Ideal Submodules
versus
Ternary Ideals
versus
Linking Ideals

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Abstract

We show that ideal submodules and closed ternary ideals in Hilbert modules are the same; this contradicts the result of [Kol17]. We use this insight as a peg on which to hang a short note about interrelations with other notions regarding Hilbert modules. We point out that everything can be understood in an effortless way in terms of the linking algebra. We briefly sketch how this insight works for extensions of Hilbert modules and their Busby invariants. We also pose some open problems that naturally arise adopting our point if view.

0 Ideal submodules *versus* ternary ideals

Since the abstract and the section introductions are quite enough of introduction, let us right start with our *peg*: Ideal submodule=closed ternary ideal. After the *peg* and drawing some obvious consequences, we introduce a third equivalent notion, *linking ideal*, that will be our guide.

Definition. Let I be a closed ideal in a C^*–algebra $\mathcal{B}$, and let E be a Hilbert $\mathcal{B}$–module. The ***ideal submodule*** of E associated with I is $\overline{\mathsf{span}}\, EI$. More generally, we say $K \subset E$ is an ***ideal submodule*** of E if $K = \overline{\mathsf{span}}\, EI$ for some ideal I in $\mathcal{B}$.

Definition. A linear subspace K of a Hilbert $\mathcal{B}$–module E is a ***ternary ideal*** if $E\langle K, E\rangle \subset K$.

Definition. For a Hilbert $\mathcal{B}$–module E we denote by $\mathcal{B}_E := \overline{\mathsf{span}}\langle E, E\rangle$ its ***range ideal***.

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Proposition. *For a subset K of a Hilbert $\mathcal{B}$–module E the following conditions are equivalent:*

1. *K is an ideal submodule of E.*
2. *K is a closed ternary ideal in E.*

Proof. Of course, if K is an ideal submodule, it is a closed $\mathcal{B}$–submodule of E. Like for any closed submodule of E, this implies $\overline{\operatorname{span}}\, K\mathcal{B}_K = K$ and $\overline{\operatorname{span}}\langle K, E\rangle = \mathcal{B}_K = \overline{\operatorname{span}}\langle K, E\rangle$. From the first property we infer that $\mathcal{B}_K$ is the unique smallest ideal in $\mathcal{B}$ for which K is the associated ideal submodule. (Indeed, $K = \overline{\operatorname{span}}\, EI = \overline{\operatorname{span}}\, EI\mathcal{B}_K$. So, $\overline{\operatorname{span}}\, I\mathcal{B}_K = I \cap \mathcal{B}_K$ is a smaller ideal with which K is associated. Since $\mathcal{B}_K = \overline{\operatorname{span}}\langle K, K\rangle = \overline{\operatorname{span}}\langle KI, KI\rangle$, the ideal $I \cap \mathcal{B}_K$ cannot be smaller than $\mathcal{B}_K$.) From the second property we infer $\langle K, E\rangle \subset \mathcal{B}_K$, so, $E\langle K, E\rangle \subset E\mathcal{B}_K \subset K$.

Vice versa if K is a closed ternary ideal, then

$$\langle E, E\rangle\langle K, E\rangle \subset \langle E, K\rangle.$$

Making use of an approximate unit for $\mathcal{B}_E$, we get $\langle K, E\rangle \subset \overline{\operatorname{span}}\langle E, K\rangle$, and by taking adjoints $\langle E, K\rangle \subset \overline{\operatorname{span}}\langle K, E\rangle$, so $\overline{\operatorname{span}}\langle K, E\rangle = \overline{\operatorname{span}}\langle E, K\rangle$. Since $\langle K, E\rangle\langle E, E\rangle \subset \langle K, E\rangle$, we see that $I := \overline{\operatorname{span}}\langle K, E\rangle$ is an ideal in $\mathcal{B}_E$ and, further, in $\mathcal{B}$. Clearly, $EI \subset K$. On the other hand, $\overline{\operatorname{span}}\, EI = \overline{\operatorname{span}}\, E\langle E, K\rangle = \overline{\operatorname{span}}\, \mathcal{K}(E)K \supset K$. (Use an approximate unit for $\mathcal{K}(E)$.) In conclusion, $K = \overline{\operatorname{span}}\, EI$ (and, of course, $I = \overline{\operatorname{span}}\langle K, E\rangle = \mathcal{B}_K$). ■[1]

Conventions. Here and in the sequel, we assume that notions like *Hilbert modules*, $E^* \ni x^* = \langle x, \bullet\rangle$, *rank-one* operators $xy^*\colon z \mapsto x\langle y, z\rangle$, *compact operators* $\mathcal{K}(E, F) = \overline{\operatorname{span}}\, FE^*$, and their basic properties are known. We do **not** adopt the common standard convention according to which writing a product of spaces would mean the closure of the linear span: AB for subsets of spaces for which a product ab is defined, means exactly the set $AB = \{ab\colon a \in A, b \in B\}$ and nothing else. $\mathcal{B}^a(E, F)$ means the set of *adjointable* (automatically bounded) operators between Hilbert modules E and F. When necessary, we add a superscript ^{bil} to mean *$\mathcal{A}$–$\mathcal{B}$–linear maps* between $\mathcal{A}$–$\mathcal{B}$–bimodules.

Looking at the proof of the proposition, in either direction is was important to note (for different reasons in each case) that $\overline{\operatorname{span}}\langle K, E\rangle = \overline{\operatorname{span}}\langle E, K\rangle$. This is true for all submodules of E, so we get this as soon as we established that K is a submodule. While in the forward direction this was obvious, for the backward direction we used some case specific trickery. One can also argue (as we do below) referring to ternary subspaces. And one can argue, as we do in the following corollary, appealing to the general theorem that a two-sided closed ideal in a C^*–algebra is a $*$–ideal (see, for instance, Murphy [Mur90, Theorem 3.1.3]).

[1] This contradicts [Kol17, Theorem 4.3]. Indeed, the example proposed in its proof has the general structure $E = \mathcal{B} \oplus \mathcal{B}$ and $K = I_1 \oplus I_2$ for distinct closed ideals $I_1 \neq I_2$ of $\mathcal{B}$. (Direct sum of Hilbert $\mathcal{B}$–modules. In the example we even have $I_1 \supset I_2$.) Since $\overline{\operatorname{span}}\, EI = I \oplus I$ for all I, there is no I such that $\overline{\operatorname{span}}\, EI = K$, so K is not an ideal submodule. But, also $\overline{\operatorname{span}}\, E\langle K, E\rangle = \overline{\operatorname{span}}\, E(I_1 + I_2)$ (not $\oplus$!) is not contained in K. So, unlike claimed in [Kol17], K is not a closed ternary ideal, and the example is not a counter example.

Corollary. *If K is a closed ternary ideal, then $E\langle E,K\rangle \subset K$ and $K\langle E,E\rangle \subset K$. On the contrary, none of the two conditions alone implies that K is a ternary ideal. However, both conditions together* **do** *imply that K is a ternary ideal.*

Proof. The second condition is true for every submodule K. If K is a closed ternary ideal then the first condition follows from $\overline{\operatorname{span}}\langle E,K\rangle = \overline{\operatorname{span}}\langle K,E\rangle$ from the preceding proof.

Conversely, suppose both conditions hold. Then $\overline{\operatorname{span}}\langle E,K\rangle$ is a closed two-sided ideal of $\mathcal{B}$, hence, a $*$–ideal. Therefore, $\overline{\operatorname{span}}\langle E,K\rangle = \overline{\operatorname{span}}\langle K,E\rangle$, hence, $E\langle K,E\rangle \subset \overline{\operatorname{span}}\, E\langle E,K\rangle \subset K$.

Finally, taking $E = \mathcal{B} \ni \mathbf{1}$, the subspaces K fulfilling only the first and only the second condition are the left and the right ideals in $\mathcal{B}$, respectively, while (we just used that) both conditions together mean K is an ideal. Of course, there are left ideals that are not right ideals, so no two of these three conditions coincide. ∎

Let us complete the picture (without any reference to [Kol17]) by adding a third point of view. Recall that the ***linking algebra*** of a Hilbert $\mathcal{B}$–module E is $\mathcal{K}\binom{\mathcal{B}}{E} = \begin{pmatrix} \mathcal{K}(\mathcal{B}) & \mathcal{K}(E,\mathcal{B}) \\ \mathcal{K}(\mathcal{B},E) & \mathcal{K}(E) \end{pmatrix} = \begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{K}(E) \end{pmatrix}$, while the ***reduced linking algebra*** of E is $\mathcal{K}\binom{\mathcal{B}_E}{E} = \begin{pmatrix} \mathcal{B}_E & E^* \\ E & \mathcal{K}(E) \end{pmatrix}$. The two coincide if and only E is ***full***, that is, if $\mathcal{B}_E = \mathcal{B}$.

Definition. A subset K of a Hilbert $\mathcal{B}$–module E is a ***linking ideal*** if $\begin{pmatrix} \overline{\operatorname{span}}\langle K,K\rangle & K^* \\ K & \overline{\operatorname{span}}\, KK^* \end{pmatrix}$ is an ideal in the reduced linking algebra of E.

Theorem. *For a subset K of a Hilbert $\mathcal{B}$–module E the following conditions are equivalent:*

1. *K is an ideal submodule of E.*

2. *K is a closed ternary ideal in E.*

3. *K is a closed linking ideal in E.*

We prefer to discuss a part of the proof in a more general situation. Recall that a ***ternary subspace*** of a Hilbert module E is a linear subspace $F \subset E$ such that $F\langle F,F\rangle \subset F$.

