Tensor Product Systems of Hilbert Modules and Dilations of Completely Positive Semigroups *

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Abstract

In these notes we study the problem of dilating unital completely positive (CP) semigroups (quantum dynamical semigroups) to weak Markov flows and then to semigroups of endomorphisms (E_0–semigroups) using the language of Hilbert modules. This is a very effective, representation free approach to dilation. This way we are able to identify the right algebra (maximal in some sense) for endomorphisms to act. We are lead inevitably to the notion of tensor product systems of Hilbert modules and units for them, generalizing Arveson's notions for Hilbert spaces.

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In the course of our investigations we are not only able to give new natural and transparent proofs of well-known facts for semigroups on $\mathcal{B}(H)$. The results extend immediately to much more general set-ups. For instance, Arveson classifies E_0–semigroups on $\mathcal{B}(H)$ up to cocycle conjugacy by product systems of Hilbert spaces [Arv89]. We find that conservative CP-semigroups on arbitrary unital C^*–algebras are classified up to cocycle conjugacy by product systems of Hilbert modules. Looking at other generalizations, it turns out that the role played by E_0–semigroups on $\mathcal{B}(H)$ in dilation theory for CP-semigroups on $\mathcal{B}(G)$ is now played by E_0–semigroups on $\mathcal{B}^a(E)$, the full algebra of adjointable operators on a Hilbert module E. We have CP-semigroup versions of many results proved by Paschke [Pas73] for CP maps.

Contents

1 Introduction

The basic theorem in dilation theory for completely positive mappings or semigroups of completely positive mappings on a unital C^*–algebra $\mathcal{B}$ (CP-semigroups, quantum dynamical semigroups) is the Stinespring construction; see Example 2.16. The Stinespring construction is, however, based on the fact that $\mathcal{B} \subset \mathfrak{B}(G)$ is represented as an algebra of operators on a Hilbert space G, usually refered to as the *initial space*. This makes it, in general, impossible to recover the ingredients of the Stinespring construction for a composition $S \circ T$ of completely positive mappings in terms of the Stinespring constructions for the single mappings T and S.

On the contrary, making use of Hilbert modules it is very easy to express the GNS-construction of $S \circ T$ in terms of the GNS-constructions for the mappings T and S. The result of the GNS-constructions for T and S are Hilbert $\mathcal{B}$–$\mathcal{B}$–modules E and F with cyclic vectors ξ and ζ, respectively, such that

$$T(b) = \langle \xi, b\xi \rangle \quad \text{and} \quad S(b) = \langle \zeta, b\zeta \rangle;$$

see Example 2.14. The composition of T and S can be found with the help of the tensor product $E \odot F$. We find

$$S \circ T(b) = \langle \xi \odot \zeta, b\xi \odot \zeta \rangle$$

so that the the GNS-module of $S \circ T$ may be identified as the $\mathcal{B}$–$\mathcal{B}$–submodule of $E \odot F$ which is generated by the cyclic vector $\xi \odot \zeta$. In Example 2.16 we point out that this possibility is due to a functorial behaviour of two-sided Hilbert modules. A Hilbert $\mathcal{A}$–$\mathcal{B}$–module may be considered as a functor sending representations of $\mathcal{B}$ to representations of $\mathcal{A}$ and the composition of two such functors is just the tensor product.

In usual dilation theorems for CP-semigroups $T = \bigl(T_t\bigr)$, inner products are written down in form of *correlation kernels* and the representation space is realized by a *Kolmogorov decomposition*. In contrast to that, we are able to construct the representation space, starting from the GNS-modules of each T_t by an inductive limit over *insertion of time points*. These insertions are realized, roughly speaking, by continued splitting of elements belonging to the GNS-module *at time* t into tensors belonging to GNS-modules at smaller times.

These notes are organized as follows. Section 2 is devoted to introduce the basic notations. We explain the essence of what we need later on for semigroups in simple examples whithout being disturbed by lots of indices. Because we intend to show that most in these notes works purely algebraically, we need well-known notions in a version for pre-Hilbert modules. This makes Section 2 rather long. As an advantage most of these notes is almost self-contained. Only basic knowledge in C^*–algebra theory (and Cauchy-Schwarz inequality for semi-Hilbert modules) is required.

In Section 3 we define what we understand by a weak Markov flow and a dilation to an e_0–semigroup (i.e. to a semigroup of not necessarily unital endomorphisms) in terms of operators on a (pre-)Hilbert module E. If T is a conservative completely positive semigroup on a unital C^*–algebra $\mathcal{B}$, then a weak Markov flow is a family j of (usually non-unital) homomorphisms j_t from $\mathcal{B}$ into another (pre-)C^*–algebra $\mathcal{A} \subset \mathfrak{B}^a(E)$ fulfilling $j_s(\mathbf{1})j_t(b)j_s(\mathbf{1}) = j_s(T_{t-s}(b))$ $(b \in \mathcal{B}, s \leq t)$. A dilation is an e_0–semigroup ϑ on $\mathcal{A}$ fulfilling

$\vartheta_t \circ j_s = j_{t+s}$. These definitions parallel completely those given in [BP94, BP95, Bha96] in terms of Hilbert spaces. In Section 6 we will see that the definitions fit perfectly into the algebraic set-up of [Acc78].

Sections 4 and 5 may be considered as the heart of these notes. In Section 4 we construct the representation module E_t until time t. We obtain E_t as an inductive limit over all possibilities for splitting the interval $[0,t]$ into smaller intervals $[0,t_i]$ whose lengths t_i sum up to t, by *inserting* the algebra $\mathcal{B}$ in between the intervals; see the crucial Observation 4.2. We find the factorization

$$E_s \odot E_t = E_{s+t}.$$

In other words, we are lead to the notion of *tensor product systems* of two-sided (pre-) Hilbert modules. The cyclic vectors ξ_t of the GNS-constructions for the T_t survive the inductive limit. The corresponding elements $\xi^t \in E_t$ form a *unit*, i.e.

$$\xi^s \odot \xi^t = \xi^{s+t}.$$

Both notions parallel the notions for Hilbert spaces introduced by Arveson [Arv89].

E_t contains E_s $(t \geq s)$ in a natural way. This allows to construct a second inductive limit E. The embedding $E_s \to E_t$ is, however, only right linear, not bilinear. Consequently, on E there does not exist a unique left multiplication by elements of $\mathcal{B}$. There exists, however, a natural projection onto the range of the canonical embedding $E_t \to E$. In other words, the left multiplication on E_t gives rise to a representation j_t of $\mathcal{B}$ on E. The collection of all j_t turns out to be a weak Markov flow. We remark that existence of projections onto (closed) submodules is a rare thing to happen in the context of (pre-)Hilbert modules.

Also the factorization $E_s \odot E_t = E_{s+t}$ carries over to the second inductive limit. We find

$$E \odot E_t = E.$$

We may define the semigroup $\vartheta_t(a) = a \odot \mathsf{id} \in \mathcal{B}^a(E \odot E_t) = \mathcal{B}^a(E)$ of endomorphisms of $\mathcal{B}^a(E)$. In this way we do not only recover the e_0–semigroup constructed in [Bha96] which arises just by restricting ϑ to the algebra $\mathcal{A}_\infty$ generated by all $j_t(b)$. We also show how it may be extended to an E_0–semigroup. The approach in [Bha96] is based on the Stinespring construction, so that $\mathcal{A}_\infty$ is identified as a subalgebra of some $\mathcal{B}(H)$; see Example 2.16. In this identification $\mathcal{B}^a(E)$ lies somewhere in between $\mathcal{A}_\infty$ and $\mathcal{B}(H)$. Only the approach by Hilbert modules made it possible to identify the correct subalgebra $\mathcal{B}^a(E)$ of $\mathcal{B}(H)$ to which the e_0–semigroup from [Bha96] extends as an E_0–semigroup. This also shows that we may expect that in the classification of CP-semigroups on general C^*–algebras E_0–semigroups on $\mathcal{B}^a(E)$ play the role which is played by E_0–semigroups on $\mathcal{B}(H)$ in the classification of CP-semigroups on $\mathcal{B}(G)$, when G, H are Hilbert spaces.

We remark that the construction of the weak Markov flow is also possible in the nonstationary case (i.e. we are concerned rather with families $\left(T_{t,s}\right)_{t \geq s}$ of transition operators fulfilling $T_{t,r} \circ T_{r,s} = T_{t,s}$ $(t \geq r \geq s)$). Of course, here we do not have a time shift semigroup ϑ. Such a construction was already done for more general indexing sets in [Bel85] in terms of Stinespring construction, however, based on the hypothesis that some kernel be positive definite. The methods in [Bel85] are also restricted to normal mappings on von Neumann algebras.

In Section 6 we analyze the notion of weak Markov flow from the algebraical point of view. We show that existence of certain conditional expectations which, usually, forms a part of the definition (see [Acc78, AFL82]) follows automatically from our definition. It turns out that an *essential* weak Markov flow (i.e. the GNS-representation of the conditional expectation $\varphi(\bullet) = j_0(\mathbf{1}) \bullet j_0(\mathbf{1})$ is faithful) lies in between two universal flows which are determined completely by the CP-semigroup T. Like in [AFL82], the crucial role is played by a *correlation kernel* $\mathscr{T}$ which is, however, $\mathcal{B}$–valued (roughly speaking the moments of the process j in the conditional expectation φ). The second inductive limit E may be considered as both the *Kolmogorov decomposition* for the correlation kernel in the sense of Murphy [Mur97] and as the GNS-module of φ. Doing the the Stinespring construction, we recover the $\mathbb{C}$–valued correlation kernels as used in [AFL82, Bel85, BP94].

In Section 7 we reverse the proceeding and start with a pair consisting of a product system and a unit. We associate with each such pair a CP-semigroup and show that we can recover the pair from the CP-semigroup, if the unit is *generating* in a suitable sense. (This seems to be close to what Arveson calls a type I product system.) We find that CP-semigroups are classified by pairs of product systems and generating units. Like Arveson's classification of E_0–semigroups on $\mathscr{B}(H)$ by product systems of Hilbert spaces up to cocycle conjugacy [Arv89], we find that conservative CP-semigroups are classified by their product system of Hilbert modules up to cocycle conjugacy. The cocycles which appear here are, in general, not unitary, but partially isometric. However, if we restrict our classification to E_0–semigroups, then our cocycles are unitary, too. In Section 13 we show that in the case $\mathcal{B} = \mathscr{B}(G)$ the two classifications coincide. Thus, we obtain a generalization of Arveson's classification to E_0–semigroups on arbitrary unital C^*–algebras $\mathcal{B}$.

