = \frac{1}{\sqrt{2}n}$. For any $x \in H$ we find $\|a^*(x)y_n\|_{\mathcal{I}} = \|x\| \sqrt{\frac{1}{n}}$

$$\frac{\|a^*(x)y_n\|_{\mathcal{I}}}{\|y_n\|_{\mathcal{I}}} = \|x\| \sqrt{2n},$$

that is, despite L is bounded, the operator $a^*(x)$ is unbounded whenever $x \neq 0$.

We see, looking directly at boundedness of the operators L or Λ is not promising. So, in the sense of concluding from boundedness of something to boundedness of all $a^*(x)$, the following obvious theorem in terms of $\varkappa$ is the best we can do.

6.2 Theorem. *Let $\mathcal{I} = (H, \varkappa)$ be a $\varkappa$–interacting Fock space. If $\varkappa$ is bounded, then the creator map a^* is bounded by $\|a^*\| \leq \|\varkappa\|$.*

Proof. $\|a^*(x)\| = \|\varkappa\ell^*(x)\| \leq \|\varkappa\| \, \|x\|$. ■

The condition that $\varkappa$ be bounded is not necessary.

6.3 Example. Returning to Example 5.6, we consider the interacting Fock space $\mathcal{I} = \Omega\mathbb{C} \oplus H \oplus \Omega_2\mathbb{C}$ based on H as embedded by choosing for Ω_2 a unit vector in $H \otimes H$. The norm of $a^*(x)$ is the norm of x, so the creator map a^* is an isometry.

$\lambda_1(x) = a^*(x)\Omega = x$, so $\varkappa_1 = \lambda_1 = \mathsf{id}_H$. For $\varkappa_2$ we compute $\lambda_2(x \otimes y) = a^*(x)a^*(y)\Omega = \langle \overline{x}, y \rangle \Omega_2$, so

$$\varkappa_2(x \otimes y) \;=\; \varkappa_2(x \otimes \varkappa_1 y) \;=\; \lambda_2(x \otimes y) \;=\; \Omega_2 \langle \overline{x}, y \rangle.$$

If $\dim H \geq \infty$, we may choose a self-adjoint orthonormal sequence e_n. Since $\left\| \sum_{n=1}^N \frac{e_n \otimes e_n}{n} \right\|^2$ is bounded uniformly by $\sum_n \frac{1}{n^2} < \infty$, but $\sum_{n=1}^N \frac{\langle e_n, e_n \rangle}{n} = \sum_{n=1}^N \frac{1}{n}$ diverges, the map $\varkappa_2$, hence, $\varkappa$, is unbounded.

Note that λ_2 is not weakly adjointable. (The linear functional $\langle \Omega_2, \lambda_2 \bullet \rangle$ is unbounded, so there is no vector $Z = \lambda_2^* \Omega_2 \in \overline{H \otimes H}$ generating it as $\langle Z, \bullet \rangle$.) That is, $\mathcal{I}$ is not regular. Note, too, that there is no difference if we assume H is a Hilbert space. In Example 6.7, we will see a regular example.

The preceding example is based on (and an example for) the fact that the tensor product of Hilbert spaces does not share the usual universal property of tensor product: Not every bounded bilinear map $j \colon H \times H \to \mathbb{C}$ gives rise to a linear map $\breve{j} \colon H \bar{\otimes} H \to \mathbb{C}$ satisfying $\breve{j}(x \otimes y) = j(x, y)$. This gives the right idea. For boundedness of $a^*(x)$ or $a^* \colon x \mapsto a^*(x)$ not boundedness of $\varkappa$ is the relevant question, but boundedness of the bilinear map $(x, X) \mapsto \varkappa(x \otimes X)$. (We could replace the pre-Hilbert norm on $H \otimes \mathcal{F}(H)$ with the projective norm on the tensor product, that has the universal property. But it would not give any better insight, so we dispense with this idea.) Keeping this in mind, the following improvement of Theorem 6.2 is immediate.