Lemma. *If F is a closed ternary subspace of a Hilbert $\mathcal{B}$–module E, then (by abuse of notation) $\mathcal{B}_F := \overline{\operatorname{span}}\langle F,F\rangle$ is a C^*–subalgebra of $\mathcal{B}$ and F is a (full) Hilbert $\mathcal{B}_F$–module. The linking algebra of F is a* **blockwise** *subalgebra of the linking algebra of E (that is, $\mathcal{K}\binom{\mathcal{B}_F}{F}$ is a $*$–subalgebra of $\mathcal{K}\binom{\mathcal{B}}{E}$ with respect to the canonical identifications $\mathcal{B}_F \subset \mathcal{B}$, $F^* \subset E^*$, $F \subset E$, and $\mathcal{K}(F) \subset \mathcal{K}(E)$).*

Moreover, the C^*–subalgebra generated by F is

$$\begin{pmatrix} \overline{\operatorname{span}}\langle F,F\rangle & F^* \\ F & \overline{\operatorname{span}}\, FF^* \end{pmatrix}$$

and coincides with the reduced linking algebra $\mathcal{K}\binom{\mathcal{B}_F}{F}$.

F is a Hilbert submodule of E (if and only) if $\mathcal{B}_F$ is an ideal in $\mathcal{B}$.

Of course, every Hilbert submodule of E is a ternary subspace.

PROOF. Straightforward verification. ■

Remark. The lemma applies, in particular, to a ternary subspace F of a C^*–algebra $\mathcal{A}$, a so-called ***ternary ring of operators*** (***TRO***) in the definition of Blecher and Neal [BN07]. Note, however, that the C^*–subalgebra of $\mathcal{A}$ generated by F may but need not be isomorphic to the linking algebra. (For instance, let $\mathcal{A} = O_n$ be a Cuntz algebra and take for F the closed $\mathcal{A}$–subbimodule generated by the generating isometries s_k. Then the C^*–subalgebra generated by F is $\mathcal{A}$ and F is not a corner in $\mathcal{A}$.) The statement in the lemma about the linking algebras is a statement about an off-diagonal corner of $M_2(\mathcal{A})$, not about $\mathcal{A}$.

Observation. Matrices like linking algebras have been discussed in a more general context under the name of *generalized matrix algebras* in Skeide [Ske00]. The properties regarding *blockwise* subalgebra in the lemma are not automatic for subalgebras. For instance, if the linking algebra is unital (say, if E is finite-dimensional), then the subalgebra $\mathbb{C}\mathbf{1}$ is not a *blockwise* or *matrix subalgebra*. A unital matrix subalgebra must contain at least the two units of each entry in the diagonal, separately. The more important is to observe that an ideal of a matrix algebra is a matrix subalgebra, automatically. (Indeed, using approximate units for each diagonal entry from the left and from the right, we get projection maps $P_{i,j}$ onto each matrix entry.)

PROOF OF THE THEOREM. Since (1) and(2) are equivalent by our proposition, we only show equivalence of (2) and(3). Again, for both directions ((2)⇒(3) also involving the corollary) we are left with straightforward verifications. ■

Definition. After the theorem, we say a ***closed ideal*** in a Hilbert module E is a subset K fulfilling one (hence, all) of the conditions in the theorem.

We add some *folklore* regarding uniqueness properties among among the elements discussed in this section. By the observation, an ideal $\mathcal{I}$ in the linking algebra of E is a matrix subalgebra, that is, it has the form $\mathcal{I} = \begin{pmatrix} I & K^* \\ K & J \end{pmatrix} \subset \mathcal{K}\begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{K}(E) \end{pmatrix}$. If $\mathcal{I}$ is closed, then necessarily: I is a closed ideal in $\mathcal{B}$; K is a closed ternary ideal in E and the ideal submodule associated with I; by symmetry between E and E^* (considered as a full Hilbert $\mathcal{K}(E)$–module) and the uniqueness statements in the first part of the proof of the proposition, K^* is a closed ternary ideal in E^* and $J = \mathcal{K}(K)$. Summing up, every closed ideal $\mathcal{I}$ in $\mathcal{K}\binom{\mathcal{B}}{E}$ gives rise to a unique closed ideal $I = P_{1,1}(\mathcal{I})$ in $\mathcal{B}$ and every closed ideal I in $\mathcal{B}$ gives rise to a unique ideal $\mathcal{I}$ in $\mathcal{K}\binom{\mathcal{B}}{E}$ such that $I = P_{1,1}(\mathcal{I})$. Requiring that E is full, respectively, restricting to the reduced linking algebra, the symmetry between E and E^* as well as the correspondences among the corners become perfect. We, thus, proved, the following:

Supplement. *Let E be a Hilbert $\mathcal{B}$–module*

1. *The formula $I = P_{1,1}(\mathcal{I})$ establishes a one-to-one correspondence between*

   (a) *closed ideals I in $\mathcal{B}$ and*

   (b) *closed ideals $\mathcal{I}$ in $\mathcal{K}\binom{\mathcal{B}}{E}$,*

   also fulfilling $P_{2,1}(\mathcal{I}) = \overline{\text{span}}\, KI$ and $P_{2,2}(\mathcal{I}) = \mathcal{K}(\overline{\text{span}}\, KI)$.

2. *There are one-to-one correspondences between:*

   (a) *Closed ideals $\mathcal{I}$ in the reduced linking algebra $\mathcal{K}\binom{\mathcal{B}_E}{E}$ of E.*

   (b) *Closed ideals I in $\mathcal{B}_E$.*

   (c) *Closed ideals K in E.*

   (d) *Closed ideals J in $\mathcal{K}(E)$.*

   The correspondences satisfy

$$
\begin{aligned}
P_{1,1}(\mathcal{I}) &= I &&= \overline{\text{span}}\langle K, K\rangle, \\
P_{2,1}(\mathcal{I}) &= K &&= \overline{\text{span}}\, KI = \overline{\text{span}}\, JK, \\
P_{2,2}(\mathcal{I}) &= J &&= \overline{\text{span}}\, KK^*,
\end{aligned}
$$

   and are determined by (the appropriate subsets of) them (completed by $P_{1,2}(\mathcal{I}) = K^*$).

The symmetric situation in the second part is crucial for almost all that follows; not only because of the stated uniqueness properties (most of them desirable, if not indispensable), but also (and maybe even more importantly) by existence theorems for certain "good" maps. Not relevant for this note, the perfect symmetry between E and E^*, $\mathcal{B}_E = \mathcal{K}(E^*)$ and $\mathcal{K}(E)$ $(= (\mathcal{K}(E))_{E^*})$ is also the situation in an important notion like *Morita equivalence*. (Optically, the symmetry is disturbed by writing $\mathcal{B}_E$ for one diagonal entry, and $\mathcal{K}(E)$ for the other; but this is how it occurs the first definition of Morita equivalence, which avoids the use of correspondences. Another definition, more symmetric and quite common, avoids the asymmetry caused by writing down the right module E^*; instead, it defines a left Hilbert modules structure on the same E. But, honestly, we think there is no real symmetry between the theory of Hilbert right modules (module map condition=assosiativity condition) and Hilbert left modules (module map conditions have to flip symbols in formulae); so we are not entirely happy calling this symmetric. In Skeide [Ske16, Section 2] we propose a perfectly symmetric definition (using correspondences), the way it is done in abstract algebra, and show its equivalence with (one of) the usual definitions.)

1 Ideals and maps

In Section 0, we have defined three notions of closed ideal in a Hilbert module and we showed, in the theorem, the three notions coincide. In this section we wish to justify the name ideal in Hilbert modules for this structure, by comparing it with what ideals are in C^*–algebra theory. (The closely related notion of extension we postpone to Section 2.)

Closed ideals in a C^*–algebra are precisely:

- The subspaces that may be divided out, with a quotient in the same category.
- The kernels of the homomorphisms of the category.

The two are intimately related by:

- If I is a closed ideal in $\mathcal{B}$, then the canonical map $b \mapsto b+I$ is homomorphism with kernel I.
- If $\varphi\colon \mathcal{B} \to \mathcal{A}$ is a homomorphism, then $\ker\varphi$ is a closed ideal and $\varphi(\mathcal{B}) \cong \mathcal{B}/\ker\varphi$ via $\varphi(b) \mapsto b+I$.

(The whole situation is also captured by the statement that we have a so-called *short exact sequence*

$$0 \xrightarrow{\text{can.}} I \xrightarrow{\text{can.}} \mathcal{B} \xrightarrow{\text{can.}} \mathcal{B}/I \xrightarrow{\text{can.}} 0, \qquad (*)$$

that is, $\mathcal{B}$ is what is called an *extension* of $\mathcal{B}/I$ by I. We discuss this later in Section 2.)

So, our job is to find a quotienting procedure for closed ideals of Hilbert modules and the right sort of (homo)morphisms. Actually, the search is not limited to the morphisms, but starts earlier with the question what is the class of objects. Are we considering a category of Hilbert $\mathcal{B}$–modules for fixed $\mathcal{B}$, or are we looking at a category of Hilbert modules over arbitrary C^*–algebras? Are the modules required full or not? And is there a point of view where the question of fullness does not play a role? This is, obviously, related also to the question on which structure elements the three notions of ideal put emphasis.

Ideal submodules and the linking algebra put emphasis on $\mathcal{B}$. For the former, everything is expressed and determined explicitly by choosing an ideal I of $\mathcal{B}$; the latter contains full information about $\mathcal{B}$ in the 11–corner. Different choices for I giving the same ideal submodule, are visible only if we keep this information. This suggests that ideal submodules, with full knowledge about the chosen ideal I, is best adapted in the situation where we wish to fix the C^*–algebra $\mathcal{B}$ and consider only Hilbert $\mathcal{B}$–modules. However, this is not really compatible with the idea of dividing out ideal submodules.