Contractive CP-semigroups T on $\mathcal{B}$ may be turned into conservative CP-semigroups $\widetilde{T}$ on $\widetilde{\mathcal{B}} = \mathcal{B} \oplus \mathbb{C}\mathbf{1} \cong \mathcal{B} \oplus \mathbb{C}$. In Section 8 we investigate how the dilation of the original semigroup T sits inside the dilation on the module $\widetilde{E}$ constructed from $\widetilde{T}$. We show that $\widetilde{E}$ is "precisely one vector too big" to be generated by $\widetilde{j}(\mathcal{B})$ alone. Finally, we demonstrate in the simplest possible non-trivial example what the construction really does. In this way we also obtain an explicit non-trivial example for a product system.

In Section 9 we recover in a particularly transparent way the classical Markov process on the center of $\mathcal{B}$ which was discovered in [Bha93]. This Section gives a first hint why, in general, in our construction we may not expect to find unital Markov flows j.

Until Section 9 we stayed at an algebraic level where we did not complete pre-Hilbert modules. In Section 11 we need for the first time *completed versions* of our results. Section 10 provides the necessary remarks. In this context we show our first continuity result. If T is a $c+0$–semigroup, then ϑ is a strictly continuous E_0–semigroup on $\mathscr{B}^a(E)$.

In Section 11 we investigate dilations of CP-semigroups with bounded generators (Christensen-Evans generators [CE79]) with the help of the calculus on the full Fock module developed in [Ske99]. (There is also a dilation on a symmetric Fock module discovered earlier in [GS99] also with the help of a quantum stochastic calculus. A weak Markov flow was also constructed in [PS].) We show that the *time ordered Fock modules* until time t, which are contained in the full Fock modules until time t as submodules, form a product system and that their vacua form a unit. The time shift endomorphism constructed from this unit on the time ordered Fock module (see Section 7) is just the restriction from the natural time shift endomorphism on the full Fock module. We construct a partially isometric cocycle with respect to the time shift which shows that CP-semigroups with bounded generators are *cocycle subconjugate* (in the sense of Definition 7.7) to the trivial semigroup. This shows that in our theory flows constructed on the time ordered Fock module play the role of flows constructed on the symmetric Fock space with an initial space in the theory of E_0–semigroups on $\mathcal{B}(H)$, the so-called *CCR-flows*; see [Bha98a]. This is even more satisfactory as it is well-known that the symmetric Fock space and the time ordered Fock space are canonically isomorphic.

In the last three Sections we study normal CP-semigroups on von Neumann algebras. In Section 12 we explain based on Appendix C and [Ske97] how our constructions extend to strong closures of Hilbert modules, so-called *von Neumann modules*. In Theorem 12.1 we obtain the positive answer to the yet open question, whether the e_0–semigroup constructed in [Bha96] is strongly continuous. In Section 13 we study the special case $\mathcal{B} = \mathcal{B}(G)$. The most important result is probably Theorem 13.11 which asserts that any von Neumann $\mathcal{B}(G)$–$\mathcal{B}(G)$–module is *centered*. Among the two-sided Hilbert modules the centered modules introduced in [Ske98] form a particularly well behaved subclass. As (topological) modules they are generated by the subspace of those elements which *commute* with $\mathcal{B}$. The results in Section 13 explain to some extent why so much can be said in the case $\mathcal{B}(G)$, whereas the same methods fail for more general algebras $\mathcal{B}$.

In Section 14 we generalize a result on the order structure of the set of normal CP-semigroups on $\mathcal{B}(G)$ dominated by a fixed normal E_0–semigroup, obtained in [Bha98a], to the case of normal CP-semigroups on arbitrary von Neumann algebras dominated by a fixed conservative normal CP-semigroup (not necessarily an E_0–semigroup). The result from [Bha98a] plays a crucial role in deciding, whether a given dilation is minimal, or not. We hope that we will be able to generalize also these methods from $\mathcal{B}(G)$ to arbitrary von Neumann algebras (or, more generally, multiplier algebras).

In Appendix A we provide the necessary facts about inductive limits of pre-Hilbert modules. We put some emphasis on the difference between one-sided and two-sided modules. This distinction is crucial as it makes the difference between the first inductive limit in Section 4 (which is a limit of two-sided pre-Hilbert modules) and the second inductive limit in Section 5 (which is only one-sided).

Appendix B is the basis for our notion of essential weak Markov flows. A weak Markov flow is essential, if the closed ideal generated by $j_0(\mathbf{1})$ is essential in $\mathcal{B}^a(\overline{E})$. In this case, the closed ideal may be identified with the compact operators on $\overline{E}$ so that $\mathcal{B}^a(\overline{E})$ is just its multiplier algebra. Example B.3 shows that we cannot drop the completions in this definition.

The exposition of basic facts about von Neumann modules is postponed to Appendix C, because we need them only in Sections 12 – 14.

2 Preliminaries and conventions

In this section we collect the preliminary notions and results which are essential for the rest of these notes. Since we intend to keep the level of discussion up to a certain extent algebraical, we give the definitions in a form refering to pre–C^*–algebras rather to C^*–algebras. This causes that some well-known notions will come along in an unusual shape. Therefore, we decided to be very explicit, making this section somewhat lengthy.

The changes to the well-known versions can be summarized in that homomorphisms between C^*–algebras always are contractive, whereas homomorphisms between pre–C^*–algebras need not be contractive. Consequently, whenever the word 'contractive' appears in the context of homomorphisms, this is in order to assure that these homomorphisms may be extended to the C^*–completions of the pre–C^*–algebras under consideration. A pay-off of this strict distinction between algebraic constructions and their topologic extension is that most of the constructions extend directly to more general $*$–algebras.

2.1 Conventions. Mappings between vector spaces, usually, are assumed to be linear. The unit of an algebra $\mathcal{A}$, usually, we denote by $\mathbf{1}$. Only when confusion can arise, we will write $\mathbf{1}_{\mathcal{A}}$. We follow the same convention with the identity mapping id on a space. Mappings between unital $*$–algebras are called *unital*, if they respect the unit. Homomorphisms between unital $*$–algebras are not necessarily assumed to be unital; however, cf. also Definition 2.13. The constructions $\oplus$, $\otimes$, $\odot$, etc. are understood algebraically, unless stated otherwise, explicitly. Completions or closures are indicated by $\bar{\ }$.

Let $\mathcal{A}$ denote a pre–C^*–algebra, no matter whether unital or not. Then its *unitization* is $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}\widetilde{\mathbf{1}}$ equipped with the unique C^*–norm of $\overline{\mathcal{A}} \oplus \mathbb{C}\widetilde{\mathbf{1}}$. (We remark that, if $\mathcal{A}$ is unital, then $\widetilde{\mathcal{A}}$ is isomorphic to $\mathcal{A} \oplus \mathbb{C}$.) If $L\colon \mathcal{A} \to \mathcal{B}$ is a mapping between pre–C^*–algebras, then its *unitization* is defined as the extension of L to a unital mapping $\widetilde{L}\colon \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$. Of course, $\|\widetilde{L}\| \leq 1 + \|L\|$. If both $\mathcal{A}$ and $\mathcal{B}$ already have a unit, then $\|\widetilde{L}\| = \max\bigl(1, \|L\|\bigr)$.

2.2 Completely positive mappings. Let $\mathcal{A}$ and $\mathcal{B}$ denote pre–C^*–algebras. A mapping $T\colon \mathcal{A} \to \mathcal{B}$ is *completely positive*, if

$$\sum_{i,j} b_i^* T(a_i^* a_j) b_j \geq 0$$

for all choices of finitely many $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$. Usually, we will assume that completely positive mappings are *contractive*, i.e. $\|T\| \leq 1$.

2.3 Conditional expectations. A mapping φ from a pre–C^*–algebra $\mathcal{A}$ onto a pre–C^*–subalgebra $\mathcal{B} \subset \mathcal{A}$ is called a *conditional expectation*, if $\widetilde{\varphi}$ is a projection of norm 1. This is equivalent to say that $\widetilde{\varphi}$ is a bounded positive $\widetilde{\mathcal{B}}$–$\widetilde{\mathcal{B}}$–linear mapping; see Takesaki [Tak79]. A conditional expectation φ is called *faithful*, if $\varphi(a^*a) = 0$ implies $a = 0$.

2.4 Semigroups of completely positive mappings. Let $\mathcal{B}$ be a pre–C^*–algebra and $\mathbb{T} = \mathbb{R}^+$ or $\mathbb{T} = \mathbb{N}_0$ an index set. A *completely positive semigroup on* $\mathcal{B}$, or *CP-semigroup* for short, is a semigroup $T = \bigl(T_t\bigr)_{t\in\mathbb{T}}$ of completely positive contractions T_t on $\mathcal{B}$. If $\mathcal{B}$ is unital and all T_t are unital, then we say the CP-semigroup is *conservative*. By the *trivial CP-semigroup* on $\mathcal{B}$ we mean $T_t = \mathsf{id}$.

With few exceptions in this section, and in Sections 8 and 14, we assume that $\mathcal{B}$ is a *unital C^*–algebra* and that CP-semigroups on $\mathcal{B}$ are *conservative*. If $\mathcal{B}$ is supposed to act as an algebra of operators on a Hilbert space then we denote this Hilbert space by G.

2.5 Semigroups of endomorphisms. Let $\mathcal{A}$ denote a pre–C^*–algebra. An *e_0–semigroup on* $\mathcal{A}$ is a semigroup $\vartheta = \bigl(\vartheta_t\bigr)_{t\in\mathbb{T}}$ of contractive endomorphisms of $\mathcal{A}$. If $\mathcal{A}$ is unital and ϑ is unital, we say ϑ is an *E_0–semigroup*. Usually, neither $\mathcal{A}$ nor ϑ need to be unital. We can, however, always pass to the E_0–semigroup $\widetilde{\vartheta} = \bigl(\widetilde{\vartheta}_t\bigr)_{t\in\mathbb{T}}$ on $\widetilde{\mathcal{A}}$.

2.6 Observation. In these notes, usually, $\mathcal{A}$ is a pre–C^*–algebra which is generated by a collection of C^*–subalgebras. Therefore, $\mathcal{A}$ is spanned linearly by its *quasiunitaries* (i.e. elements v fulfilling $v^*v + v + v^* = 0 = vv^* + v^* + v$) and possibly $\mathbf{1}$, if $\mathcal{A}$ is unital, so that all representations of $\mathcal{A}$ map into some set of bounded operators.