6.4 Theorem. *Let $\mathcal{I} = (H, \varkappa)$ be a $\varkappa$–interacting Fock space. Then:*

1. *$a^*(x)$ is bounded if and only if there exists a constant M_x such that $\|\varkappa(x \otimes X)\| \leq M_x \|X\|$ for all $X \in \mathcal{D}_{\mathcal{I}}$.*

2. *a^* is bounded if and only if there exists a constant M such that $\|\varkappa(x \otimes X)\| \leq M \|x\| \|X\|$ for all $x \in H$ and $X \in \mathcal{D}_{\mathcal{I}}$.*

Recalling the properties of $\varkappa$ and the interrelation of $\varkappa$ with λ, we observe that $\|\varkappa(x \otimes X)\|$, for fixed x, takes its supremum varying over vectors of the form λX ($X \in \mathcal{F}(H)$). The first condition transforms into

$$\|\lambda\ell^*(x)X\| \;=\; \|\lambda(x\otimes X)\| \;=\; \|\varkappa(x\otimes\lambda X)\| \;\leq\; M_x\,\|\lambda X\|\,,$$

and analogously for the second condition. Recalling that the Λ of an interacting Fock space $\mathcal{I}$ embeddably based on H is related to the λ, when we actually identify $\mathcal{I}$ as a $\varkappa$–interacting Fock via the embedding ξ, by $\lambda = \xi\Lambda$, we obtain the following criterion in terms of Λ, which is independent of how we actually embedded $\mathcal{I}$. The nice thing is that (as the equation $\Lambda(x\otimes X) = a^*(x)X$, which we used already so many times and which holds for arbitrary interacting Fock space based on H) the inequalities expressed in terms of Λ hold independently on whether $\mathcal{I}$ is based embeddably or non-embeddably.

6.5 Corollary. *Let $\mathcal{I}$ be an interacting Fock space based on H. Then:*

1. *$a^*(x)$ is bounded if and only if there exists a constant M_x such that $\|\Lambda\ell^*(x)X\| \leq M_x\,\|\Lambda X\|$ for all $X\in\mathcal{F}(H)$.*

2. *a^* is bounded if and only if there exists a constant M such that $\|\Lambda\ell^*(x)X\| \leq M\,\|x\|\,\|\Lambda X\|$ for all $X\in\mathcal{F}(H)$.*

Now suppose $\mathcal{I}$ is regular, that is, Λ has a weak adjoint so that $L := \Lambda^*\Lambda \geq 0$ induces $\mathcal{I}$ as POI-interacting Fock space. Then

$$\|\Lambda\ell^*(x)X\|^2 \;=\; \langle X,(\ell(x)L\ell^*(x))X\rangle, \qquad\qquad \|\Lambda X\|^2 \;=\; \langle X, LX\rangle.$$

This allows, finally, to answer the long standing question, when a POI-interacting Fock spaces has bounded creators, in terms of operator inequalities.

6.6 Theorem. *Let $\mathcal{I}$ be a POI-interacting Fock space induced by the positive Fock operator $L\in\mathcal{L}(\mathcal{F}(H),\overline{\mathcal{F}(H)})$. Then:*

1. *$a^*(x)$ is bounded if and only if there exists a constant M_x such that*

$$\ell(x)L\ell^*(x) \;\leq\; M_x^2 L.$$

2. *a^* is bounded if and only if there exists a constant M such that*

$$\ell(x)L\ell^*(x) \;\leq\; M^2\,\|x\|^2\,L.$$

It is noteworthy that for the components L_n of L, the inequalities read

$$\ell(x)L_{n+1}\ell^*(x) \;\leq\; M_x^2 L_n, \qquad\qquad \ell(x)L_{n+1}\ell^*(x) \;\leq\; M^2\,\|x\|^2\,L_n$$

(with M_x and M, respectively, independent of n; in fact, if the constants exist, but dependent on n, this means, the restriction of $a^*(x)$ and a^*, respectively, to H_n are bounded).

We know from Example 6.1 that boundedness of L is not sufficient for L to fulfill the conditions in Theorem 6.6. The following example shows that boundedness of L is also not necessary.