In fact, the quotient of a Hilbert space and a Hilbert subspace (so, $\mathcal{B} = \mathbb{C}$) is isomorphic to the orthogonal complement of the subspace. This complement is a concrete subspace of the Hilbert space which, therefore, inherits an inner product from the containing space (in which, as we all know, the complement is closed). But for Hilbert modules over general $\mathcal{B}$ this fails as soon as the submodule is not complemented.

1.1 Example. Let E be a (nonzero) full Hilbert $\mathcal{B}$–module and $K = \overline{\operatorname{span}} EI \neq E$ be the ideal submodule associated with a proper essential closed ideal I in $\mathcal{B}$ (so that $K^\perp = \{0\}$). What would be a $\mathcal{B}$–valued inner product on the quotient E/K? (The quotient **is**, in fact, a Banach $\mathcal{B}$–module. By (a simple application of) Lance's [Lan95, Theorem 3.5], there is at most one inner product on a Banach module that turns it into a Hilbert module.) Suppose we have such an inner product. Choose a nonzero $x + K$ in the quotient, so, $b := \sqrt{\langle x + K, x + K\rangle} \neq 0$. Since I is essential, we may choose $i \in I$ such that $bi \neq 0$. Then

$$\|(x + K)i\|^2 \;=\; \|i^*\langle x + K, x + K\rangle i\| \;=\; \|(bi^*)(bi)\| \;\neq\; 0,$$

while $(x + K)i = xi + K = 0 \in E/K$. Consequently, there is no inner product on E/K turning it into a Hilbert $\mathcal{B}$–module.

(Note: A moments thought shows that for a general ideal submodule $K = \overline{\operatorname{span}} EI$, the closed ideal I is essential in $\mathcal{B}_{K^{\perp\perp}}$. (Indeed, if $x \in E$ such that $xi = 0$ for all i, then $\langle y, x\rangle = 0$ for all $y \in K$, so, $x \in K^\perp$. If x is also in $K^{\perp\perp}$, then $x = 0$. Repeating the same for $E^n \supset K^n$, shows that I is essential for $I_0 := \operatorname{span}\langle K^{\perp\perp}, K^{\perp\perp}\rangle$. Using that I_0 is a dense ideal in $\mathcal{B}_{K^{\perp\perp}}$, we see that I is essential also in $\mathcal{B}_{K^{\perp\perp}}$: For $b_0 \in I_0$ we find $bi = 0\,\forall i \Rightarrow b(b_0 i) = (bb_0)i = 0\,\forall i \Rightarrow bb_0 = 0$. So, $bb_0 = 0 \forall b_0$, that is, $b = 0$.) Repeating the argument of the example for $x \in K^{\perp\perp}\backslash K$ as soon as $K^{\perp\perp} \supsetneq K$, shows that the example captures the general situation when K is non-complemented.)

The example shows that as soon as the ideal submodule K is not complemented (typically, as soon as $\mathcal{B}_K = \mathcal{B}_E \cap I$ is non-unital), the quotient $\mathcal{B}$–module E/K does not admit an inner product. The key point in the contradiction is that a $\mathcal{B}$–valued inner product has no choice but assigning to certain elements $x + K$ with $x \in K$, so $x + K = 0$, non-zero $\langle x + K, x + K\rangle$. On the other hand, these elements are, actually, in I. If we divide out I from $\mathcal{B}$, then the preceding contradiction disappears. In fact, it is easy to show, that the $\mathcal{B}/I$–valued inner product $\langle x + K, x' + K\rangle = \langle x, x'\rangle + I$ is well-defined. Also, E/K may be viewed as $\mathcal{B}/I$–module via $(x + K)(b + I) = xb + K$, and E/K with this inner product is a Hilbert $\mathcal{B}/I$–module. (See Bakic and Guljas [BG02], where ideal submodules have been defined, for details.) So, in order to quotient out an ideal submodule, we also have to take a quotient of the algebra $\mathcal{B}$.

The quotient map $v\colon x \mapsto x + K$ shows us a sort of homomorphisms which have ideal submodules as kernels. In fact, denote by $\varphi\colon b \mapsto b + I$ the quotient map for the quotient $\mathcal{B}/I$. Then

$$\langle vx, vx'\rangle \;=\; \varphi(\langle x, x'\rangle),$$

as observed in [BG02], who called such a map a *φ–morphism*. Following Abbaspour and Skeide [Ske06, AS07], we prefer to call a map v from a Hilbert $\mathcal{B}$–module to a Hilbert C–module fulfilling the preceding property for a homomorphism $\varphi\colon \mathcal{A} \to C$ a ***φ–isometry***; we say v is a ***generalized isometry*** if there exists a homomorphism φ turning it into a φ–isometry. (Note that a φ–isometry is norm-preserving if and only if φ is faithful on the range ideal of the $\mathcal{B}$–module.) We are sure that there will be earlier instances of φ–isometries and that most of their properties are *folklore*; in any case, most of their properties are easy exercises.

Clearly, the maps that occur in the short exact sequence (∗) that characterizes the quotient procedure, are generalized isometries. This depends, however, on the special choice of the C^*–algebra over which the quotient is a Hilbert module. We will see in Observation 1.7 that, when C is chosen unfortunately, there are maps between Hilbert modules having ideal submodules as kernels, which are not generalized isometries.

Conclusion for *ideal submodules*.

- The notion of ideal submodule, like the notion of generalized isometries as the corresponding morphisms, puts much emphasis on the algebra $\mathcal{B}$. This might be felt as suggesting that we are looking at $\mathcal{B}$–modules for fixed $\mathcal{B}$. On the contrary, the quotient by an ideal submodule does (almost always) not admit an inner product with values in $\mathcal{B}$.

- While being an ideal submodule does not depend on the choice of the ideal I that makes it explicit, the C^*–algebra $\mathcal{B}/I$ over which the quotient is a Hilbert module does depend on the choice of I. This is quite unfortunate.

- The derived notion of generalized isometry does not characterize all maps between Hilbert modules whose kernels appear as ideal submodules.

In conclusion, ideal submodule is not the most fortunate choice to illustrate that ideals in Hilbert modules merit their name. (Actually, in order to give a completely asymmetric definition, according to the corollary in Section 0, we might add to the theorem in Section 0 the fourth equivalent property that K be a closed $\mathcal{B}$–submodule of E satisfying $\mathcal{K}(E)K \subset K$.)

The notion of (closed) ternary ideal refers to the ternary product $(x, y, z) \mapsto (x.y.z) := x\langle y, z\rangle$ on a Hilbert $\mathcal{B}$–module E. Like the ternary product itself, the condition to be a ternary ideal is an intrinsic condition to the space; no reference to the C^*–algebra $\mathcal{B}$ (or its ideals) is made. The fact that each Hilbert module sits as closed ternary subspace in its (reduced) linking algebra, shows that every Hilbert module is a TRO as discussed in the remark in Section 0. The corollary in Section 0 shows that our definition of closed ternary ideal coincides with the definition of *ternary ideal* in [BN07]. In Corollary 1.8, we will see that the ternary product seen as abstract map $V \times V \times V \to V$ on a linear space determines a potential structure of full Hilbert module up to ***generalized unitary*** (a bijective generalized isometry): There is, up to generalized unitary, at most one possibility to identify V with a TRO.

Unlike the point of view of Hilbert module (where we started with ideal submodules as spaces to be divided out and and ended up only with an imperfect correspondence with generalized isometries as morphisms), here, putting emphasis on the ternary product, both questions have quite clear answers: The natural morphism are *ternary homomorphisms*. A ***ternary homomorphism*** is a linear map v from a TRO E to a TRO F satisfying $v(x.y.z) = (vx.vy.vz)$. In terms of the inherited Hilbert module structure this is $v(x\langle y, z\rangle) = (vx)\langle vy, vz\rangle$. And, by a simple application of the corollary in Section 0, ternary ideals K in a TRO E are precisely the subspaces that can be divided out, if we wish that the ternary product

$$(x + K.y + K.z + K) \; = \; (x.y.z) + K$$

is well-defined on the quotient E/K. There are a number of different ways to show that the ternary product comes from a semiinner product E/K and that (as usual with quotients of normed spaces) this semiinner product is inner, if (and only if) the ternary ideal K is closed. Like above for uniqueness of the TRO structure, we postpone the proof of this fact, until we uncover a better point of view.

Conclusion for *closed ternary ideals*.

- The TRO-structure of a Hilbert module is perfect to capture its structure intrinsically, by which we mean without explicit reference to $\mathcal{B}$.
- It is crystal what the corresponding morphisms and and closed ideals are, providing a perfect analogy with homomorphism and closed ideals in C^*–algebras.
- But we left open to prove a number of statements, because the proofs fit better into the next case.

In conclusion, closed ternary ideals resolve perfectly the problem we posed in the beginning of this section; but, the notion is not "suggestive" regarding how to, actually, prove it does.

We now come to the third point of view, linking ideals, where everything is defined in terms of the reduced linking algebra – or, alternatively, of the linking algebra but requiring full modules. First of all, note that we could have defined linking ideals also by making reference to the linking algebra instead of the reduced linking algebra. As a condition this is the same:

1.2 Proposition. *For a subset K of a Hilbert module E the following conditions are equivalent:*

- *K is a closed linking ideal of E.*
- $\begin{pmatrix} \overline{\operatorname{span}}\langle K, K\rangle & K^* \\ K & \overline{\operatorname{span}}\, KK^* \end{pmatrix}$ *is a closed ideal in the linking algebra of E.*

(We omit the simple proof that follows simply by observing that if E is Hilbert $\mathcal{B}$–module, then $\mathcal{B}_E$ is an ideal in $\mathcal{B}$ and, therefore, the reduced linking algebra of E is an ideal in the linking algebra.) Also the proof of the theorem in Section 0 does not depend on this choice. However, if we opted for the linking algebra instead of the reduced linking algebra we would run into precisely the same difficulties that make ideal submodules to be not the most convenient choice: Too much emphasis on the 11–corner, when E is non-full.