2.7 Hilbert modules. See [Pas73, Ske97]. Let $\mathcal{B}$ denote a unital pre–C^*–algebra. A *pre-Hilbert $\mathcal{B}$–module* is a right $\mathcal{B}$–module E with a sesquilinear *inner product* $\langle \bullet, \bullet \rangle \colon E \times E \to \mathcal{B}$, fulfilling $\langle x, x \rangle \geq 0$ $(x \in E)$ (*positivity*), $\langle x, x \rangle = 0$ implies $x = 0$ (*strict* positivity), and $\langle x, yb \rangle = \langle x, y \rangle b$ $(x, y \in E; b \in \mathcal{B})$ (*right linearity*). If strict positivity is missing, then we speak of a *semi-inner product* and a *semi-Hilbert $\mathcal{B}$–module*.

On a semi-Hilbert $\mathcal{B}$–module E we have $\langle x, y \rangle = \langle y, x \rangle^*$, $\langle xb, y \rangle = b^* \langle x, y \rangle$, and *Cauchy-Schwarz inequality*

$$\langle x, y \rangle \langle y, x \rangle \leq \| \langle y, y \rangle \| \, \langle x, x \rangle. \tag{2.1}$$

From Cauchy-Schwarz inequality it follows that $\|x\| = \sqrt{\|\langle x, x \rangle\|}$ defines a semi-norm on E. This semi-norm is a norm, if and only if E is a pre-Hilbert $\mathcal{B}$–module. If a pre-Hilbert $\mathcal{B}$–module E is complete in this norm, then we say E is a *Hilbert $\mathcal{B}$–module*.

Let E be a semi-Hilbert $\mathcal{B}$–module and denote by $\mathcal{N}_E = \{x \in E \colon \langle x, x \rangle = 0\}$ the submodule consisting of length-zero elements. By the pre-Hilbert $\mathcal{B}$–module and Hilbert $\overline{\mathcal{B}}$–module *associated with* E, we mean $E/\mathcal{N}_E$ and $\overline{E/\mathcal{N}_E}$, respectively. Notice that the completion of any pre-Hilbert $\mathcal{B}$–module is a Hilbert $\overline{\mathcal{B}}$–module in a natural fashion.

If $(E_t)_{t \in \mathbb{L}}$ is a family of non-trivial pre-Hilbert $\mathcal{B}$–modules (where $\mathbb{L}$ is some indexing set), then also the direct sum $E = \bigoplus_{t \in \mathbb{L}} E_t$ is a pre-Hilbert $\mathcal{B}$–module in an obvious way. Suppose that all E_t are Hilbert modules. Then E is a Hilbert module, if and only if $\mathbb{L}$ is a finite set.

2.8 Example. Any pre–C^*–algebra $\mathcal{B}$ is a pre-Hilbert $\mathcal{B}$–module with inner product $\langle b, b' \rangle = b^* b'$. It is a Hilbert $\mathcal{B}$–module, if and only if $\mathcal{B}$ is complete.

More generally, a right ideal I in $\mathcal{B}$ is a pre-Hilbert $\mathcal{B}$–module (actually, a pre-Hilbert I–module) in the same way. It can be shown that any pre-Hilbert $\mathcal{B}$–module can be embedded into a certain completion of the direct sum of such ideals; see [Pas73, Ske97].

2.9 Example. Let G and H be Hilbert spaces and let $\mathcal{B} \subset \mathcal{B}(G)$ be a $*$–algebra of bounded operators on G. Then any subspace $E \subset \mathcal{B}(G, H)$, for which $E\mathcal{B} \subset E$ and $E^*E \subset \mathcal{B}$ becomes a pre-Hilbert $\mathcal{B}$–module with inner product $\langle x, y \rangle = x^* y$. Obviously, operator norm and Hilbert module norm coincide, so that E is a Hilbert $\mathcal{B}$–module, if and only if E is a norm closed subset of $\mathcal{B}(G, H)$.

2.10 Operators on Hilbert modules. Let E and F be pre-Hilbert $\mathcal{B}$–modules. By $\mathcal{L}^r(E, F)$ ($\mathcal{B}^r(E, F)$) we denote the sets of (*bounded*) *right module homomorphisms* $E \to F$. A mapping $a \colon E \to F$ is called *adjointable*, if there is an *adjoint* mapping $a^* \colon F \to E$ fulfilling $\langle x, ay \rangle = \langle a^* x, y \rangle$ $(x \in F, y \in E)$. By $\mathcal{L}^a(E, F)$ ($\mathcal{B}^a(E, F)$) we denote the sets of (*bounded*) *adjointable* mappings $E \to F$. We have $\mathcal{L}^a(E, F) \subset \mathcal{L}^r(E, F)$ and $\mathcal{B}^a(E, F) \subset \mathcal{B}^r(E, F)$. If E is complete, then $\mathcal{L}^a(E, F) = \mathcal{B}^a(E, F)$. With one exception in the proof Theorem 10.2, we only speak of right linear mappings.

The sets $\mathcal{L}^a(E) = \mathcal{L}^a(E, E)$ and $\mathcal{B}^a(E) = \mathcal{B}^a(E, E)$ form a $*$–algebra and a pre–C^*–algebra, respectively. Moreover, $\overline{\mathcal{B}^a(E)} = \mathcal{B}^a(\overline{E})$. In particular, if E is complete, then $\mathcal{B}^a(E)$ is a C^*–algebra.

An operator of the form $|x\rangle\langle y|$ $(x, y \in E)$ is called *rank-one operator*. The linear span $\mathcal{F}(E)$ of all rank-one operators is called the pre–C^*–algebra of *finite rank operators*, its completion $\mathcal{K}(E)$ is called the C^*–algebra of *compact operators*. Notice, however, that the elements of $\mathcal{K}(E)$ can be considered as operators on E, in general, only if E is complete. Notice that these operators, in general, are not compact in the usual sense as operators between Banach spaces.

A *projection* on a pre-Hilbert module is a mapping p fulfilling $p^2 = p = p^*$. By definition p is adjointable and, obviously, p is bounded. An *isometry* between pre-Hilbert modules is a mapping ξ which preserves inner products, i.e. $\langle \xi x, \xi y\rangle = \langle x, y\rangle$. A *unitary* is a surjective isometry. Obviously, projections, isometries, and unitaries extend as projections, isometries, and unitaries, respectively, to the completions. Moreover, if an isometry has dense range, then its extension to the completions is a unitary.

isoob **2.11 Observation.** A unitary u is adjointable where the adjoint is $u^* = u^{-1}$. An isometry ξ need not be adjointable (but always right linear). If it is adjointable, then $\xi^*\xi = \mathsf{id}$ and $\xi\xi^*$ is a projection onto the range of ξ. Conversely, if there exists a projection onto the range of ξ, then ξ is adjointable.

semi **2.12 Observation.** If E and F are semi-Hilbert $\mathcal{B}$–modules, and if $a\colon E \to F$ is a mapping which is adjointable in the above sense, then $x + \mathcal{N}_E \mapsto ax + \mathcal{N}_F$ is a well-defined element in $\mathcal{L}^a(E/\mathcal{N}_E, F/\mathcal{N}_F)$.

modrep **2.13 Representations on Hilbert modules.** A *representation* of a pre–C^*–algebra $\mathcal{A}$ on a pre-Hilbert $\mathcal{B}$–module E is a homomorphism $j\colon \mathcal{A} \to \mathcal{L}^a(E)$ of $*$–algebras. In particular, if E is an $\mathcal{A}$–$\mathcal{B}$–module, such that $\langle x, ay\rangle = \langle a^*x, y\rangle$ (i.e. $a \mapsto (x \mapsto ax)$ defines a *canonical homomorphism*), then we say E is a *pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module*. If $\mathcal{A}$ has a unit and we refer to $\mathcal{A}$ as unital, explicitly, then we assume that the unit of $\mathcal{A}$ acts as a unit on E.

Clearly, a homomorphism j extends to a homomorphism $\overline{\mathcal{A}} \to \mathcal{B}^a(\overline{E})$, if and only if it is contractive. We say a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module E is *contractive*, if the canonical homomorphism is contractive. In particular, if $\mathcal{A}$ is a C^*–algebra, then E is contractive, automatically.

GNS **2.14 Example.** Let $\mathcal{A}$ and $\mathcal{B}$ be unital pre–C^*–algebras and let $T\colon \mathcal{A} \to \mathcal{B}$ a completely positive mapping. Then $\mathcal{A} \otimes \mathcal{B}$ with inner product defined by setting

$$\langle a \otimes b, a' \otimes b'\rangle = b^* T(a^* a') b'$$

is a semi-Hilbert $\mathcal{A}$–$\mathcal{B}$–module in a natural way. Setting $E = \mathcal{A} \otimes \mathcal{B}/\mathcal{N}_{\mathcal{A}\otimes\mathcal{B}}$ and $\xi = \mathbf{1} \otimes \mathbf{1} + \mathcal{N}_{\mathcal{A}\otimes\mathcal{B}} \in E$, we have $T(a) = \langle \xi, a\xi\rangle$. Moreover, ξ is *cyclic* in the sense that $E = \mathsf{span}(\mathcal{A}\xi\mathcal{B})$. The pair (E, ξ) is called the *GNS-representation* of T. The pre-Hilbert module E is called *GNS-module*. If T is bounded, then the construction extends to $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$, so that E is contractive and we may consider also $\overline{E}$. Obviously, T is conservative (i.e. $T(\mathbf{1}) = \mathbf{1}$), if and only if $\langle \xi, \xi\rangle = \mathbf{1}$.

If $\mathcal{A}$ or $\mathcal{B}$ are non-unital and T is contractive, then we can do the construction for $\widetilde{T}$ (or, more generally, for $\widetilde{T/\|T\|}$, if T is bounded). However, the statement that also $\widetilde{T}$ is completely positive, actually, is equivalent to construct the GNS-module with a cyclic vector; see the discussion in Ske97p2 [Ske97].

2.15 Tensor product of Hilbert modules. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be pre–C^*–algebras. Let E be a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module and let F be a pre-Hilbert $\mathcal{B}$–$\mathcal{C}$–module. Then the tensor product $E \otimes F$ with inner product defined by setting

$$\langle x \otimes y, x' \otimes y' \rangle = \big\langle y, \langle x, x' \rangle y' \big\rangle$$

is a semi-Hilbert $\mathcal{A}$–$\mathcal{C}$–module in a natural way; see [AS98] for an elementary proof of positivity. The *interior tensor product* of Hilbert modules or shortly *tensor product* is the pre-Hilbert $\mathcal{A}$–$\mathcal{C}$–module $E \odot F = E \otimes F / \mathcal{N}_{E \otimes F}$. By $E \bar{\odot} F$ we denote the completion of $E \odot F$. (There is also an *exterior* tensor product of pre-Hilbert modules; see [Lan95, Ske98].)