6.7 Example. The construction of a counter example is based on the following computation. Denote by $e_1, \ldots, e_n$ the standard ONB of $\mathbb{C}^n$, and define the unit vector $e^n := \sum_i \frac{e_i \otimes e_i}{\sqrt{n}} \in \mathbb{C}^n \otimes \mathbb{C}^n$. Then $\langle e^n, x \otimes y \rangle = \frac{1}{\sqrt{n}} \sum_i x_i y_i$. With the projection $p_n := e^n e^{n*}$, it follows that

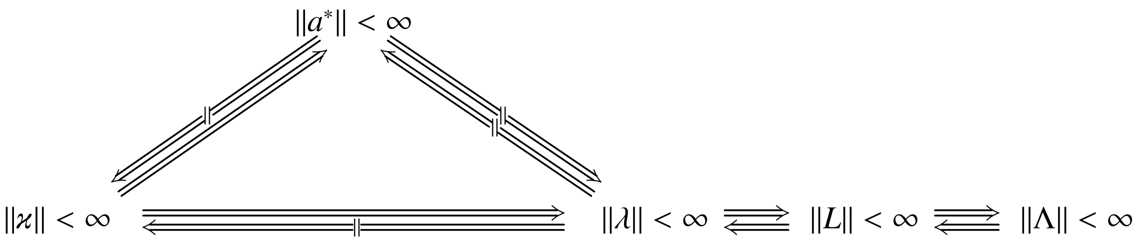
$$\langle x^n \otimes y^n, (np_n)(x^n \otimes y^n) \rangle \;\leq\; \|x^n\|^2 \|y^n\|^2, \quad \text{so,} \quad (x^n \otimes \mathsf{id}_{\mathbb{C}^n})^*(np_n)(x^n \otimes \mathsf{id}_{\mathbb{C}^n}) \;\leq\; \|x^n\|^2 \,\mathsf{id}_{\mathbb{C}^n}.$$

Consequently, if we define $H := \bigoplus_{n \in \mathbb{N}} \mathbb{C}^n$ and the unbounded operator $L_2 := \bigoplus_{m,n \in \mathbb{N}} \delta_{m,n} n p_n$ on $H \otimes H$, then for $x = \bigoplus_{n \in \mathbb{N}} x^n$ we get

$$\begin{aligned}(x \otimes \mathsf{id}_H)^* L_2 (x \otimes \mathsf{id}_H) \;&=\; \bigoplus_{n \in \mathbb{N}} (x^n \otimes \mathsf{id}_{\mathbb{C}^n})^*(np_n)(x^n \otimes \mathsf{id}_{\mathbb{C}^n}) \\ &\leq\; \bigoplus_{n \in \mathbb{N}} \|x^n\|^2 \,\mathsf{id}_{\mathbb{C}^n} \;\leq\; \sup_{n \in \mathbb{N}} \|x^n\|^2 \bigoplus_{n \in \mathbb{N}} \mathsf{id}_{\mathbb{C}^n} \;\leq\; \|x\|^2 \,\mathsf{id}_H.\end{aligned}$$

Therefore, putting $L_1 := \mathsf{id}_H$ and $L_n = 0$ for $n \geq 3$, we get a POI-interacting Fock space with bounded creator map but unbounded $L_2 \leq L$.

Let us collect the (non)implications we have in a diagram.