In the observation in Section 0, we have learned that ideals in matrix algebras like the (reduced) linking algebra, automatically are matrix subalgebras: The ranges of the projections onto the corners are (closed) subspaces of the ideal. Also, a range of a homomorphism Φ into a C^*–algebra inherits the structure of a matrix algebra by simply decomposing it into the images of the corners; Φ, in that way, becomes by definition a blockwise map between matrix algebras. But if $\mathcal{A}$ has already a given matrix algebra structure, it is not said that the inherited matrix algebra structure of the range of Φ respects the given one of $\mathcal{A}$.

1.3 Example. Let $\mathcal{B}$ be unital and consider the matrix subalgebra $C = \begin{pmatrix} 1\mathbb{C} & \\ & 1\mathbb{C} \end{pmatrix} \subset \mathcal{A} := M_2(\mathcal{B})$. Let $U \in M_2$ be any unitary (scalar) matrix such that $U \bullet U^*$ does not leave invariant the diagonal. Then $\Phi := (U \bullet U^*) \restriction C$ is a homomorphism from the matrix algebra C into the matrix algebra $\mathcal{A}$ that is not onto a matrix subalgebra of $\mathcal{A}$. Moreover, $U \bullet U^*$ itself is an automorphism of $\mathcal{A}$ that is not blockwise.

Our scope is to infer properties of maps between Hilbert modules from properties of homomorphisms between linking algebras (and, then, to derive from these informations about kernels and quotients). To that goal it is necessary that the homomorphisms between the reduced linking algebras we take into consideration, induce in the first place maps between the underlying Hilbert modules. We cannot think of any more natural condition that the restriction sends the 21–corner to the 21–corner. But this already enough to force the homomorphism be blockwise.

1.4 Proposition. *Let* $\Phi\colon \begin{pmatrix} \mathcal{B}_E & E^* \\ E & \mathcal{K}(E) \end{pmatrix} \to \begin{pmatrix} C_F & F^* \\ F & \mathcal{K}(F) \end{pmatrix}$ *be a homomorphism between the reduced linking algebras of a Hilbert* $\mathcal{B}$*–module* E *and a Hilbert* C*–module* F *such that* $\Phi(E) \subset F$. *Then* Φ *is blockwise.*

Proof. Clearly, $\Phi(E^*) = \Phi(E)^* \subset F^*$ and $\Phi(\mathcal{K}(E)) = \overline{\operatorname{span}}\, \Phi(E)\Phi(E)^* \subset \overline{\operatorname{span}}\, FF^* = \mathcal{K}(F)$. Likewise, $\Phi(\mathcal{B}_E) \subset C_F$. ■

1.5 Remark. More generally, by the same proof, a homomorphism between the linking algebras, that preserves the 21–corner, preserves the reduced linking algebras (and is blockwise on them). However, for the linking linking algebras, the proposition may fail. Indeed, suppose $\mathcal{B}_E \neq \mathcal{B}$ is unital so that $\mathcal{B} = \mathcal{B}_E \oplus J$, and suppose that $\Phi(E) \neq F$ is complemented in F. Then we may extend a homomorphism between the reduced linking algebras by any homomorphism from J to $\mathcal{K}(\Phi(E)^\perp) \subset \mathcal{K}\begin{pmatrix} \Phi(E) \\ \Phi(E)^\perp \end{pmatrix} = \mathcal{K}(F)$.

So, speaking about linking ideals, we now examine homomorphisms between reduced linking algebras that are blockwise. The following theorem is *folklore*.

1.6 Theorem. *For a map* u *from a full Hilbert* $\mathcal{B}$*–module* E *to a Hilbert* C*–module the following conditions are equivalent:*

1. *u is a generalized isometry.*

2. *u is a ternary homomorphism.*

3. *u extends to a (blockwise, unique) homomorphism* $\Phi = \begin{pmatrix} \varphi & u^* \\ u & \phi \end{pmatrix}\colon \begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{K}(E) \end{pmatrix} \to \begin{pmatrix} C & F^* \\ F & \mathcal{K}(F) \end{pmatrix}$.

The equivalence between (2) *and* (3) *remains true for arbitrary* E, *if we speak about homomorphisms* Φ *between the reduced linking algebras.*

Proof. Equivalence of (1) and (2) appears, for instance, in Abbaspour and Skeide [AS07, Theorem 2.1].[2] And results allowing to conclude from (1) to (3) (the other direction is obvious) are around in [BG02, Ske06, AS07]. But probably none of these references is primary. We prefer to give an independent and streamlined proof.

Of course, (3) implies (1) (independently of whether E is full or not, but we know from the proof of Proposition 1.4 that Φ is blockwise and unique, once E is full), and (1) implies (2). So, let us assume (2) holds.

Let $a \in \mathcal{B}^a(E)$ and define an operator on the pre-Hilbert C–submodule $\operatorname{span}(uE)(uE)^*F$ of F by $(ux)(ux')^*y \mapsto (uax)(ux')^*y$. It is easy to show that the operator corresponding to a^* is a formal adjoint on the generating set $(uE)(uE)^*F$. Therefore, this well-defines a representation of the C^*–algebra $\mathcal{B}^a(E)$ by adjointable operators on $(uE)(uE)^*F$. Since the unital C^*–algebra $\mathcal{B}^a(E)$ is spanned by its unitaries, the representation operators are bounded, so that the representation extends further to a representation by bounded operators on $F_E = \overline{\operatorname{span}}(uE)(uE)^*F \subset F$. Obviously, this representation maps $\mathcal{K}(E)$ into $\mathcal{K}(F_E) \subset \mathcal{K}(F)$. We denote by ϕ the corresponding map. We do the same for ternary homomorphism $u^*\colon x^* \mapsto (ux)^*$ from the Hilbert $\mathcal{K}(E)$–module E^* to the Hilbert $\mathcal{K}(F)$–module F^*, obtaining a homomorphism φ from $\mathcal{B} = \mathcal{B}_E = \mathcal{K}(E^*)$ into $\mathcal{K}(F^*) = C_F \subset C$. It is routine to show that the matrix Φ is a homomorphism. ■

1.7 Observation. If E is not full, then (2) or (3) do not imply (1). For instance, if I is a proper essential ideal in $\mathcal{B}$ then the identity on I does not extend to a homomorphism $\mathcal{B} \to I$. The ternary homomorphism $u = \mathsf{id}_I$ from the Hilbert $\mathcal{B}$–module I to the Hilbert I–module I does not extend to a homomorphism between the linking algebras.

1.8 Corollary. *Suppose the linear space V has a ternary product $(\bullet.\bullet.\bullet)$ and suppose we have two injective maps u_i from V into C^*–algebras $\mathcal{A}_i$ such that u_iV are TROs and $u_i(x.y.y) = (u_ix)(u_iy)^*(u_iz)$. Then $u\colon u_1x \mapsto u_2x$ is a ternary unitary and a φ–unitary for the φ from Theorem 1.6. So, up to ternary or generalized isomorphism there is at most one TRO structure on V reproducing the given ternary product.*

1.9 Observation. Generalized homomorphisms $E \to F$ correspond one-to-one with blockwise homomorphisms between the linking algebras, in a way not dissimilar as ideals in $\mathcal{B}$ correspond to ideals in the linking algebra of E in the first part of the supplement in Section 0. However, while the 11–corner I alone determines the ideal in the linking algebra, for homomorphisms between the linking algebras, we need to know the 11–corner φ and the 21–corner u of Φ. Given only φ, there need not even exist a φ–isometry u for given E and F, hence, no Φ; see [Ske06] for more details. This applies also to the reduced linking algebras, in the second part of the supplement the Conditions (b) and (c) that refer to the diagonal corners, have no counterpart for homomorphisms. But we do have a one-to-one correspondence between ternary homomorphisms and blockwise homomorphisms between the reduced linking algebras.

[2] We mentioned already in Skeide and Sumesh [SS14, Footnote 2] that, unlike stated in [AS07, Theorem 2.1], the hypotheses that a ternary homomorphisms has to be linear cannot be dropped.

Conclusion for *closed linking ideals*.

- Linking ideals with maps that extend as blockwise homomorphisms between the reduced linking algebras perfectly resolve the problem of dividable submodule and homomorphisms posed in the beginning of this section.

- However, it turns out that when we want to characterize the homomorphisms without actually having to construct their extensions, then we end up with exactly the ternary homomorphisms.

Summary of conclusions.

- The notions that perfectly resolves the quotient/homomorphism-problem in an intrinsic way, that is, requiring not more than looking at the modules and maps between them, are the notions of closed ternary ideal and of ternary homomorphism.

- The notions of ideal submodule and generalized isometry explicitly involve other corners of the linking algebra and, at least for non-full E, there may be unfortunate choices that ruin a satisfactory solution of our problem.

- While the ternary notions resolve our problem intrinsically, when we wish to show that they do so, we best run through linking notations (closed linking ideal and blockwise homomorphisms), based on our theorems that establish their equivalence.

In many situations it has shown fruitful, to introduce properties of Hilbert modules and of maps between Hilbert modules by passing to the (reduced) linking algebras and examine how "good" blockwise properties of the linking algebras are reflected by those of their 21–corners. (Only for instance: *Von Neumann modules* are Hilbert modules over a von Neumann algebra whose linking algebra is a von Neumann algebra and maps between them are *normal* if they possess a normal (in particular, positive!) blockwise extension; see [Ske00]. Only the definition of *dynamical systems on Hilbert modules* as one parameter groups of ternary automorphisms allowed to provide a neat characterization of their generators as *ternary derivations*, while their characterization as *generalized derivation* fails due to domain problems; see [AS07]. Only studying the extension of so-called τ–maps (a τ–isometry but with τ a CP-map) to a blockwise map between the reduced linking algebras allows to give an intrinsic characterization without reference to τ; see [SS14].)