2.16 Example. Let G be a pre-Hilbert space and $\mathcal{B} \subset \mathcal{B}(G)$ a $*$–algebra of operators on G. In other words, G is a pre-Hilbert $\mathcal{B}$–$\mathbb{C}$–module. Let E be a pre-Hilbert $\mathcal{B}$–module. Then $H = E \odot G$ is another pre-Hilbert space. Moreover, any element x in E gives rise to a mapping $L_x \colon g \mapsto x \odot g$ in $\mathcal{B}(G, H)$ such that $\langle x, y \rangle = L_x^* L_y$ and $L_{xb} = L_x b$. We see that any pre-Hilbert module may be identified as a submodule of some $\mathcal{B}(G, H)$ as in Example 2.9. For reasons, which we clarify immediately, we refer to this construction as the *Stinespring construction*.

If E is a contractive pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module, then any element a in $\mathcal{A}$ gives rise to an operator $\rho(a) \colon x \odot g \mapsto ax \odot g$ in $\mathcal{B}(H)$; cf. Observation 2.20. Clearly, ρ is a contractive representation of $\mathcal{A}$ on H. If we apply this construction to the GNS-module of a completely positive mapping in Example 2.14, then $T(a) = L_\xi^* \rho(a) L_\xi$. In other words, we obtain the usual Stinespring construction. (Observe that L_ξ is an isometry in $\mathcal{B}(G, H)$, if and only if $\langle \xi, \xi \rangle = \mathbf{1}$, i.e. if T is conservative.)

The same construction works, if we start with an arbitrary contractive representation of a pre–C^*–algebra $\mathcal{B}$ on a pre-Hilbert space G. In other words, a contractive pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module may be considered as a functor which sends contractive representations of $\mathcal{B}$ to contractive representations of $\mathcal{A}$. It is easy to check that the composition of two such functors amounts to construct the tensor product of the underlying pre-Hilbert modules. In this case we also have $L_{x \odot y} = L_x L_y$.

As an interesting application we will draw some consequences for compositions of completely positive mappings. The following observation is essential in understanding the first inductive limit in Section 4. This idea is already present in [Rie74].

2.17 Observation. Let $T \colon \mathcal{A} \to \mathcal{B}$ and $S \colon \mathcal{B} \to \mathcal{C}$ be contractive completely positive mappings with GNS-modules E and F and with cyclic vectors ξ and ζ, respectively. Let G be the GNS-module of the composition $S \circ T$ with cyclic vector χ. Then the mapping

$$\chi \longmapsto \xi \odot \zeta$$

extends (uniquely) as a two-sided isometric homomorphism $G \to E \odot F$. In particular, we have $S \circ T(a) = \langle \xi \odot \zeta, a\xi \odot \zeta \rangle$.

Observe that $E \odot F = \mathsf{span}(\mathcal{A}\xi\mathcal{B} \odot \mathcal{B}\zeta\mathcal{C}) = \mathsf{span}(\mathcal{A}\xi \odot \mathcal{B}\zeta\mathcal{C}) = \mathsf{span}(\mathcal{A}\xi\mathcal{B} \odot \zeta\mathcal{C})$. By the above isometry we may identify G as the submodule $\mathsf{span}(\mathcal{A}\xi \odot \zeta\mathcal{C})$ of $E \odot F$. In other words, inserting a unit $\mathbf{1}$ in $\chi = \xi \odot \zeta$ in between ξ and ζ amounts to an isometry.

Suppose that $\mathcal{B}$ and $\mathcal{C}$ are algebras of operators on some pre-Hilbert spaces. We want to emphasize that, unlike the GNS-construction, the knowledge of the Stinespring construction for the mapping T does not help in finding the Stinespring construction for $S \circ T$. What we need is the Stinespring construction for T based on the representation of $\mathcal{B}$ arising from the Stinespring construction for S. The GNS-construction, on the other hand, is *representation free*. It is sufficient to do it once for each completely positive mapping.

The importance of the following simple observation cannot be overestimated. It assures that the mappings $\gamma_{\tau\sigma}$, which mediate the second inductive limit in Section 5, and, as a consequence, also the canonical mappings k_τ appearing there have an adjoint.

2.18 Observation. Let E be a pre-Hilbert $\mathcal{B}$–module and let F be a contractive pre-Hilbert $\mathcal{B}$–$\mathcal{C}$–module. Let $x \in E$. Then

$$x \odot \mathsf{id} \colon y \longmapsto x \odot y$$

defines a mapping $F \to E \odot F$ with $\|x \odot \mathsf{id}\,\| \leq \|x\|$. The adjoint mapping is defined by

$$x^* \odot \mathsf{id} \colon x' \odot y \longmapsto \langle x, x' \rangle y$$

In the special case when $F = \mathcal{B}$ (cf. Example 2.8), whence $E \odot F = E$, we write $x^* \colon x' \mapsto \langle x, x' \rangle$.

Moreover, if $\langle x, x \rangle = \mathbf{1}$, then $x \odot \mathsf{id}$ is an isometry. More precisely, $(x^* \odot \mathsf{id})(x \odot \mathsf{id}) = \mathsf{id}_F$ and $(x \odot \mathsf{id})(x^* \odot \mathsf{id})$ is the projection $(|x\rangle\langle x|) \odot \mathsf{id}$ in $\mathcal{B}^a(E \odot F)$.

All these observations follow from the fact that the mapping $x \otimes \mathsf{id} \colon F \to E \otimes F$ has an adjoint and Observations 2.12 and 2.11.

2.19 Observation. Let E, F, F', G be pre-Hilbert modules and let $\beta \colon F \to F'$ be an isometric two-sided homomorphism of two-sided pre-Hilbert modules. Then also the mapping $\mathsf{id} \odot \beta \odot \mathsf{id} \colon E \odot F \odot G \to E \odot F' \odot G$ is an isometric two-sided homomorphism of two-sided pre-Hilbert modules.

2.20 Observation. Let E be a pre-Hilbert $\mathcal{B}$–module and let F be a contractive pre-Hilbert $\mathcal{B}$–$\mathcal{C}$–module. Let $a \in \mathcal{B}^r(E)$. Then $\|a \odot \mathsf{id}_F\| \leq \|a\|$; see [Lan95, Ske97]. In particular, the tensor product of two contractive pre-Hilbert modules is again contractive.

2.21 The strict topology. It is well-known that $\mathcal{B}^a(\overline{E})$ is the multiplier algebra of $\mathcal{K}(E)$. In other words, $\mathcal{B}^a(\overline{E})$ is the completion of $\mathcal{K}(E)$ with respect to the locally convex Hausdorff topology defined by the two families of seminorms $a \mapsto \|ak\|$ and $a \mapsto \|ka\|$ $(k \in \mathcal{K}(E))$. Another topology on $\mathcal{B}^a(E)$ is given by the two families of seminorms $a \mapsto \|ax\|$ and $a \mapsto \|a^*x\|$ $(x \in \overline{E})$. Also in this topology $\mathcal{B}^a(\overline{E})$ is complete. In general, the two topologies are different. They coincide, however, on bounded subsets. We follow the convention in [Lan95] and mean by the ***strict topology*** one of the above topologies restricted to bounded subsets of $\mathcal{B}^a(E)$. We say a bounded mapping $T \colon \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ is ***strict***, if it sends bounded strictly convergent nets to bounded strictly convergent nets. By boundedness it is sufficient to check convergence on total subsets of E.

So far, we have stated the preliminary definitions and facts which are needed in the main part of this article. Basics about inductive limits of Hilbert modules are postponed to Appendix A. We do not know a reference for this, so formal proofs are included.

The basics about von Neumann modules are needed only for Sections 12 – 14. They are postponed to Appendix C. Proofs of some results on von Neumann modules, extending those of [Ske97], are included. Like von Neumann algebras, which may be considered as concrete realizations of abstract W^*–algebras in the sense of [Sak71], von Neumann modules may be considered as concrete realizations of abstract W^*–modules in the sense of [Sch96].

3 Weak Markov flows of CP-semigroups and dilations to e_0–semi-groups: Module version

In this section we give the definition of weak Markov flow and dilation to an e_0–semigroup for a CP-semigroup T similar to that given in [BP94] and [Bha96], however, in terms of operators acting on a Hilbert module rather than on a Hilbert space. In Sections 4 and 5 we construct a Hilbert module E on which a *minimal* and a *maximal* weak Markov flow may be realized. In Section 6 we show that among all weak Markov flows the minimal and the maximal are distinguished by *universal properties*. In this way, we also recover the uniqueness result for the minimal weak Markov flow obtained in [BP94].

3.1 Definition. Let $\mathbb{T}$ be $\mathbb{R}^+$ (the non-negative reals) or $\mathbb{N}_0$ (the non-negative integers). Let $T = \big(T_t\big)_{t\in\mathbb{T}}$ be a conservative CP-semigroup on a unital C^*–algebra $\mathcal{B}$. A *weak Markov flow* of T on a pre-Hilbert $\mathcal{B}$–module E is a pair $(\mathcal{A}, j)$, where $\mathcal{A}$ is a pre–C^*–subalgebra of $\mathcal{B}^a(E)$ and $j = \big(j_t\big)_{t\in\mathbb{T}}$ is a family of homomorphisms $j_t\colon \mathcal{B} \to \mathcal{A}$, fulfilling the *Markov property*

$$j_s(\mathbf{1})j_t(b)j_s(\mathbf{1}) = j_s(T_{t-s}(b)) \quad \text{for all} \quad t, s \in \mathbb{T}; t \geq s; b \in \mathcal{B}, \tag{3.1}$$

and j_0 is injective. We use the abbreviation $p_t := j_t(\mathbf{1})$. (See also [Acc78, Bel84, Bel85].)

A weak Markov flow $(\mathcal{A}, j)$ on E is *cyclic*, if there is a *cyclic unit vector* $\xi \in E$ (i.e. $E = \mathcal{A}\xi$ and $\langle \xi, \xi\rangle = \mathbf{1}$), such that $j_0(b) = |\xi\rangle b\langle\xi|$.

By $\mathcal{A}_\infty$ we denote the ($*$–)algebra generated by $j_\mathbb{T}(\mathcal{B})$. A cyclic weak Markov flow is *essential*, if ξ is cyclic already for $\mathcal{A}_\infty$, i.e. $E = \mathcal{A}_\infty\xi$.

An essential weak Markov flow $(\mathcal{A}, j)$ is *minimal*, if $\mathcal{A} = \mathcal{A}_\infty$. It is *maximal*, if $\mathcal{A} = \mathcal{B}^a(E)$.