The tail that starts from $\|\lambda\| < \infty$ to the right, needs a comment. Clearly, a bounded Λ is weakly adjointable, so there exists $L = \Lambda^*\Lambda$ and, necessarily, is bounded, too. And if L exists, so that $\mathcal{I}$ is embeddable, then also λ exists (and is bounded, if L is). If λ exists (because we started with an embeddable interacting Fock space based on H), then λ is just the Λ for an isomorphic interacting Fock space based on H; again λ bounded implies existence of L, which is bounded, too. So, bounded λ and bounded Λ are "the same", but only the situtation with Λ is one that does not come along with an explicitly chosen embedding; and if Λ is not bounded, then the situation is more general in that $\mathcal{I}$ need not be embeddably based. So, it would add to the diagram if we made the same non-arrows which are there between λ and a^* also between Λ and a^*. Last but not least, also the the non-arrow from $\|\lambda\| < \infty$ to $\|\varkappa\| < \infty$ requires a word; indeed if $\|\lambda\| < \infty$ implied $\|\varkappa\| < \infty$, then together with the arrow from $\|\varkappa\| < \infty$ to $\|a^*\| < \infty$ we would get the arrow from $\|\lambda\| < \infty$ to $\|a^*\| < \infty$, which , as we know, is not true.

7 Subproduct systems: A class of examples

A class of operator algebras ($*$ or not) generated by creators on Fock type spaces arises from so-called *subproduct systems*. Subproduct systems (even of correspondences) have been introduced by Shalit and Solel [SS09] and, independently, (under the name of *inclusion systems* and limited to Hilbert spaces) by Bhat and Mukherjee [BM10]. The operator algebras of our interest in these notes, have been introduced by Davidson, Ramsey, and Shalit [DRS11] and led to several forthcoming papers by Shalit and his collaborators. During the 2011 Spring School and Conference on "Product Systems and Independence in Quantum Dynamics" in Greifswald, when listening to Shalit's talk, several participants noted instantaneously, that the Fock type spaces of subproduct systems are interacting Fock spaces; this also includes the same set of creators in a canonical basing.

The scope of this section is to examine the structure of these interacting Fock spaces arising from subproduct systems (namely, $\varkappa$–interacting Fock spaces, where $\varkappa = \pi$ is a projection, apart from being a squeezing, fulfilling an extra condition). On the fly, we examine the general structure of $\varkappa$–interacting Fock spaces, where $\varkappa = \pi$ is a projection.

A (***discrete***) ***subproduct system*** (of Hilbert spaces) is a family $H^{\circledS} = (H_n)_{n\in\mathbb{N}_0}$ of Hilbert spaces H_n with isometric ***coproduct*** maps $w_{m,n}\colon H_{m+n} \to H_m \bar{\otimes} H_n$ iterating coassociatively, and with $H_0 = \mathbb{C}$ such that the marginal maps $v_{n,0}, v_{0,n}$ become the canonical identifications $H_n \otimes \mathbb{C} \cong H_n \cong \mathbb{C} \otimes H_n$. (In several places, there occurred also *superproduct systems*, replacing the isometries with coisometries. A far reaching generalization of both (arising in the dilation theory of multi-parameter CP-semigroups) is work in progress; Shalit and Skeide [SS18].)

For our purposes, it is better to pass to the ***product*** maps $v_{m,n} := w_{m,n}^*\colon H_m \bar{\otimes} H_n \to H_{m+n}$, which are coisometries. The associativity condition, then, really means that the ***product*** $(x_m, y_n) \mapsto x_m y_n := v_{m,n}(x_m \otimes y_n)$ is associative.

If $H^{\circledS}$ is a subproduct systems, then the ***Fock space*** over $H^{\circledS}$ is $\mathcal{F}(H^{\circledS}) := \overline{\bigoplus}_{n\in\mathbb{N}_0} H_n$. For each $x \in H_1$, we define the creator $a^*(x) \in \mathcal{B}(\mathcal{F}(H^{\circledS}))$ by $a^*(x)X_n := xX_n$ for all $n, X_n \in H_n$; see, for instance, [DRS11].