In the following section we add as an example the notion of *extensions* for Hilbert modules. Extensions have been introduced by Bakic and Guljas [BG04] (examined in their forthcoming papers and also by Kolarec), based on the notion from [BG02], which correspond to the first of the three points of view we discussed until here. Without any ambition to be complete, we recover some of the results on extensions of Hilbert modules as corollaries of the known results on extensions of C^*–algebras.

2 Extensions

An *extension* of C^*–algebras is essentially a short exact sequence

$$0 \xrightarrow{0} \mathcal{A} \xrightarrow{\psi} \mathcal{B} \xrightarrow{\varphi} C \xrightarrow{0} 0. \qquad (**)$$

A sequence of maps is ***exact*** if the range of each map coincides with the kernel of the next. An exact sequence of three, augmented by the 0s in the diagram, is ***short***. Since we are speaking of C^*–algebras, the maps are homomorphisms. 0 as C^*–algebra is short for $\{0\}$. And the only (linear) maps that have $\{0\}$ as domain or as codomain, are the zero-maps 0 that map everything to 0. (Usually, we will not write the 0s on arrows.) The situation of this short exact sequence is (usually) referred to by saying, $\mathcal{B}$ is an ***extension*** of C by $\mathcal{A}$.

Since, by exactness, the kernel of ψ is $\{0\}$, the homomorphism ψ is an embedding and $\mathcal{A}$ is isomorphic to the subalgebra $\psi(\mathcal{A})$ of $\mathcal{B}$. Since, by exactness, $\psi(\mathcal{A}) = \ker\varphi$, $\psi(\mathcal{A})$ is an ideal in $\mathcal{B}$. Since, by exactness, φ is surjective, we see that $\varphi(b) \mapsto b + \psi(\mathcal{A})$ establishes an isomorphism $\varphi(\mathcal{B}) \cong \mathcal{B}/\psi(\mathcal{A})$. So, $(*)$ does, indeed, capture (up to isomorphism) the general situation of extension. However, while $(*)$ puts emphasis on the algebra called extension, $\mathcal{B}$, in extension theory one would rather fix $\mathcal{A}$ and C and analyze the class of all extensions of C by $\mathcal{A}$. (Also for this reason it is reasonable to use for $\mathcal{A}$ a letter for generic C^*–algebras, not a letter for ideals.)

2.1 Definition. An ***extension*** (or ***ternary extension***) of Hilbert modules is a short exact sequence

$$0 \longrightarrow G_{\mathcal{A}} \xrightarrow{v} E_{\mathcal{B}} \xrightarrow{u} F_C \longrightarrow 0 \qquad (***)$$

of ternary homomorphisms. We say E is an extension of F by G.

(The notation $E_{\mathcal{B}}$ means E is a Hilbert $\mathcal{B}$–module. We choose a "lettering", that is most compatible with both the notation from the preceding sections and the above short exact sequence of C^*–algebras.)

The modules are not required full; fullness does not play a role in our definitions. However, we have the following obvious consequence of our theorem in Section 0 and of Theorem 1.6:

2.2 Proposition. *For full Hilbert modules the definition of extension is equivalent to the definition in [BG04].*

2.3 Theorem. *Extensions*

$$0 \longrightarrow G \xrightarrow{v} E \xrightarrow{u} F \longrightarrow 0$$

of Hilbert modules are in one-to-one correspondence with ***blockwise*** *extensions (that is, all homomorphisms are blockwise)*

$$0 \longrightarrow \mathcal{K}\begin{pmatrix} \mathcal{A}_G \\ G \end{pmatrix} \xrightarrow{\Psi} \mathcal{K}\begin{pmatrix} \mathcal{B}_E \\ E \end{pmatrix} \xrightarrow{\Phi} \mathcal{K}\begin{pmatrix} C_F \\ F \end{pmatrix} \longrightarrow 0$$

of the corresponding reduced linking algebras, via the requirement that the (co)restriction of Φ *and* Ψ *to maps between the* 21*–corners are* u *and* v*, respectively,* Proof. This is essentially just Theorem 1.6 plus the observation that the correspondence between ternary homomorphisms and blockwise homomorphisms respect properties such as injectivity and surjectivity of these maps. ■

2.4 Corollary. *By (co)restricting the corresponding blockwise extension of an extension E of F by G to the two diagonal corners, this extension gives rise to an associated extension $\mathcal{B}_E$ of C_F by $\mathcal{A}_G$ and an associated extension $\mathcal{K}(E)$ of $\mathcal{K}(F)$ by $\mathcal{K}(G)$.*

We follow, as we said without any ambition of being complete, through (very) few basic properties of extensions of C^*–algebras and their module analogues. This should be well readable together with Blackadar [Bla06, Section II.8.4]. It is understood that when we say blockwise extension we have Hilbert modules around as in Theorem 2.3, and we are referring to the line with the corresponding reduced linking algebras.

2.5 Split extensions. The extension $(**)$ of C^*–algebras is ***split*** if there exists a homomorphism $s\colon C \to \mathcal{B}$ (called a ***splitting***) that is a right inverse of φ. (So, $s \circ \varphi$ is a conditional expectation onto $s(C) \cong C$; see Question 3.1.) This definition makes sense for extensions of Hilbert modules, if we replace the homomorphism s for $(**)$ with a ternary homomorphism $s\colon F \to E$ for $(***)$. It is an immediate consequence of Theorem 2.3 that this requirement is equivalent to requiring that the corresponding blockwise extension is split via a blockwise splitting (being the unique blockwise homomorphism corresponding to s via Theorem 1.6).

Also the ***super-trivial***[3] extension $C \oplus \mathcal{A}$ of C by $\mathcal{A}$ (with φ being the canonical projection and ψ being the canonical embedding) generalizes. To that goal recall that the ***external*** direct sum of a Hilbert C–module F and a Hilbert $\mathcal{A}$–module G is $F \oplus G$ with its natural Hilbert $C \oplus \mathcal{A}$–module structure (so that also $\mathcal{K}(F \oplus G) = \mathcal{K}(F) \oplus \mathcal{K}(G)$). It follows that $F \oplus G$ is an extension (obviousuly split) of F by G.

2.6 Busby invariant. The ***Busby invariant*** of an extension $\mathcal{B}$ of C by $\mathcal{A}$ as in $(**)$ is the canonical homomorphism

$$\tau\colon C \xrightarrow{\cong} \mathcal{B}/\psi(\mathcal{A}) \longrightarrow M(\mathcal{A})/\mathcal{A} =: Q(\mathcal{A}),$$

where the second arrow refers to the image of the canonical homomorphism from $\mathcal{B}$ into the multiplier algebra of its ideal $\psi(\mathcal{A}) \cong \mathcal{A}$. (Recall: if $\vartheta\colon \mathcal{D} \to \mathcal{E}$ is a homomorphism and if I is an ideal in $\mathcal{D}$ such that $\vartheta(I)$ is an ideal in $\mathcal{E}$, then $d + I \mapsto \vartheta(d) + \vartheta(I)$ well-defines a homomorphism $\mathcal{D}/I \to \mathcal{E}/\vartheta(I)$. Here, I is $\psi(\mathcal{A})$ and the image of I in $M(\mathcal{A})$ under ϑ is $\mathcal{A}$.)

The only thing it takes to make work the definition of Busby invariant for an extension $(***)$ of Hilbert modules as the 21–corner of the Busby invariant of the corresponding blockwise extension of the reduced linking algebras, is to get hold of the blockwise structure of the multiplier algebra $M\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right)$ and the ***corona*** $Q\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right)$ of the reduced linking algebra $\mathcal{K}\binom{\mathcal{A}_G}{G}$. So, starting

[3] Unfortunately, *trivial extension*, usually, refers to split extensions. So, we had to invent something to mean *more trivial than trivial*.

from an extension (∗∗∗), by Theorem 2.3 we get the (unique) blockwise extension

$$0 \longrightarrow \mathcal{K}\binom{\mathcal{A}_G}{G} \xrightarrow{\ \Psi\ } \mathcal{K}\binom{\mathcal{B}_E}{E} \xrightarrow{\ \Phi\ } \mathcal{K}\binom{C_F}{F} \longrightarrow 0$$

(such that the (co)restriction to the 21–corner of Φ and Ψ are u and v, respectively). For calculating its Busby invariant $\mathcal{T}\colon \mathcal{K}\binom{C_F}{F} \to Q\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right) = M\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right)/\mathcal{K}\binom{\mathcal{A}_G}{G}$, recall that for any Hilbert module E we have $M(\mathcal{K}(E)) = \mathcal{B}^a(E)$. (Kasparov [Kas80]; see Skeide [Ske01, Corollary 1.7.14] for a simple proof.) So,

$$M\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right) = \mathcal{B}^a\binom{\mathcal{A}_G}{G} = \begin{pmatrix} M(\mathcal{A}_G) & \mathcal{B}^a(G,\mathcal{A}_G) \\ \mathcal{B}^a(\mathcal{A}_G,G) & \mathcal{B}^a(G) \end{pmatrix} \supset \begin{pmatrix} \mathcal{A}_G & G^* \\ G & \mathcal{K}(G) \end{pmatrix} = \mathcal{K}\binom{\mathcal{A}_G}{G}.$$

It follows that

$$Q\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right) = \begin{pmatrix} M(\mathcal{A}_G)/\mathcal{A}_G & \mathcal{B}^a(G,\mathcal{A}_G)/G^* \\ \mathcal{B}^a(\mathcal{A}_G,G)/G & \mathcal{B}^a(G)/\mathcal{K}(G) \end{pmatrix} = \begin{pmatrix} Q(\mathcal{A}_G) & Q(G)^* \\ Q(G) & Q(\mathcal{K}(G)) \end{pmatrix},$$

where we defined $Q(G) := \mathcal{B}^a(\mathcal{A}_G, G)/G$. Note that this is really the quotient of the Hilbert $M(\mathcal{A}_G)$–module $M(G) := \mathcal{B}^a(\mathcal{A}_G, G)$ and its ternary ideal G.