The cyclic vector ξ *intertwines* j_0 and $\mathsf{id}_\mathcal{B}$ in the sense that $j_0(b)\xi = \xi b$, so that $E = \mathcal{A}p_0\xi = \mathcal{A}j_0(\mathcal{B})\xi = \mathcal{A}\xi\mathcal{B}$. It follows that $\varphi(a) = p_0 a p_0$ defines a conditional expectation $\varphi\colon \mathcal{A} \to j_0(\mathcal{B})$. The cyclicity condition just means that E is the GNS-module of φ. Essential means that this GNS-module is generated by $\mathcal{A}_\infty$ alone.

It is not difficult to check that a minimal weak Markov flow is determined up to unitary equivalence. For instance, doing the Stinespring construction (Example 2.16) we are reduced to [BP94, Theorem 2.7]. (Notice that the formulation in [BP94] does not require existence of a cyclic vector. It is, however, replaced by the requirement that j_0 is, in our language, the left-regular representation of $\mathcal{B}$ on itself. If we remove existence of the cyclic vector, then an arbitray direct sum of minimal weak Markov flows would also be minimal.) We give a different proof of uniqueness in Section 6. Additionally, we show that also the maximal weak Markov flow is unique.

Many authors require that j_t sends the unit of $\mathcal{B}$ to the unit of $\mathcal{A}$; see e.g. [EL77, Küm85, Sau86]. However, in our setting this contradicts the Markov property (3.1), unless T is an E_0–semigroup.

3.2 Example. Suppose T consists of unital endomorphisms (i.e. T is an E_0–semigroup). Set $E = \mathcal{A} = \mathcal{B}$ (cf. Example 2.8). Define j_t by setting $j_t(b)x = T_t(b)x$. Then $(\mathcal{A}, j)$ is a minimal *and* a maximal weak Markov flow of T on E with cyclic vector $\xi = \mathbf{1}$.

Conversely, if $j_t(\mathbf{1}) = \mathbf{1}$ for all $t \in \mathbb{T}$, then we easily conclude from the Markov property and injectivity of j_0 that T is an E_0–semigroup.

3.3 Definition. Let E be a pre-Hilbert $\mathcal{B}$–module and let $j_0 \colon \mathcal{B} \to \mathcal{B}^a(E)$ be a homomorphism. An e_0–semigroup $\vartheta = \big(\vartheta_t\big)_{t\in\mathbb{T}}$ on a pre–C^*–subalgebra $\mathcal{A}$ of $\mathcal{B}^a(E)$ is called an *e_0–dilation* of T on E, if $(\mathcal{A}, j)$ with $j_t = \vartheta_t \circ j_0$ is a weak Markov flow. It is said to be an *E_0–dilation*, if ϑ is an E_0–semigroup.

A dilation is called *minimal* and *maximal*, if the weak Markov flow $(\mathcal{A}, j)$ is minimal and maximal, respectively. In either case, there exists an element ξ in E, such that

$$\langle \xi, \vartheta_t \circ j_0(b)\xi \rangle = T_t(b).$$

The results on uniqueness in Section 6 imply that both the minimal and the maximal dilation of T are unique, too. In particular, the maximal dilation is always an E_0–dilation, and by Example 3.2 the minimal dilation is only an e_0–dilation, unless T is an E_0–semigroup.

4 The first inductive limit: Product systems

In this section we construct for each $\tau \in \mathbb{T}$ a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module E_τ which is more or less the GNS-module of T_τ enlarged by *inserting* the algebra $\mathcal{B}$ at each time σ in between 0 and τ. Suppose $\tau = \sigma_2 + \sigma_1$. By Observation 2.17 the GNS-modules $\mathcal{E}_\tau$, $\mathcal{E}_{\sigma_1}$, and $\mathcal{E}_{\sigma_2}$, at times τ, σ_1, and σ_2, respectively, are related by the tensor product $\mathcal{E}_\tau \subset \mathcal{E}_{\sigma_2} \odot \mathcal{E}_{\sigma_1}$. In order to have equality we could try to replace the GNS-module at time τ by $\mathcal{E}_{\sigma_2} \odot \mathcal{E}_{\sigma_1}$. In other words, we inserted $\mathcal{B}$ at time σ_1. However, for a different choice of σ_1 and σ_2, in general, we obtain different modules. Also splitting $[0, \tau]$ into three or more subintervals will destroy the desired factorization. In order to be stable under any further splitting we have to perform an inductive limit over all possible partitions of the interval $[0, \tau]$. Concerning inductive limits we use the notations from Appendix A.

There exist, essentially, two ways of looking at an interval partition. Firstly, with emphasis on the end points of each subinterval. Secondly, with emphasis on the length of each subinterval. The different pictures are useful for different purposes. In these notes we will concentrate on the second point of view. Whereas, we need the first in order to see that the interval partitions form a lattice.

Let $\tau > 0$ in $\mathbb{T}$. We define $\mathbb{I}_\tau$ to be the set of all *finite ordered tuples* $\big\{(t_n, \ldots, t_1) \in \mathbb{T}^n \colon n \in \mathbb{N}, \tau = t_n > \ldots > t_1 > 0\big\}$. On $\mathbb{I}_\tau$ we have a natural notion of *inclusion*, *union*, and *intersection* of tuples. By inclusion we define a *partial order* on $\mathbb{I}_\tau$.

We define $\mathbb{J}_\tau$ to be the set of all *finite tuples* $\mathfrak{t} = (t_n, \ldots, t_1) \in \mathbb{T}^n$ $(n \in \mathbb{N}, t_i > 0)$ having *length*

$$|\mathfrak{t}| := \sum_{i=1}^{n} t_i = \tau.$$

For two tuples $\mathfrak{s} = (s_m, \ldots, s_1) \in \mathbb{J}_\sigma$ and $\mathfrak{t} = (t_n, \ldots, t_1) \in \mathbb{J}_\tau$ we define the *joint tuple* $\mathfrak{s} \smile \mathfrak{t} \in \mathbb{J}_{\sigma+\tau}$ by

$$\mathfrak{s} \smile \mathfrak{t} = ((s_m, \ldots, s_1), (t_n, \ldots, t_1)) = (s_m, \ldots, s_1, t_n, \ldots, t_1).$$

We equip $\mathbb{J}_\tau$ with a partial order by saying $\mathfrak{t} \geq \mathfrak{s} = (s_m, \ldots, s_1)$, if for each j $(1 \leq j \leq m)$ there are (unique) $\mathfrak{s}_j \in \mathbb{J}_{s_j}$ such that $\mathfrak{t} = \mathfrak{s}_m \smile \ldots \smile \mathfrak{s}_1$.

We extend the definitions of $\mathbb{I}_\tau$ and $\mathbb{J}_\tau$ to $\tau = 0$, by setting $\mathbb{I}_0 = \mathbb{J}_0 = \{()\}$, where $()$ is the *empty tuple*. For $\mathfrak{t} \in \mathbb{J}_\tau$ we put $\mathfrak{t} \smile () = \mathfrak{t} = () \smile \mathfrak{t}$.

orderprop **4.1 Proposition.** *The mapping* $\mathfrak{o}\colon (t_n, \ldots, t_1) \mapsto \Bigl(\sum_{i=1}^{n} t_i, \ldots, \sum_{i=1}^{1} t_i\Bigr)$ *is an order isomorphism* $\mathbb{J}_\tau \to \mathbb{I}_\tau$.

Proof. Of course, $\mathfrak{o}$ is bijective. Obviously, the image in $\mathbb{I}_\tau$ of a tuple $(|\mathfrak{s}_m|, \ldots, |\mathfrak{s}_1|)$ in $\mathbb{J}_\tau$ is *contained* in the image of $\mathfrak{s}_m \smile \ldots \smile \mathfrak{s}_1$. Conversely, let $(s_m, \ldots, s_1)$ be a tuple in $\mathbb{I}_\tau$ and $(t_n, \ldots, t_1) \geq (s_m, \ldots, s_1)$. Define a function $\mathfrak{n}\colon \{0, \ldots, m\} \to \{0, \ldots, n\}$ by requiring $t_{\mathfrak{n}(j)} = s_j$ $(j \geq 1)$ and $\mathfrak{n}(0) = 0$. Set $\mathfrak{t} = \mathfrak{o}^{-1}(t_n, \ldots, t_1)$ and $\mathfrak{s} = \mathfrak{o}^{-1}(s_m, \ldots, s_1)$. Furthermore, define $\mathfrak{s}_j = \mathfrak{o}^{-1}(t_{\mathfrak{n}(j)}, \ldots, t_{\mathfrak{n}(j-1)+1})$ $(j \geq 1)$. Then $\mathfrak{t} = \mathfrak{s}_m \smile \ldots \smile \mathfrak{s}_1 \geq (|\mathfrak{s}_m|, \ldots, |\mathfrak{s}_1|) = \mathfrak{s}$. ■

latob **4.2 Observation.** $\mathbb{I}_\tau$ is a lattice with the union of two tuples being their unique least upper bound and the intersection of two tuples being their unique greatest lower bound. In particular, $\mathbb{I}_\tau$ is directed increasingly. Observe that (τ) is the unique minimum of $\mathbb{I}_\tau$ $(\tau > 0)$. By Proposition 4.1 all these assertions are true also for $\mathbb{J}_\tau$.

The reason why we use the lattice $\mathbb{J}_\tau$ instead of $\mathbb{I}_\tau$ is the importance of the operation $\smile$. Notice that $\smile$ is an operation not on $\mathbb{J}_\tau$, but rather an operation $\mathbb{J}_\sigma \times \mathbb{J}_\tau \to \mathbb{J}_{\sigma+\tau}$. We can say two tuples $\mathfrak{s} \in \mathbb{J}_\sigma$ and $\mathfrak{t} \in \mathbb{J}_\tau$ are just *glued together* to a tuple $\mathfrak{s} \smile \mathfrak{t} \in \mathbb{J}_{\sigma+\tau}$. Before we can *glue together* the corresponding tuples $\mathfrak{o}(\mathfrak{s}) \in \mathbb{I}_\sigma$ and $\mathfrak{o}(\mathfrak{t}) \in \mathbb{I}_\tau$, we first must *shift* all points in $\mathfrak{o}(\mathfrak{s})$ by the time τ. (This behaviour is not surprising. Recall that the t_i in a tuple in $\mathbb{J}_\tau$ stand for time differences. These do not change under time shift. Whereas the t_i in a tuple in $\mathbb{I}_\tau$ stand for time points, which, of course, change under time shift.) Hence, in the description by $\mathbb{I}_\tau$ the time shift must be acted out explicitly, whereas in the description by $\mathbb{J}_\tau$ the time shift is intrinsic and works automatically. Our decision to use $\mathbb{J}_\tau$ instead of the more common $\mathbb{I}_\tau$ is the reason why, in the sequel, in many formulae where one intuitively would expect a time shift, no explicit time shift appears. It is, however, always encoded in our notation. (Cf., for instance, Equations (5.2), (8.1), and (11.1).)