Since $v_{1,n}$ is a coisometry, it is surjective. More precisely, it maps the Hilbert space $H_1 \bar{\otimes} H_n$ onto the Hilbert space H_{n+1}. If we take only the algebraic tensor product $H_1 \otimes H_n$, then (as soon as as H_{n+1} is not finite-dimensional) it is no longer surjective, but only with dense range. So, thinking of $\mathcal{F}(H^{\circledS})$ as an interacting Fock space (writing also $H_0 = \Omega\mathbb{C}$ with $\Omega = 1 \in \mathbb{C} = H_0$), we are in the situation sketched in the end of Section 5, where $(***)$ is replaced by the weaker (5.4). As explained there, we know how to pass to the proper interacting Fock space $\underline{\mathcal{F}(H^{\circledS})} := \bigoplus_{n\in\mathbb{N}_0} \underline{H}_n$ determined by the family of dense pre-Hilbert subspaces

$$\underline{H}_n \;:=\; \operatorname{span} a^*(H_1)^n\Omega \;\subset\; H_n.$$

Roughly, we started with a subproduct system (that is, by definition) of Hilbert spaces and obtained the topological version of interacting Fock space as discussed in the end of Section 5. The reduction, there, to a proper interacting Fock space (with only bounded creators) can be interpreted, in the context of subproduct systems, as the passage to the ***algebraic subproduct system*** of (dense) pre-Hilbert (sub)spaces and their algebraic tensor products ***generated*** by H_1. That it actually is the one generated by H_1, follows clearly from writing the structure with (coisometric) product maps. The nth pre-Hilbert space is just what is spanned by n–fold products of elements from H_1; it is clear by construction that the iterated products $v_{m,n}$ leaves these algebraic domains invariant. If we insisted to work with the (isometric) coproduct maps $w_{m,n}$, then it would not at all be clear if we could find **dense** pre-Hilbert subspaces so that the restriction of $w_{m,n}$ would map into their algebraic tensor product. (It is *a priori* not even clear for $w_{1,1}$. But, while for the products $v_{m,n}$ the problem is solved inductively, here, for the coproducts $w_{m,n}$ no inductive solution is possible, because with each new level $N + 1$, the possible solution for $n, m \leq N$ will be affected.) For this the following observation, which tells that by the co/isometric property we actually do obtain an algebraic subproduct system $(\underline{H}_n)_{n\in\mathbb{N}_0}$ with respect to the (co)restricted coproduct maps $w_{m,n}$, is remarkable:

7.1 Observation. Suppose we have (pre-)Hilbert spaces $H \supset H'$ and $G \supset G'$, and suppose we have a (necessarily adjointable) coisometry $w\colon H \to G$ that (co)restricts to a surjective map $w'\colon H' \to G'$. Then the adjoint w^* of w (co)restricts, too, to a map $G' \to H'$, necessarily the adjoint of w'. (Indeed, by replacing H with the range of the projection w^*w (so that, in particular, surely w^* maps G into that space no matter how small or big the subspace G' is), we may assume that w is actually unitary. Then, like for every invertible map, the restriction of the inverse map w^* to the image G' of a restriction of the map w to H', sends G' into (hence, onto) H'. If we add again what we cut away to make w unitary, we see that w^* maps G' onto $H' \cap (w^*wH)$. Of course, ww^* (co)restricts to $\mathsf{id}_{G'}$; the only question was if the first map w^* of the product ww^* does lead or does not lead out of H'.) Consequently, the (coisometric!) product maps of the algebraic subproduct system $(\underline{H}_n)_{n\in\mathbb{N}_0}$ have (isometric) adjoints for the algebraic (co)domains. Therefore, while in the general case considered in the end of Section 5 the restrictions of the creators to dense interacting Fock space need not be adjointable, in our case here the (co)restrictions of the $a^*(x)$ remain adjointable. (Indeed, $a^*(x)$, on the algebraic domain, is adjointable if and only if each $a^*(x) \upharpoonright \underline{H}_n$ (considered as map into $\underline{H}_{n+1}$) is adjointable, and $a^*(x) \upharpoonright \underline{H}_n = w_{1,n}(x \otimes \mathsf{id}_{\underline{H}_n})$ has an adjoint, namely, $(x \otimes \mathsf{id}_{\underline{H}_n})^* v_{1,n}$.) Therefore, the (proper) interacting Fock space of a subproduct system $H^{\circledcirc}$ is adjointable.