Observation. If $\mathcal{A}_G$ is nonunital, then $M(G)$ need not be full but only what we call ***strictly full***. The most drastic way to see this, is if $\mathcal{K}(G)$ is unital (that is, if G is algebraically finitely generated; for instance, if G is the full Hilbert $\mathcal{K}(H)$–module H^* for some infinite-dimensional Hilbert space H). In that case, $M(G) = \mathcal{B}^a(\mathcal{A}_G, G) = G$. (The easiest way to see this is, to consider the Hilbert module G^* over the unital C^*–algebra $\mathcal{K}(G)$.) Then, $M(\mathcal{A}_G)_{M(G)} = \overline{\mathsf{span}}\langle M(G), M(G)\rangle = \overline{\mathsf{span}}\langle G, G\rangle = \mathcal{A}_G \subsetneq M(\mathcal{A}_G)$. So, looking at $M\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right)$, we are definitely leaving the situation where our matrix algebras are reduced linking algebras. Even worse, in the same situation we have $Q\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right) = \begin{pmatrix} Q(\mathcal{A}_G) & 0 \\ 0 & 0 \end{pmatrix}$. A similar thing happens, if G is full over a unital algebra with nonunital $\mathcal{K}(G)$ (for instance $G = H$); just that now the zero-place and nonzero-place in the diagonal switch. Taking the external direct sum of these two cases, we get an example where both $\mathcal{A}_G$ and $\mathcal{K}(G)$ are nonunital and where $Q(G) = \{0\}$ so that $Q\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right)$ is diagonal with both places in the diagonal nonzero.

In view of this observation, it appears appropriate to introduce the ***reduced multiplier linking algebra*** $M^{red}(G)$ of a Hilbert $\mathcal{A}$–module G as the C^*–subalgebra of $M\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right)$ generated by $M(G)$. Of course, $M^{red}(G) \supset \mathcal{K}\binom{\mathcal{A}_G}{G}$ sits as an ideal in $M\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right)$ and $Q^{red}(G) := M^{red}(G)/\mathcal{K}\binom{\mathcal{A}_G}{G} \subset Q(\mathcal{K}\binom{\mathcal{A}_G}{G})$ sits as an ideal in $Q(\mathcal{K}\binom{\mathcal{A}_G}{G})$ and coincides with the C^*–subalgebra generated by $Q(G)$. Obviously, $M^{red}(G)$ is the reduced linking algebra of $M(G)$ and $Q^{red}(G)$ is the reduced linking algebra of $Q(G)$. Clearly, a (blockwise) homomorphism $\mathcal{T}\colon \mathcal{K}\binom{C_F}{F} \to Q^{red}(G)$ maybe considered a (blockwise) homomorphism $\mathcal{T}\colon \mathcal{K}\binom{C_F}{F} \to Q\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right)$.

Summarizing: We have a one-to-one correspondence between between

(i) ternary homomorphisms $T\colon F \to Q(G)$ and

(ii) blockwise homomorphisms $\mathcal{T} = \begin{pmatrix} \tau & T^* \\ T & \vartheta \end{pmatrix}\colon \mathcal{K}\binom{C_F}{F} \to Q^{red}(G) \subset Q\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right)$

such that the 21–corner of $\mathcal{T}$ is T. If $\mathcal{T}$, considered as map into $Q\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right)$, is the Busby invariant of the blockwise extension as in Theorem 2.3, then we say T is the ***Busby invariant*** of the corresponding extension of Hilbert modules. Noteworthy are the following observations:

- The *multiplier space* $M(G)$ of G shares (expressed in terms of essential ternary ideals) the same universal property as the multiplier algebra of a C^*–algebra. Therefore, the homomorphism $T\colon F \to E/vG \to Q(G)$ maybe defined directly from an extension $(***)$ without passing through the blockwise extension, and the result, obviously, coincides.

- Every homomorphism $C \to Q(\mathcal{A})$ is the Busby invariant of an extension $\mathcal{B}$ of C by $\mathcal{A}$ and the Busby invariant determines the extension up to suitable equivalence. Adding the one-to-one correspondence between extension of Hilbert modules and blockwise extensions of the reduced linking algebras, we get the same statement for the Busby invariants for extensions of Hilbert modules.

We do not deepen this further.

Observation. We briefly address the the situation when the Busby invariant T of an extension of Hilbert modules $(***)$ is 0, and use this to be more explicit about how the *canonical homomorphism* from E into the closed ternary ideal $M(G)$ looks like. By $(***)$ being a short exact sequence, it follows that vG sits as a closed ternary ideal in E. So let us assume for the time being, that G actually is a closed ternary ideal in E and that v is the canonical identification of $G \subset E$. Recall that then $\mathcal{A}_G$ is a closed ideal in $\mathcal{B}_E$, and $G = \overline{\mathsf{span}}\, E\mathcal{A}_G$. The multiplier space of G is $M(G) = \mathcal{B}^a(\mathcal{A}_G, G)$ and every element x in E gives rise to a map $\mathcal{B}^a(\mathcal{A}_G, G) \ni x\colon a \to xa$. The Busby invariant is is 0 if and only the map $E \to \mathcal{B}^a(\mathcal{A}_G, G)$ is into $G \subset \mathcal{B}^a(\mathcal{A}_G, G)$. This means for each $x \in E$ there is a unique (since G is full) $g_x \in G$ such that $xa = g_x a$ for all $a \in \mathcal{A}_G$. By setting $px := g_x$, we define a map $p\colon E \to G$. Since $\langle E, G\rangle \subset \mathcal{A}_G$, by choosing an approximate unit $(u_\lambda)_{\lambda\in\Lambda}$ for $\mathcal{A}_G$, we get

$$\langle x, px'\rangle \;=\; \lim_\lambda \langle xu_\lambda, px'\rangle \;=\; \lim_\lambda \langle p_x u_\lambda, px'\rangle \;=\; \langle px, px'\rangle.$$

It is an intriguing exercise to show that this implies that p is adjointable (hence, linear) and fulfills $p^*p = p$; in other words, p is a projection. It follows that G is a direct summand in E. From $xa \in G$ for all $a \in \mathcal{A}_G$and $G^\perp$ being a submodule, it follows that $ya = 0$ for all $y \in G^\perp$. ($G \cap G^\perp = \{0\}$.) Therefore, $yg^* = 0 = gy^*$ for all $g \in G$, $y \in G^\perp$. We get

$$(g' + y')\langle y, g'' + y''\rangle \;=\; g'\langle y, g'' + y''\rangle + y'\langle y, g''\rangle + y'\langle y, y''\rangle \;=\; 0 + 0 + y'\langle y, y''\rangle \;\in\; G^\perp,$$

since $G^\perp$, being a submodule, is also a ternary subspace. So, also $G^\perp$ is a ternary ideal. From $yg^* = 0 = gy^*$ it also follows that $\mathcal{B}_E \supset \mathcal{A}_G\mathcal{A}_{G^\perp} = \{0\}$ and since E is full in $\mathcal{B}_E$, we have $\mathcal{B}_E = \mathcal{A}_G \oplus \mathcal{A}_{G^\perp}$. Therefore, the direct sum $E = G \oplus G^\perp$ is, actually, external. Returning to the general situation, we have the closed ternary ideal vG of E, where G is ternary isomorphic to vG via v. It follows that $\mathcal{A}_G \cong \mathcal{B}_{vG}$ via $\psi\colon \langle g, g'\rangle \mapsto \langle vg, vg'\rangle$. Composing $x \in \mathcal{B}^a(\mathcal{A}_G, G)$ to $vx\psi^{-1} \in \mathcal{B}^a(\mathcal{B}_{vG}, vG)$ establishes a ternary (and strict!) isomorphism $M(G) \to M(vG)$ which, of course, survives the quotient to $Q(G) \cong Q(vG)$.

So, summarizing, the Busby invariant of an extension E of F via G is 0 if and only if E is isomorphic to the external direct sum $F \oplus G$ and the ternary homomorphism u and v correspond, under this isomorphism, to the canonical maps making $F \oplus G$ an extension. So, the trivial Busby invariant corresponds exactly to the super-trivial extension. As a corollary we recover the statement, well-known for extensions of C^*–algebras, that extensions by a Hilbert module G with trivial corona space $Q(G) = \{0\}$ (that is, $M(G) = G$) are super-trivial.