Let $T = (T_t)_{t \in \mathbb{T}}$ be a conservative CP-semigroup on a unital C^*–algebra $\mathcal{B}$. For each t let $\mathcal{E}_t$ denote the GNS-module of T_t and $\xi_t \in \mathcal{E}_t$ the cyclic vector. (Observe that $\mathcal{E}_0 = \mathcal{B}$ and $\xi_0 = \mathbf{1}$.) Let $\mathfrak{t} = (t_n, \ldots, t_1) \in \mathbb{J}_\tau$. We define

$$E_{\mathfrak{t}} = \mathcal{E}_{t_n} \odot \ldots \odot \mathcal{E}_{t_1} \quad \text{and} \quad E_{()} = \mathcal{E}_0.$$

In particular, we have $E_{(\tau)} = \mathcal{E}_\tau$. By Observations 2.17 and 2.19

$$\xi_\tau \longmapsto \xi_{\mathfrak{t}} := \xi_{t_n} \odot \ldots \odot \xi_{t_1}$$

defines an isometric two-sided homomorphism $\beta_{\mathfrak{t}(\tau)} \colon \mathcal{E}_\tau \to E_{\mathfrak{t}}$.

Now suppose that $\mathfrak{t} = (t_n, \ldots, t_1) = \mathfrak{s}_m \smile \ldots \smile \mathfrak{s}_1 \geq \mathfrak{s} = (s_m, \ldots, s_1)$ with $|\mathfrak{s}_j| = s_j$. By

$$\beta_{\mathfrak{t}\mathfrak{s}} = \beta_{\mathfrak{s}_m(s_m)} \odot \ldots \odot \beta_{\mathfrak{s}_1(s_1)}$$

we define an isometric two-sided homomorphism $\beta_{\mathfrak{t}\mathfrak{s}} \colon E_{\mathfrak{s}} \to E_{\mathfrak{t}}$. Obviously, $\beta_{\mathfrak{t}\mathfrak{r}}\beta_{\mathfrak{r}\mathfrak{s}} = \beta_{\mathfrak{t}\mathfrak{s}}$ for all $\mathfrak{t} \geq \mathfrak{r} \geq \mathfrak{s}$. All this follows by repeated application of Observation 2.19. We obtain the following result.

1stil **4.3 Proposition.** *The family $\big(E_{\mathfrak{t}}\big)_{\mathfrak{t} \in \mathbb{J}_\tau}$ together with $\big(\beta_{\mathfrak{t}\mathfrak{s}}\big)_{\mathfrak{s} \leq \mathfrak{t}}$ forms an inductive system of pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules. Hence, also the inductive limit $E_\tau = \liminda_{\mathfrak{t} \in \mathbb{J}_\tau} E_{\mathfrak{t}}$ is a $\mathcal{B}$–$\mathcal{B}$–pre-Hilbert module and the canonical mappings $i_{\mathfrak{t}} \colon E_{\mathfrak{t}} \to E_\tau$ are isometric two-sided homomorphisms.*

This is the first step of the inductive limit where the involved isometries preserve left multiplication. In other words, if we restrict to a fixed endpoint τ, then we are concerned with a well-defined left multiplication, no matter how many time points in the interval $[0, \tau]$ are involved.

Before we investigate the connections among the E_τ, we observe that E_τ contains a distinguished element.

unit **4.4 Proposition.** *Let $\xi^\tau = i_{(\tau)}\xi_\tau$. Then $i_{\mathfrak{t}}\xi_{\mathfrak{t}} = \xi^\tau$ for all $\mathfrak{t} \in \mathbb{J}_\tau$. Moreover, $\langle \xi^\tau, b\xi^\tau \rangle = T_\tau(b)$. In particular, $\langle \xi^\tau, \xi^\tau \rangle = \mathbf{1}$.*

Proof. Let $\mathfrak{s}, \mathfrak{t} \in \mathbb{J}_\tau$ and choose $\mathfrak{r}$, such that $\mathfrak{r} \geq \mathfrak{s}$ and $\mathfrak{r} \geq \mathfrak{t}$. Then $i_{\mathfrak{s}}\xi_{\mathfrak{s}} = i_{\mathfrak{r}}\beta_{\mathfrak{r}\mathfrak{s}}\xi_{\mathfrak{s}} = i_{\mathfrak{r}}\xi_{\mathfrak{r}} = i_{\mathfrak{r}}\beta_{\mathfrak{r}\mathfrak{t}}\xi_{\mathfrak{t}} = i_{\mathfrak{t}}\xi_{\mathfrak{t}}$.

Moreover, $\langle \xi^\tau, b\xi^\tau \rangle = \langle i_{(\tau)}\xi_\tau, b i_{(\tau)}\xi_\tau \rangle = \langle i_{(\tau)}\xi_\tau, i_{(\tau)} b\xi_\tau \rangle = \langle \xi_\tau, b\xi_\tau \rangle = T_\tau(b)$. ■

4.5 Corollary. $(\xi^\tau)^* i_{\mathfrak{t}} = \xi_{\mathfrak{t}}^*$ *for all $\mathfrak{t} \in \mathbb{J}_\tau$. Therefore, $\xi_{\mathfrak{t}}^*\beta_{\mathfrak{t}\mathfrak{s}} = \xi_{\mathfrak{s}}^*$ for all $\mathfrak{s} \leq \mathfrak{t}$.*

1tensor **4.6 Remark.** Clearly, $E_0 = \mathcal{E}_0 = \mathcal{B}$ and $\xi^0 = \xi_0 = \mathbf{1}$. In particular, $E_\tau = E_0 \odot E_\tau = \xi^0 \odot E_\tau$ where $\mathsf{id} = \xi^0 \odot \mathsf{id}$ gives the identification.

In [Arv89] Arveson defined the notion of *tensor product system of Hilbert spaces.* We generalize this, on the one hand, dropping measurability and separability conditions and, on the other hand, considering pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules instead of Hilbert spaces. The difference to a version for Hilbert modules is only marginal, because the completions always can be performed. (Recall that a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module is contractive, automatically, if $\mathcal{B}$ is a C^*–algebra; see also Section 10.) On the other hand, it may be of some interest to know that certain operators leave invariant the algebraic domain, or even have adjoints on this domain. So it is important to consider pre-Hilbert module versions.

psdef **4.7 Definition.** Let $\mathcal{B}$ be a C^*–algebra. A family $E^\odot = \big(E_t\big)_{t \in \mathbb{T}}$ of pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules is called a *tensor product system of pre-Hilbert modules* or shortly a *product system*, if

$E_0 = \mathcal{B}$, and if there exists a family $\big(u_{st}\big)_{s,t\in\mathbb{T}}$ of two-sided unitaries $u_{st}\colon E_s \odot E_t \to E_{s+t}$, fulfilling the *associativity condition*

$$u_{r(s+t)}(\mathsf{id} \odot u_{st}) = u_{(r+s)t}(u_{rs} \odot \mathsf{id})$$

for all $r, s, t \in \mathbb{T}$.

A family $\xi^\odot = \big(\xi_t\big)_{t\in\mathbb{T}}$ of vectors $\xi_t \in E_t$ with $\xi_0 = \mathbf{1}$ is called a *unit* for the product system, if $u_{st}(\xi_s \odot \xi_t) = \xi_{s+t}$. A unit is called *unital*, if it consists of *unit vectors* (i.e. $\langle \xi_t, \xi_t\rangle = \mathbf{1}$). A unit is called *generating*, if E_t is spanned by images of elements $b_n\xi_{t_n} \odot \ldots \odot b_1\xi_{t_1}b_0$ ($\mathfrak{t} \in \mathbb{J}_t, b_i \in \mathcal{B}$) under successive applications of appropiate mappings $\mathsf{id} \odot u_{ss'} \odot \mathsf{id}$.

prodsys **4.8 Theorem.** *The family $E^\odot = \big(E_\tau\big)_{\tau\in\mathbb{T}}$ (with E_τ as in Proposition 4.3) forms a product system. The family $\xi^\odot = \big(\xi^\tau\big)_{\tau\in\mathbb{T}}$ (with ξ^τ as in Proposition 4.4) forms a generating unital unit for this product system.*

Proof. Let $\sigma, \tau \in \mathbb{T}$ and choose $\mathfrak{s} \in \mathbb{J}_\sigma$ and $\mathfrak{t} \in \mathbb{J}_\tau$. Then the proof that the E_τ form a product system is almost done by observing that

$$E_\mathfrak{s} \odot E_\mathfrak{t} = E_{\mathfrak{s}\smile\mathfrak{t}}. \tag{4.1}$$ preasscond

From this, intuitively, the mapping $u_{\sigma\tau}\colon i_\mathfrak{s}x_\mathfrak{s} \odot i_\mathfrak{t}y_\mathfrak{t} \mapsto i_{\mathfrak{s}\smile\mathfrak{t}}(x_\mathfrak{s} \odot y_\mathfrak{t})$ should define a surjective isometry. Surjectivity is clear, because elements of the form $i_{\mathfrak{s}\smile\mathfrak{t}}(x_\mathfrak{s} \odot y_\mathfrak{t})$ are total in $E_{\sigma+\tau}$. To see isometry we observe that $i_\mathfrak{s}x_\mathfrak{s} = i_{\hat{\mathfrak{s}}}\beta_{\hat{\mathfrak{s}}\mathfrak{s}}x_\mathfrak{s}$ and $i_\mathfrak{t}y_\mathfrak{t} = i_{\hat{\mathfrak{t}}}\beta_{\hat{\mathfrak{t}}\mathfrak{t}}y_\mathfrak{t}$ for $\hat{\mathfrak{t}} \geq \mathfrak{t}$ and $\hat{\mathfrak{s}} \geq \mathfrak{s}$. Similarly, $i_{\mathfrak{s}\smile\mathfrak{t}}(x_\mathfrak{s} \odot y_\mathfrak{t}) = i_{\hat{\mathfrak{s}}\smile\hat{\mathfrak{t}}}(\beta_{\hat{\mathfrak{s}}\mathfrak{s}}x_\mathfrak{s} \odot \beta_{\hat{\mathfrak{t}}\mathfrak{t}}y_\mathfrak{t})$. Therefore, for checking the equation

$$\langle i_\mathfrak{s}x_\mathfrak{s} \odot i_\mathfrak{t}y_\mathfrak{t}, i_{\mathfrak{s}'}x'_{\mathfrak{s}'} \odot i_{\mathfrak{t}'}y'_{\mathfrak{t}'}\rangle = \langle i_{\mathfrak{s}\smile\mathfrak{t}}(x_\mathfrak{s} \odot y_\mathfrak{t}), i_{\mathfrak{s}'\smile\mathfrak{t}'}(x'_{\mathfrak{s}'} \odot y'_{\mathfrak{t}'})\rangle$$

we may assume that $\mathfrak{t}' = \mathfrak{t}$ and $\mathfrak{s}' = \mathfrak{s}$. (This is also a key observation in showing that $E_\sigma \odot E_\tau = \liminf_{(\mathfrak{s},\mathfrak{t})\in\mathbb{J}_\sigma\times\mathbb{J}_\tau} E_\mathfrak{s} \odot E_\mathfrak{t}$.) Now isometry is clear, because both $i_\mathfrak{s} \odot i_\mathfrak{t}\colon E_\mathfrak{s} \odot E_\mathfrak{t} \to E_\sigma \odot E_\tau$ and $i_{\mathfrak{s}\smile\mathfrak{t}}\colon E_{\mathfrak{s}\smile\mathfrak{t}} = E_\mathfrak{s} \odot E_\mathfrak{t} \to E_{\sigma+\tau}$ are (two-sided) isometries. The associativity condition follows directly from associativity of (4.1).