We now wish to understand the structure of interacting Fock spaces derived from subproduct systems. More precisely, we wish to understand them as $\varkappa$–interacting Fock spaces, and distinguish those $\varkappa$ that lead to interacting Fock spaces coming from subproduct systems. The following (partially well-known) result shows that not only the basing is embeddably, but that there is actually a very canonical embedding into $\mathcal{F}(H_1)$.

Here and in the sequel, we denote by $v_{n_1,\ldots,n_k}\colon H_{n_1}\bar{\otimes}\ldots\bar{\otimes}H_{n_k}\to H_{n_1+\ldots+n_k}$ the iterated product of k factors (which, by associativity, does not depend on how we iterate), and we denote the special case of n factors from H_1 as $v_{(n)} := v_{1,\ldots,1}$.

7.2 Theorem. *1. Suppose H is a Hilbert space and π_n are projections in $\mathcal{B}(H^{\bar{\otimes}n})$ (with $\pi_0 = \mathsf{id}_{\mathbb{C}}$). Then the maps $v_{m,n}\colon (\pi_m X_m)\otimes(\pi_n Y_n)\mapsto \pi_{m+n}(X_m\otimes Y_n)$ turn the family $(\pi_n H^{\bar{\otimes}n})_{n\in\mathbb{N}_0}$ into a subproduct system if and only if the projections π_n satisfy*

$$\mathsf{id}_H\otimes\pi_n \;\geq\; \pi_{n+1} \;\leq\; \pi_n\otimes\mathsf{id}_H \tag{7.1}$$

for all $n\in\mathbb{N}$.

2. Suppose $H^{\circledcirc}$ is a subproduct system, and put $\pi_n := v_{(n)}^* v_{(n)} \in \mathcal{B}(H_1^{\bar{\otimes}\,n})$. Then the π_n fulfill (7.1) and

$$X_n \;\longmapsto\; v_{(n)}^* X_n$$

is an isomorphism of subproduct systems from $H^{\circledcirc}$ to $(\pi_n H_1^{\bar{\otimes}n})_{n\in\mathbb{N}_0}$.

Proof. 1. Associativity is manifest, once the $v_{m,n}$ are well-defined. It is clear that $v_{m,n}$ is well-defined if and only if the kernel of $\pi_m\otimes\pi_n$ is contained in the kernel of π_{m+n}, that is, if and only if

$$\pi_m\otimes\pi_n \;\geq\; \pi_{m+n}. \tag{7.2}$$

What remains is to show that the necessary conditions in (7.1) (they form a subset of the conditions in (7.2)) are also sufficient. Note that (7.1) may also be written as $(\mathsf{id}_H\otimes\pi_n)\pi_{n+1} = \pi_{n+1} = \pi_{n+1}(\pi_n\otimes\mathsf{id}_H)$. We find

$$\begin{aligned}\pi_{m+n} \;&=\; (\mathsf{id}_H\otimes\pi_{m-1+n})\pi_{m+n} \;=\; (\mathsf{id}_{H^{\bar{\otimes}2}}\otimes\pi_{m-2+n})(\mathsf{id}_H\otimes\pi_{m-1+n})\pi_{m+n}\\ &=\; (\mathsf{id}_{H^{\bar{\otimes}2}}\otimes\pi_{m-2+n})\pi_{m+n} \;=\; \ldots \;=\; (\mathsf{id}_{H^{\bar{\otimes}m}}\otimes\pi_n)\pi_{m+n},\end{aligned}$$

that is, $\mathsf{id}_{H^{\bar{\otimes}m}}\otimes\pi_n \geq \pi_{m+n}$, and, similarly, $\pi_{m+n} = \pi_{m+n}(\pi_m\otimes\mathsf{id}_{H^{\bar{\otimes}n}})$, that is, $\pi_m\otimes\mathsf{id}_{H^{\bar{\otimes}n}} \geq \pi_{m+n}$. Both together give (7.2).