2.7 "Adding" extensions. Homomorphisms can be added in several ways involving direct sums. For adding the Busby invariants $\tau_i : C \to Q(\mathcal{A})$ $(i = 1, 2)$ of two extensions of C by $\mathcal{A}$ (both fixed), one takes the direct sum $\tau_1 \oplus \tau_2 : C \to Q(\mathcal{A}) \oplus Q(\mathcal{A}) = Q(\mathcal{A} \oplus \mathcal{A})$ that sends c to $\tau_1(c) \oplus \tau_2(c)$. Without any further assumptions on $\mathcal{A}$, this is the Busby invariant of an extension of C by $\mathcal{A} \oplus \mathcal{A}$. One might think about the case when $\mathcal{A} \oplus \mathcal{A}$ is isomorphic to $\mathcal{A}$. But this would, for instance, exclude simple C^*–algebras. But, $\mathcal{A} \oplus \mathcal{A}$ is the diagonal subalgebra of $M_2(\mathcal{A})$, and one may view $\tau_1 \oplus \tau_2$ as homomorphism from C into $M_2(\mathcal{A})$. (At least, $M_2(\mathcal{A})$ is simple, if $\mathcal{A}$ is; also, the centers of $M_2(\mathcal{A})$ and $\mathcal{A}$ coincide.) It is not so rare (for instance, if $\mathcal{A}$ is *stable*) that $\mathcal{A} \cong M_2(\mathcal{A})$. So, at least in that case, we may add Busby invariants up to automorphisms of $M_2(\mathcal{A}) \cong \mathcal{A}$, and passing to the corresponding (equivalence class of) extension(s), we have a "sum" operation among extensions of C by $\mathcal{A}$ (actually, a monoid). The restriction $M_2(\mathcal{A}) \cong \mathcal{A}$ is not as harsh as it might sound. Anyway, we know that the extensions are super-trivial, if $\mathcal{A}$ is unital. And if $\mathcal{A}$ is nonunital, even if $\mathcal{A}$ should not be isomorphic to $M_2(\mathcal{A})$, we may pass to the ***stabilization*** $\mathcal{K} \otimes \mathcal{A}$ (where $\mathcal{K} := \mathcal{K}(\mathbb{K})$ for an infinite-dimensional separable Hilbert space $\mathbb{K}$), which preserves many of the properties of $\mathcal{A}$ (properties up to *Morita equivalence*, to be precise).

We can do the same for extensions of F by G; and, again, we do it via the corresponding blockwise extensions of the linking algebras by Theorem 2.3. In order to proceed that way, we have to hypothesize that $\mathcal{K}\binom{\mathcal{A}_G}{G} \cong M_2\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right)$. And in order to speak about blockwise extensions, we have to interpret the latter as reduced linking algebra via $M_2\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right) = \mathcal{K}\binom{M_2(\mathcal{A}_G)}{M_2(G)}$. Here, $M_2(G)$ is a Hilbert $M_2(\mathcal{A})$–module in an obvious fashion, full over $M_2(\mathcal{A}_G)$, so that $\mathcal{K}\binom{M_2(\mathcal{A}_G)}{M_2(G)}$ is, indeed, its reduced linking algebra. (It is important to note that the blockwise structure of the matrix $M_2\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right)$ has absolutely **nothing** to do with the blockwise structure of the reduced linking algebra $\mathcal{K}\binom{M_2(\mathcal{A}_G)}{M_2(G)}$! The former is a 2×2–matrix of reduced linking algebras; the latter is a reduced linking algebra whose blocks consist of 2×2–matrices.)

If we have $\mathcal{K}\binom{\mathcal{A}_G}{G} \cong M_2\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right) = \mathcal{K}\binom{M_2(\mathcal{A}_G)}{M_2(G)}$, then this isomorphism turns over to the multiplier algebras $\mathcal{B}^a\binom{\mathcal{A}_G}{G} = M\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right) \cong M\left(M_2\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right)\right) = M\left(\mathcal{K}\binom{M_2(\mathcal{A}_G)}{M_2(G)}\right) = \mathcal{B}^a\binom{M_2(\mathcal{A}_G)}{M_2(G)}$ and the corona algebras $Q\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right) \cong Q\left(\mathcal{K}\binom{M_2(Q(\mathcal{A}_G))}{M_2(Q(G))}\right)$. The direct sum $Q\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right) \oplus Q\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right) =: \begin{pmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{pmatrix} \oplus \begin{pmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{pmatrix}$ sits in $Q\left(\mathcal{K}\binom{M_2(Q(\mathcal{A}_G))}{M_2(Q(G))}\right)$ as

$$\begin{pmatrix} \begin{pmatrix} Q_{1,1} & \\ & \end{pmatrix} & \begin{pmatrix} Q_{1,2} & \\ & \end{pmatrix} \\ \begin{pmatrix} Q_{2,1} & \\ & \end{pmatrix} & \begin{pmatrix} Q_{2,2} & \\ & \end{pmatrix} \end{pmatrix} \oplus \begin{pmatrix} \begin{pmatrix} & \\ & Q_{1,1} \end{pmatrix} & \begin{pmatrix} & \\ & Q_{1,2} \end{pmatrix} \\ \begin{pmatrix} & \\ & Q_{2,1} \end{pmatrix} & \begin{pmatrix} & \\ & Q_{2,2} \end{pmatrix} \end{pmatrix}.$$

(Likewise for $\mathcal{K}\binom{\mathcal{A}_G}{G}$ and $\mathcal{B}^a\binom{\mathcal{A}_G}{G}$, but for the Busby invariant we need this only for Q.) So it is clear, how we have to add two blockwise Busby invariants $\mathcal{T}_i = \begin{pmatrix} \tau_i & T_i^* \\ T_i & \vartheta_i \end{pmatrix}$ to obtain their "sum"

$$\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2 : \mathcal{K}\binom{C_F}{F} \longrightarrow Q\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right) \oplus Q\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right) \subset Q\left(\mathcal{K}\binom{M_2(Q(\mathcal{A}_G))}{M_2(Q(G))}\right) \cong Q\left(\mathcal{K}\binom{\mathcal{A}_G}{G}\right).$$

The 21–corner of $\mathfrak{T}$ is a map $T\colon F \to Q(G) \oplus Q(G) \subset M_2(Q(G)) \cong Q(G)$, where the last isomorphism is the ternary isomorphism obtained as the 21–corner of the blockwise isomorphism $Q\left(\mathcal{K}\left(\begin{smallmatrix}\mathcal{A}_G\\ G\end{smallmatrix}\right)\right) \cong Q\left(\mathcal{K}\left(\begin{smallmatrix}M_2(Q(\mathcal{A}_G))\\ M_2(Q(G))\end{smallmatrix}\right)\right)$. This map T is the ***sum*** (up to suitable equivalence) of the Busby invariants T_1 and T_2.

The sum of the Busby invariants may be defined directly by the preceding line, as soon as we have a ternary isomorphism $Q(G) \cong M_2(Q(G))$. However, only the derivation via the corresponding block-wise extensions makes such a definition not appear to be *ad hoc* and, more importantly, makes available in a blink of an eye all the related structures (like *Ext*–semigroups) and the results on them, well known for extensions of C^*–algebras. We briefly address the question, when such an isomorphism $Q(G) \cong M_2(Q(G))$ exists.

Necessarily, $\mathcal{A}_G \cong M_2(\mathcal{A}_G)$ (and also $\mathcal{K}(G) \cong M_2(\mathcal{K}(G))$), because by Theorem 1.6 a ternary isomorphism lifts to a blockwise isomorphism between the reduced linking algebras. However, also an opposite statement is true when we start with $\mathcal{A}_G \cong M_2(\mathcal{A}_G)$ (or $\mathcal{K}(G) \cong M_2(\mathcal{K}(G))$). Indeed, if we fix an isomorphism $\varphi\colon \mathcal{A}_G \to M_2(\mathcal{A}_G)$, then $ga \mapsto g\odot a \mapsto g\odot\varphi(a) \mapsto \left(\begin{smallmatrix}g\odot\varphi(a)_{1,1} & g\odot\varphi(a)_{1,2}\\ g\odot\varphi(a)_{2,1} & g\odot\varphi(a)_{2,2}\end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix}g\varphi(a)_{1,1} & g\varphi(a)_{1,2}\\ g\varphi(a)_{2,1} & g\varphi(a)_{2,2}\end{smallmatrix}\right)$ is a ternary isomorphism $G = G \odot \mathcal{A} \cong G \odot M_2(\mathcal{A}) \cong M_2(G \odot \mathcal{A}) = M_2(G)$. (The first and last equality refer to the canonical (unitary) identification of a Hilbert $\mathcal{B}$–module E with $E\odot G$, and also the isomorphism $E\odot M_2(\mathcal{B}) \cong M_2(E\odot\mathcal{B})$ is unitary. (Exercise: Find the correct action of $\mathcal{B}$ on $M_2(\mathcal{B})$ that makes tensor product $E \odot M_2(\mathcal{B})$ over $\mathcal{B}$ isomorphic to $M_2(E \odot \mathcal{B})$.) Only the step from $G \odot \mathcal{A}$ to $G \odot M_2(\mathcal{A})$ is a ternary isomorphism that goes beyond unitaries; see, for instance, [Ske06] for more details.) Note that we may replace $\mathcal{A}$ with $\mathcal{A}_G$. (The statement starting from an isomorphism $\mathcal{K}(G) \cong M_2(\mathcal{K}(G))$ follows by symmetry of the reduced linking algebra under the exchange $G \leftrightarrow G^*$.) We just proved the following result:

Proposition. $\mathcal{A}_G \cong M_2(\mathcal{A}_G) \iff G \cong M_2(G) \iff \mathcal{K}(G) \cong M_2(\mathcal{K}(G))$.

So addition of extensions of F_C by $G_{\mathcal{A}}$ makes sense under exactly the same conditions (on $\mathcal{A}_G$, only) under which addition of extensions of C_F by $\mathcal{A}_G$ makes sense.