The fact that the ξ^τ form a unit follows by similar arguments from Proposition 4.4. Obviously, this unit is unital. It is also generating, because E_τ is generated by vectors of the form $i_\mathfrak{t}(b_n\xi_{t_n} \odot \ldots \odot b_1\xi_{t_1}b_0)$ ($b_i \in \mathcal{B}$). ■

iident **4.9 Remark.** In the sequel, we always make the identification

$$E_\sigma \odot E_\tau = E_{\sigma+\tau}. \tag{4.2}$$ tensorid

We, actually, have shown, using this identification and (4.1), that $i_\mathfrak{s} \odot i_\mathfrak{t} = i_{\mathfrak{s}\smile\mathfrak{t}}$. Thanks to this identification we have a natural embedding of $\mathcal{B}^a(E_\sigma)$ into $\mathcal{B}^a(E_{\sigma+\tau})$ by sending a to $a \odot \mathsf{id}$. By Observation 2.20 this embedding is contractive. It need not be faithful.

In a certain sense, product systems of Hilbert modules with units for them are in one-to-one correspondence with CP-semigroups. This correspondence is more specific than the correspondence of product systems of Hilbert spaces with E_0–semigroups discovered by Arveson [Arv89], which is only up to cocycle conjugacy. We investigate this more systematically in Section 7. The paradigm example of a product system of Hilbert spaces is the family $\Gamma(L^2([0,\tau], H))$ of symmetric Fock spaces which is well-known to be isomorphic to a corresponding family of time ordered Fock spaces. In Section 11 we will see that the time ordered Fock module (Theorem 11.4) plays the same distinguished role for product systems of Hilbert modules.

5 The second inductive limit: Dilations and flows

2nd

Now we are going to glue together the E_τ in a second inductive limit, mediated by mappings $\gamma_{\tau\sigma}$ ($\tau \geq \sigma$). Since these mappings no longer preserve left multiplication, we no longer have a unique left multiplication on the inductive limit $E = \liminf_{\tau\to\infty} E_\tau$. It is, however, possible to define on E for each time τ a different left multiplication, which turns out to be more or less the left multiplication from E_τ. This family of left multiplications will be the weak Markov flow. Also the identification by (tensorid 4.2) has a counter part obtained by sending, formally, σ to ∞. The embedding of $\mathcal{B}^a(E_\sigma)$ into $\mathcal{B}^a(E_{\sigma+\tau})$, formally, becomes an embedding $\mathcal{B}^a(E_{``\infty"})$ into $\mathcal{B}^a(E_{``\infty+\tau"})$, i.e. an endomorphism of $\mathcal{B}^a(E)$. This endomorphism depends, however, on τ. The family formed by all these endomorphisms will be the dilating E_0–semigroup.

Let $\tau, \sigma \in \mathbb{T}$ with $\tau \geq \sigma$. Using the notation from Observation tensorvec 2.18, we define the isometry

$$\gamma_{\tau\sigma} = \xi^{\tau-\sigma} \odot \mathsf{id} \colon E_\sigma \longrightarrow E_{\tau-\sigma} \odot E_\sigma = E_\tau.$$

Let $\tau \geq \rho \geq \sigma$. Since $\big(\xi^\tau\big)$ is a unit, we have

$$\gamma_{\tau\sigma} = \xi^{\tau-\sigma} \odot \mathsf{id} = \xi^{\tau-\rho} \odot \xi^{\rho-\sigma} \odot \mathsf{id} = \gamma_{\tau\rho}\gamma_{\rho\sigma}.$$

That leads to the following result.

5.1 Proposition. *The family $\big(E_\tau\big)_{\tau\in\mathbb{T}}$ together with $\big(\gamma_{\tau\sigma}\big)_{\sigma\leq\tau}$ forms an inductive system of right pre-Hilbert $\mathcal{B}$–modules. Hence, also the inductive limit $E = \liminf_{\tau\to\infty} E_\tau$ is a right pre-Hilbert $\mathcal{B}$–module. Moreover, the canonical mappings $k_\tau \colon E_\tau \to E$ are isometries.*

Also E contains a distinguished element.

5.2 Proposition. *Let $\xi = k_0\xi^0$. Then $k_\tau\xi^\tau = \xi$ for all $\tau \in \mathbb{T}$. Moreover, $\langle\xi, \xi\rangle = \mathbf{1}$.*

Proof. Precisely, as in Proposition unit 4.4. ■

j0def **5.3 Corollary.** *By $j_0(b) = |\xi\rangle b\langle\xi|$ we define a faithful representation of $\mathcal{B}$ by operators in $\mathcal{B}^a(E)$. Moreover, $\varphi\colon a \mapsto j_0(\mathbf{1})aj_0(\mathbf{1})$ defines a conditional expectation $\mathcal{B}^a(E) \to j_0(\mathcal{B})$.*

indtensorT **5.4 Theorem.** *For all $\tau \in \mathbb{T}$ we have*

$$E \odot E_\tau = E, \tag{5.1}$$ indtensor

extending (tensorid 4.2) in a natural way. Moreover, $\xi \odot \xi^\tau = \xi$.

Proof. The mapping $u_\tau \colon k_\sigma x_\sigma \odot y_\tau \mapsto k_{\sigma+\tau}(x_\sigma \odot y_\tau)$ defines a surjective isometry. We see that this is an isometry precisely as in the proof of Theorem prodsys 4.8.

To see surjectivity let us choose $\rho \in \mathbb{T}$ and $z_\rho \in E_\rho$. If $\rho \geq \tau$ then consider x_ρ as an element of $E_{\rho-\tau} \odot E_\tau$ and apply the prescription to see that $k_\rho x_\rho$ is in the range of u_τ. If $\rho < \tau$, then apply the prescription to $\mathbf{1} \odot \gamma_{\tau\rho}x_\rho \in E_0 \odot E_\tau$. ■

E_0cor **5.5 Corollary.** *The family $\vartheta = \big(\vartheta_\tau\big)_{\tau\in\mathbb{T}}$ of endomorphisms $\vartheta_\tau \colon \mathcal{B}^a(E) \to \mathcal{B}^a(E \odot E_\tau) = \mathcal{B}^a(E)$ defined by setting*

$$\vartheta_\tau(a) = a \odot \mathsf{id}_{E_\tau}$$

forms an E_0–semigroup.

Proof. Of course, $u_{\sigma+\tau} \circ (\mathsf{id} \odot u_{\sigma\tau}) = u_\tau \circ (u_\sigma \odot \mathsf{id})$. So, the semigroup property follows directly from $E \odot E_{\sigma+\tau} = E \odot (E_\sigma \odot E_\tau) = (E \odot E_\sigma) \odot E_\tau$. ■

kident **5.6 Remark.** Making use of the identifications given by (5.1) and (4.2), the proof of Theorem 5.4, actually, shows that, $k_\sigma \odot \mathsf{id} = k_{\sigma+\tau}$. Putting $\sigma = 0$ and making use of Remark 4.6, we find

$$k_\tau = (k_0 \odot \mathsf{id})(\xi^0 \odot \mathsf{id}) = \xi \odot \mathsf{id}\,.$$

5.7 Corollary. *k_τ is an element of $\mathcal{B}^a(E_\tau, E)$. The adjoint mapping is*

$$k_\tau^* = \xi^* \odot \mathsf{id}\colon E = E \odot E_\tau \longrightarrow E_\tau.$$

Therefore, $k_\tau^* k_\tau = \mathsf{id}_{E_\tau}$ and $k_\tau k_\tau^*$ is a projection onto the range of k_τ.

main **5.8 Theorem.** *Define the family $j = \big(j_\tau\big)_{\tau\in\mathbb{T}}$ of representations, by setting $j_\tau = \vartheta_\tau \circ j_0$ and let $\mathcal{A}_\infty$ be as in Definition 3.1. Then $(\mathcal{A}_\infty, j)$ is a minimal weak Markov flow and $(\mathcal{B}^a(E), j)$ is a maximal weak Markov flow of the CP-semigroup T with cyclic vector ξ.*

ϑ is a maximal E_0–dilation of the CP-semigroup T. The restrictions $\vartheta_\tau \upharpoonright \mathcal{A}_\infty$ form a minimal e_0–dilation.

Proof. Postponing cyclicity of ξ, the remaining statements are clear, if we show the Markov property $j_\sigma(\mathbf{1})j_\tau(b)j_\sigma(\mathbf{1}) = j_\sigma(T_{\tau-\sigma}(b))$ for $\sigma \leq \tau$. By definition of j and the semigroup property of ϑ it is enough to restrict to $\sigma = 0$. We have

$$\langle \xi, j_\tau(b)\xi\rangle = \langle \xi \odot \xi^\tau, (j_0(b) \odot \mathsf{id})(\xi \odot \xi^\tau)\rangle = \langle \xi^\tau, b\xi^\tau\rangle = T_\tau(b)$$

by Corollary 5.3 and Proposition 4.4. Hence, $p_0 j_\tau(b) p_0 = |\xi\rangle T_\tau(b)\langle\xi| = j_0(T_\tau(b))$.