2. Clearly, $v_{(n)}^*$, being an isometry, defines a unitary onto $v_{(n)}^* H_n = \pi_n H_1^{\bar{\otimes}n}$. By the family $v_{(n)}^*$ of unitaries, the product maps $v_{m,n}$ lift to the family $(\pi_n H_1^{\bar{\otimes}n})_{n\in\mathbb{N}_0}$ as

$$\begin{aligned}(\pi_m X_m)\otimes(\pi_n Y_n) \;&\longmapsto\; v_{(m)}(\pi_m X_m)\otimes v_{(n)}(\pi_n Y_n) \;=\; v_{(m)}X_m\otimes v_{(n)}Y_n\\ &\longmapsto\; v_{m,n}(v_{(m)}X_m\otimes v_{(n)}Y_n) \;=\; v_{(m+n)}(X_m\otimes Y_n)\\ &\longmapsto\; v_{(m+n)}^* v_{(m+n)}(X_m\otimes Y_n) \;=\; \pi_{(m+n)}(X_m\otimes Y_n)\end{aligned}$$

(first sending the elements $\pi_m X_m$ and $\pi_n Y_n$ of the family $(\pi_n H_1^{\bar{\otimes} n})_{n\in\mathbb{N}_0}$ to the family $H^{\circleddash}$ where, then, $v_{m,n}$ is applied to send, in the end, the result $v_{(m+n)}(X_m \otimes Y_n)$ back to $(\pi_n H_1^{\bar{\otimes} n})_{n\in\mathbb{N}_0}$). This is not only precisely the action we wish to define in Part 1. It also establishes the latter, being an image of the subproduct system structure of $H^{\circleddash}$, as a properly defined operation of a subproduct system, therefore, necessarily satisfying (7.1). By construction, the family of unitaries $v_{(n)}^*$ is an isomorphism of subproduct systems. ■

7.3 Remark. Using the conditions in (7.2), this is just a suitably reformulated version of [SS09, Lemma 6.1], referring to the family $(\pi_n H^{\bar{\otimes} n})_{n\in\mathbb{N}_0}$ as a ***standard subproduct system***. That the weaker conditions in (7.1) already suffice, is new. These conditions are modeled after and motivated by an analogue set of combinatorial conditions in the combinatorics of words systems and their associated subproduct systems, discussed in Gerhold and Skeide [GS14]

Recall that by Observation 7.1, $v_{(n)}^*$ maps $\underline{H}_n$ really into the algebraic tensor power $H_1^{\otimes n}$. Therefore, π_n (co)restricts to a projection in $\mathcal{B}^a(H_1^{\otimes n})$, which we continue denoting π_n. Their direct sum π is a Fock projection in $\mathcal{B}^a(\mathcal{F}(H_1))$. If we define $\xi := \bigoplus_{n\in\mathbb{N}_0} v_{(n)}^*$, the we embed the interacting Fock space $\mathcal{I} := \underline{\mathcal{F}(H^{\circleddash})}$ onto

$$\xi\mathcal{I} \ = \ \pi\mathcal{F}(H_1) \ \subset \ \pi\overline{\mathcal{F}(H_1)} \ \subset \ \overline{\mathcal{F}(H_1)}.$$

By definition $\xi\mathcal{I}$ is a subspace of the completion $\overline{\mathcal{F}(H_1)}$ and the complement $(\xi\mathcal{I})^\perp$ is relative to that Hilbert space. But thanks to being the range of the projection $\pi \in \mathcal{B}^a(H_1^{\otimes n})$, the subspace $\xi\mathcal{I}$ is complemented also in $\mathcal{F}(H_1)$. (The complement in this space is just the intersection of the topological complement $(\xi\mathcal{I})^\perp$ with $\mathcal{F}(H_1)$.) Then, π is literally everything we can know about that embedded interacting Fock space: $\pi = L = \lambda = \varkappa$. (Indeed, clearly, $\lambda_n = \pi_n$, so $L_n = \lambda_n^*\lambda_n = \pi_n$. Clearly, inserting π_n as candidate for $\varkappa_n$ into the recursion for λ_n, we recover $\lambda_n = \pi_n = \varkappa_n$. But for being the (uniquely determined) squeezing $\varkappa$, the resulting Fock projection π, with which we wish to identify $\varkappa$, has to be a squeezing. But, also this is true, because clearly π_{n+1} is surjective, and since $\pi_{n+1} \leq \mathsf{id}_H \otimes\pi_n$, we get that π_{n+1} is 0 on $H \otimes H_n^\perp$. We see, how nicely the algebraic invariance properties discussed in Observation 7.1 in the case of interacting Fock spaces from subproduct systems work together with the more topological definitions of π–interacting Fock space.)