• • •

Here, we stop with our brief on extensions of Hilbert modules. We also do not make any attempt to compare in detail the sketchy insights of this section with the work of Bakic and Guljas [BG02, BG04] and the forthcoming papers (also by others) for what was there and what, possibly, not; surely their work is much more complete. Our scope is to illustrate that a change of point of view can be useful: Their point of view is to incorporate in the definitions the algebra of over which the Hilbert modules are modules and only then passing sometimes to linking algebras. We gave two alternative sets of definitions; one intrinsic referring to the ternary product, not making any reference to the algebra, that is, to the 11–corner of the linking algebra; and another making reference to the linking algebras (reduced, if the modules are not full), and blockwise maps between them. In each step, we put emphasis on that the intrinsic definition is equivalent to the corresponding notion for linking algebras. However, regarding the linking algebras, all structures are well-known from C^*–algebra theory; in fact, we just restrict the occurring maps to blockwise maps, and apply theorems about C^*–algebras just by taking care that and how the blockwise structure is respected. Not only allows this to motivate well the intrinsic definitions for modules, as being equivalent to the restriction of the blockwise definitions to the 21–corner, leading in this way practically automatically to new notions and theorems about modules without any effort. But the intrinsic definitions are also easier to check; in fact, one need not worry anymore about objects and maps regarding the algebra over which the modules are modules.

We close with three (sets of) questions, that follow this philosophy and to which we do not figure out the answer in this note.

3 Questions

3.1 We mentioned that a split extension $\mathcal{B}$ is, roughly, a conditional expectation $\mathbb{E}$ $(= s \circ \varphi)$ onto a subalgebra $(s(C))$ such that $\ker \mathbb{E}$ is an ideal. (See below, for conditional expectation.) Forgetting about the condition that $\ker \mathbb{E}$ is an ideal (that arises from the special situation in split extension, where $\mathbb{E}$ has to be homomorphism), it is natural to ask what is a conditional expectation from a Hilbert module onto a closed ternary subspace. (Recall from the remark in Section 0 the way in which this is weaker a condition than requiring a closed submodule; submodules are not what corresponds to subalgebras but to right ideals; ternary subspaces are what corresponds to subalgebras.) So, accepting that a *conditional expectation* of a Hilbert module onto a closed ternary subspace should be equivalently defined as (co)restriction of a blockwise conditional expectation between the reduced linking algebras, what are its intrinsic properties (as maps between modules) that guarantee to make this happen?

Recall that a ***conditional expectation*** $\mathbb{E}$ of C^*–algebras is an idempotent from a C^*–algebra $\mathcal{A}$ **onto** a C^*–subalgebra $\mathcal{B}$ fulfilling one the following two equivalent conditions:

(i) $\mathbb{E}$ is a contraction. (More precisely, $\|\mathbb{E}\| = 1$ unless $\mathcal{B} = \{0\}$.)

(ii) $\mathbb{E}$ is a positive $\mathcal{B}$–bimodule map.

So, assuming we have a blockwise conditional expectation $\mathbb{E}\colon \mathcal{K}\binom{\mathcal{B}_E}{E} \to \mathcal{K}\binom{C_F}{F}$, the (co)restrictions $\mathbb{E}_{i,j}$ to each ij–corner are clearly idempotent contractions. So, the (co)restrictions $\mathbb{E}_{i,i}$ to the diagonal entries are conditional expectations in their own right. A whole bunch of algebraic conditions follows, when we write down what the condition to be a bimodule map means for the restriction to products from different corners. The two questions we ask here are: For an idempotent from E onto a closed ternary subspace F, is it enough to check only the "ternary" condition $\mathbb{E}(\mathbb{E}(x)\langle y, \mathbb{E}(z)\rangle) = \mathbb{E}(x)\langle \mathbb{E}(y), \mathbb{E}(z)\rangle$ to guarantee an extension to a blockwise conditional expectation? (For instance, we are not sure if this allows to conclude that this map is a contraction, before actually having the blockwise extension.) Is it possibly even enough, to require that the idempotent is a contraction?

(We should keep in mind that it might be a good idea to pass to von Neumann algebras and von Neumann modules. Also in proving equivalence of (i) and (ii), passing to the biduals is an essential step in either direction. Let us also mention that Hahn-Banach type extension of off-diagonal (complete) contractions to (completely) positive contractions are not (directly) applicable; such extensions are limited to *injective* codomains. We also should keep in mind that a conditional expectation is a CP-map; as such it has a GNS-construction – a GNS-construction with special properties. Being a blockwise CP-map between linking algebras, relates the present question to the following.)

3.2 ***Semisplit extensions*** are like split extensions, just that the *splitting* s is allowed to be CP-map instead of a homomorphism. This leads directly to the question, what is a CP-map between Hilbert modules. In Skeide and Sumesh [SS14] we proposed – surprise! – that a map between (full) Hilbert modules is ***CP*** if it extends to a blockwise CP-map between the linking algebras, to be precise between the *extended* linking algebras that is *strict* on its 22–corner. (The ***extended linking algebra*** of a Hilbert module E is $\begin{pmatrix}\mathcal{B} & E^* \\ E & \mathcal{B}^a(E)\end{pmatrix} \subset \mathcal{B}^a\begin{pmatrix}\mathcal{B} \\ E\end{pmatrix}$. It is unavoidable for the codomain, because, as Stinespring type constructions show, the 22–corner involves amplification $\mathcal{K}(E)\odot\mathsf{id}$, which, in general, are no longer compact operators. Doing this consequently (and required once we wish to compose CP-maps), also in the the domain we replace $\mathcal{K}(E)$ with $\mathcal{B}^a(E)$; and being *strict* is just the natural (and indispensable) compatibility condition (which, under certain nondegeneracy conditions, also is fulfilled automatically).) There are authors who propose Asadi's τ–maps as CP-maps between Hilbert modules. (τ–maps have been proposed in Asadi [Asa09]. Bhat, Ramesh, and Sumesh [BRS12] (correctly) proved a Stinespring type theorem suggested by Asadi, also removing a *trivializing* condition from the hypotheses. Skeide [Ske12] presented a half-a-page proof using module language. A discussion of τ–maps which is quite exhaustive in many senses followed in [SS14].)

We do not think that the restriction for CP-maps between modules to be also τ–maps is justified. Following our philosophy that a semisplit extension of (full, for safety) Hilbert modules corresponds/has to do with blockwise semisplit extensions of the (extended) linking algebras, our definition of CP-map promises to be the better suiting. (Why should we restrict the blockwise CP-maps making a blockwise extension semisplit to a subset of the blockwise CP-maps?) Independently, one may examine also the subclass of semisplit extensions where the splitting is required to be a τ–map, and analyze which special properties they share.

3.3 Ideals in a C^*–algebra $\mathcal{B}$ are linear subspaces I such that $\mathcal{B}I\mathcal{B} \subset I$. ***Hereditary subalgebras*** can be equivalently characterized as C^*–subalgebras C such that $C\mathcal{B}C \subset C$. A pragmatic (that is, blockwise) definition for a closed ternary subspace F of a Hilbert $\mathcal{B}$–module E to be ***linking hereditary*** would be the requirement that the reduced linking algebra of F is hereditary in the reduced linking algebra of E. A candidate for an intrinsic definition would be to say F is ***ternary hereditary*** in E if $F\langle E, F\rangle \subset F$. Clearly, the former implies the latter. But, the former also implies that C_F is hereditary in $\mathcal{B}_E$ and that $\mathcal{K}(F)$ is hereditary in $\mathcal{K}(E)$. However, we presently do not see a reason, why a hereditary ternary subspace would imply that $\mathcal{B}_F$ and $\mathcal{K}(F)$ are hereditary, too. On the other hand, if $\mathcal{B}_F$ or $\mathcal{K}(F)$ is hereditary, then

$$F\langle E,F\rangle \subset \overline{\operatorname{span}}\,\mathcal{K}(F)F\langle E,F\rangle\mathcal{B}_F = \overline{\operatorname{span}}\,F\langle F,F\rangle\langle E,F\rangle\langle F,F\rangle = \overline{\operatorname{span}}(FF^*)(FE^*)(FF^*)F \subset F,$$

because $\langle F,F\rangle\langle E,F\rangle\langle F,F\rangle \subset \mathcal{B}_F$ or because $(FF^*)(FE^*)(FF^*) \subset \mathcal{K}(F)$. Again we do not know if one of $\mathcal{B}_F$ and $\mathcal{K}(F)$ being hereditary would imply the other one being hereditary, too. But at least we get: A closed ternary subspace F is linking hereditary if and only of both $\mathcal{B}_F$ and $\mathcal{K}(F)$ are hereditary.

We leave open the question whether or not ternary hereditary implies linking hereditary. (Of course, if not, then we think the "right" definition of heriditary subspaces of TRO is linking hereditary.) But there is another much more interesting (and probably more far reaching) question, namely: The usual definition of heriditary subalgebra is that the positive elements of $\mathcal{C}$ form a ***hereditary subcone*** of the positive elements of $\mathcal{B}$, that is, $0 \le b \le c$ for $b \in \mathcal{B}$ and $c \in \mathcal{C}$ implies $b \in \mathcal{C}$. (We preferred to start with the equivalent condition $\mathcal{C}\mathcal{B}\mathcal{C} \subset \mathcal{C}$, because this condition immediately suggests analogue properties for Hilbert modules.) Is there a similar possibility to describe exactly the same (linking) hereditary subspaces of a Hilbert module in terms of positive cones?

There are papers about positivity in Hilbert modules. (See, for instance, Blecher and Neal [BN07] and the references therein.) But, we should note that positivity in a TRO E is referring to positivity in $E \cap E^*$ in the C^*–algebra in which the TRO sits. If this C^*–algebra is the linking algebra, then this intersection is $\{0\}$. So, not only is this positivity not compatible with our point of view to take intuition from the linking algebras, but is also depends on the choice of an embedding of E as a TRO into a C^*–algebra. What we ask for is a structure that is intrinsic to the (full) Hilbert module E, a structure that is present for every Hilbert module (and tells enough about it to determine most of it, to the same extent as the positive cone of a C^*–algebra tells about the structure of the C^*–algebra), and a structure that allows to recover what we defined to be a linking hereditary subspace.

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