The squeezing π is a projection. We ask what other properties a squeezing has to satisfy to be the one that comes from a subproduct system as described. This question requires also to understand which Fock projections are squeezings. Actually, we first need a sufficiently flexible notion of projection. We say, a map π from a pre-Hilbert space H into its completion $\overline{H}$ is a ***weak*** projection if $\langle x, \pi y\rangle = \langle \pi x, \pi y\rangle$ for all $x, y \in H$. (A weak projection extends uniquely to a projection in $\mathcal{B}(\overline{H})$, and every restriction of a projection in $\mathcal{B}(\overline{H})$ to H is a weak projection.)

For simplicity, in the following theorem we assume $\pi_1 = \mathsf{id}_H$ identifying this way, H_1 with H (otherwise being only a subspace of H). One can show that we always may replace H with $H_1 := \pi_1 H$.

7.4 Theorem. *Let $\mathcal{I} = (H, \pi)$ be a π–interacting Fock space where the squeezing $\pi\colon (H \otimes \mathcal{I}) \oplus \Omega\mathbb{C} \to \mathcal{I} \subset \overline{\mathcal{I}}$ is a weak projection with $\pi_1 = \mathsf{id}_H$. Then*

$$\pi_{n+1} \;\leq\; \mathsf{id}_H \otimes \pi_n. \tag{7.3}$$

Conversely, if H is a pre-Hilbert space and $\pi \in \mathcal{B}(\overline{\mathcal{F}(H)})$ a Fock projection such that the components π_n fulfill (7.3) *and $\pi_1 = \mathsf{id}_H$, then $\mathcal{I} := \pi\mathcal{F}(H)$ is a π–interacting Fock space.*

Moreover, in either case among summands $\pi_n H^{\otimes n}$ there exist coisometries $\pi_m H^{\otimes m} \otimes \pi_n H^{\otimes n} \to \pi_{m+n} H^{\otimes m+n}$ satisfying $\pi_m X_m \otimes \pi_n Y_n \to \pi_{m+n}(X_m \otimes Y_n)$ (so that the $\pi_n \overline{H^{\otimes n}}$ form a subproduct system and $\mathcal{I}$ is its associated interacting Fock space) if and only the π_n also fulfill

$$\pi_{n+1} \;\leq\; \pi_n \otimes \mathsf{id}_H\,. \tag{7.4}$$

Proof. As discussed two paragraphs before the theorem, if π is a squeezing, then the condition (7.3) is fulfilled. On the other hand, if a π is Fock projection in $\mathcal{B}(\overline{\mathcal{F}(H)})$, then by definition π sends $\mathcal{F}(H)$ surjectively onto $\mathcal{I}$, and if π fulfills (7.3), then π_{n+1} is 0 on $H \otimes (\pi_n H^{\otimes n})^{\perp}$, so π is a squeezing. We argued already that the last statement is true. ■

It is noteworthy that the two inequalities together imply the algebric invariance discussed in Observation 7.1.

7.5 Example. There are π–interacting Fock spaces that do not come from a subproduct system. Let H be a pre-Hilbert space with an orthonormal Hamel basis $(e_n)_{n\in\mathbb{N}}$ and put $p_n = e_n e_n^*$. Then $\pi_n = p_n \otimes \ldots \otimes p_1$ define a squeezing π that does not satisfy (7.4).

7.6 Observation. By (7.3) and Theorem 6.6, a π–interacting Fock space is a POI-interacting Fock space with bounded creator map a^*.

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