Let us come to cyclicity. It is enough to show that for each $\tau \in \mathbb{T}$, $\mathfrak{t} = (t_n, \ldots, t_1) \in \mathbb{J}_\tau$, $b_n, \ldots, b_0 \in \mathcal{B}$, and $(s_n, \ldots, s_1) = \mathfrak{o}(\mathfrak{t}) \in \mathbb{I}_\tau$ (cf. Proposition 4.1)

$$\vartheta_{s_n}(b_n) \ldots \vartheta_{s_1}(b_1)\vartheta_0(b_0)\xi = \xi \odot b_n\xi^{t_n} \odot \ldots \odot b_1\xi^{t_1}b_0, \tag{5.2}$$ cyclic

because by Remarks 4.9 and 5.6, $\xi \odot b_n\xi^{t_n} \odot \ldots \odot b_1\xi^{t_1}b_0 = k_\tau i_{\mathfrak{t}}(b_n\xi_{t_n} \odot \ldots \odot b_1\xi_{t_1}b_0)$, and E is spanned by these vectors. First, observe that

$$\vartheta_\tau(b)\xi = (|\xi\rangle b\langle\xi| \odot \mathsf{id})(\xi \odot \xi^\tau) = \xi \odot b\xi^\tau. \tag{5.3}$$ bshift

Now we proceed by induction on n. Extending (5.2) to the empty tuple (i.e. $\tau = 0$), the statement is true for $n = 0$. Let us assume that (5.2) holds for n and choose $t_{n+1} > 0$ and $b_{n+1} \in \mathcal{B}$. Then by (5.3) and Remark 5.6

$$\begin{aligned}
&\vartheta_{t_{n+1}+\tau}(b_{n+1})(\xi \odot b_n\xi^{t_n} \odot \ldots \odot b_1\xi^{t_1}b_0)\\
&\qquad = (\vartheta_{t_{n+1}}(b_{n+1}) \odot \mathsf{id}_{E_\tau})(\xi \odot b_n\xi^{t_n} \odot \ldots \odot b_1\xi^{t_1}b_0)\\
&\qquad = (\vartheta_{t_{n+1}}(b_{n+1})\xi) \odot b_n\xi^{t_n} \odot \ldots \odot b_1\xi^{t_1}b_0\\
&\qquad = \xi \odot b_{n+1}\xi^{t_{n+1}} \odot b_n\xi^{t_n} \odot \ldots \odot b_1\xi^{t_1}b_0. \blacksquare
\end{aligned}$$

In principle, our construction finishes here. It seems, however, interesting to see clearly that j_τ is nothing but the left multiplication from E_τ. The following obvious proposition completely settles this problem.

5.9 Proposition. *We have $k_\tau b k_\tau^* = \vartheta_\tau\big(|\xi\rangle b\langle\xi|\big) = j_\tau(b)$. In particular, $p_\tau = k_\tau k_\tau^*$.*

5.10 Remark. Let $(k_\tau^*)_\sigma = k_\tau^* k_\sigma$ be the family associated with k_τ^* by Proposition A.3. One may check that

$$(k_\tau^*)_\sigma = \begin{cases} \gamma_{\tau\sigma} & \text{for } \sigma \le \tau \\ \gamma_{\sigma\tau}^* & \text{for } \sigma \ge \tau. \end{cases}$$

Hence, the action of k_τ^* on an element $k_\sigma x_\sigma \in E$ coming from E_σ can be interpreted as *lifting* this element to E_τ via $\xi^{\tau-\sigma}$, if σ is too small, and *truncating* it to E_τ via $(\xi^{\sigma-\tau})^*$, if σ is too big.

5.11 Observation. Since ξ is cyclic for $\mathcal{A}_\infty$, the ideal in $\mathcal{A}_\infty$ (or in $\mathcal{B}^a(E)$) generated by p_0 consists precisely of the finite rank operators $\mathcal{F}(E)$. The question, whether $\mathcal{F}(E)$ is already all of $\mathcal{A}_\infty$, is equivalent to the question, whether $p_\tau \in \mathcal{F}(E)$ for all $\tau \in \mathbb{T}$. This is true, for instance, in all cases when $p_\tau = p_0$ (cf. Example 3.2). In general, we do not know the answer.

5.12 Proposition. *The endomorphisms ϑ_τ are strict.*

PROOF. This trivially follows from the observation that vectors of the form $x \odot x_\tau$ ($x \in E, x_\tau \in E_\tau$) form a total subset of E. ■

5.13 Conclusion. The e_0–semigroup $\vartheta \upharpoonright \mathcal{A}_\infty$ is (up to completion) the e_0–dilation constructed in [Bha96]. More precisely, if $\mathcal{B}$ is represented faithfully on a Hilbert space G, then the Stinespring construction (Example 2.16) gives rise to a (pre-)Hilbert space $H = E \odot G$ and a faithful representation ρ of $\mathcal{A}_\infty$ by operators on H. Lifting ϑ to $\rho(\mathcal{A}_\infty)$, we obtain the e_0–semigroup from [Bha96].

New in our construction is the extension to an E_0–semigroup of strict endomorphisms of $\mathcal{B}^a(E)$. Of course, $\mathcal{B}^a(E)$ also has a faithful image in $\mathcal{B}(H)$. However, it seems not possible to find this subalgebra easily without reference to the module description. The module description also allows us to show that, if T is a normal and strongly continuous CP-semigroup on a von Neumann algebra $\mathcal{B} \subset \mathcal{B}(G)$ (i.e. T is continuous in the strong topology of $\mathcal{B}$), then ϑ is normal and strongly continuous, too; see Section 12. In Section 13 we will see that in the case when $\mathcal{B} = \mathcal{B}(G)$ (and T normal), then $\rho(\mathcal{B}^a(E))$ is all of $\mathcal{B}(H)$ (Corollary 13.10). In this way, we recover a result from [Bha98a].

6 Weak Markov flows of CP-semigroups: Algebraic version

In this section we give the definition of weak Markov flow of a CP-semigroup in an algebraic fashion and answer as to how far minimal and maximal dilations are unique. The definition only refers to the family j of homomorphisms, but no longer to the representation module. Other structures like a family of conditional expectations $\varphi_\tau = p_\tau \bullet p_\tau$ can be reconstructed; see Propositions 6.2 and 6.8. This is the only section in these notes, where j_0 is not necessarily faithful; cf. Definition 6.3.

If we want to encode properties of a weak Markov flow, which are of an *essentially spatial* nature, then we have to require that the GNS-representation of the conditional expectation φ_0 is suitably faithful. This leads to the notion of an *essential* weak Markov flow. Among all such flows we are able to single out two universal objects, which are realized by the *minimal* and the *maximal* weak Markov flow in Theorem 5.8.

algMd **6.1 Definition.** Let $T = \bigl(T_t\bigr)_{t\in\mathbb{T}}$ be a conservative CP-semigroup on a unital C^*–algebra $\mathcal{B}$. A *weak Markov flow* of T is a pair $(\mathcal{A}, j)$, where a $\mathcal{A}$ is a pre–C^*–algebra and $j = \bigl(j_t\bigr)_{t\in\mathbb{T}}$ is a family of homomorphisms $j_t \colon \mathcal{B} \to \mathcal{A}$, fulfilling the Markov property (3.1)

Let I be a subset of $\mathbb{T}$. By $\mathcal{A}_I$ we denote the algebra generated by $\bigl\{j_t(b) \colon t \in I, b \in \mathcal{B}\bigr\}$. In particular, we set $\mathcal{A}_{t]} = \mathcal{A}_{[0,t]}$, $\mathcal{A}_{[t} = \mathcal{A}_{[t,\infty)}$, and $\mathcal{A}_\infty = \mathcal{A}_{[0,\infty)} = \bigcup\limits_{t\in\mathbb{T}} \mathcal{A}_{t]}$.

A *morphism* from a weak Markov flow $(\mathcal{A}, j)$ of T to a weak Markov flow $(\mathcal{C}, k)$ of T is a contractive $*$–algebra homomorphism $\alpha \colon \mathcal{A} \to \mathcal{C}$ fulfilling

$$\alpha \circ j_t = k_t \quad \text{for all} \quad t \in \mathbb{T}.$$

α is an *isomorphism*, if it is also an isomorphism between pre–C^*–algebras (i.e. α is isometric onto). The class consisting of weak Markov flows and morphisms among them forms a category.

In the original definition in [BP94] the family of projections $j_s(\mathbf{1})$ in (3.1) is replaced by a more general family of projections p_s, so that the Markov property reads $p_s j_t(b) p_s = j_s(T_{t-s}(b))$ $(s \le t)$. However, one easily checks that $p_s \ge j_s(\mathbf{1})$. (Putting $t = s$ and $b = \mathbf{1}$, we obtain $p_s j_s(\mathbf{1}) p_s = j_s(\mathbf{1})$. Multiplying this by $(\mathbf{1} - p_s) \in \widetilde{\mathcal{A}}$ from the left and from the right we obtain $j_s(\mathbf{1}) = p_s j_s(\mathbf{1})$ and $j_s(\mathbf{1}) = j_s(\mathbf{1}) p_s$, respectively.) Therefore, j fulfills (3.1).

Actually, in [BP94] the family p_t is required to be increasing. Proposition 6.2 shows that the $j_t(\mathbf{1})$ fulfill this requirement, automatically. After that, we always set $p_t := j_t(\mathbf{1})$. This is natural also in view of Proposition 6.8. The existence of conditional expectations φ_t onto $\mathcal{A}_{t]}$, guaranteed therein, usually, forms a part of the definitition of Markov process used by other authors; see e.g. [Acc78, AFL82, Küm85, Sau86].

incproj **6.2 Proposition.** *The $j_t(\mathbf{1})$ form an increasing family of projections, i.e. $j_s(\mathbf{1}) \le j_t(\mathbf{1})$ for $s \le t$.*

Proof. For arbitrary projections p, q with $pqp = p$ we have $qp = p = pq$ so that $p \le q$. (Indeed, consider the $|(\mathbf{1} - q)p|^2 = p(\mathbf{1} - q)^2 p = p(\mathbf{1} - q)p = 0$ in $\widetilde{\mathcal{A}}$. This implies $(\mathbf{1} - q)p = p - qp = 0$.) Using this, our assertion follows directly from (3.1). ■

Proposition 6.2 shows that the p_t form an approximate unit for the C^*–completion of $\mathcal{A}_\infty$. This shows, in particular, that $\mathcal{A}_\infty$ is non-unital, unless $p_t = \mathbf{1}$ for some $t \in \mathbb{T}$. In a faithful non-degenerate representation of $\mathcal{A}_\infty$ the p_t converge to $\mathbf{1}$ strongly.

faithful **6.3 Definition.** A weak Markov flow j is called *faithful*, if j_0 is injective.

Of course, the main goal in constructing a weak Markov flow is to recover T_t in terms of j_t. This is done by $p_0 j_t(b) p_0 = j_0(T_t(b))$ and, naturally, leads to the requirement that j_0 should be *injective*. Nevertheless, as the following remark shows, there are interesting examples of weak Markov flows where j_0 is not injective.

shiftM **6.4 Remark.** If j is a weak Markov flow, then also the time shifted family j^τ with $j^\tau_t = j_{t+\tau}$ for some fixed $\tau \in \mathbb{T}$ is a weak Markov flow. The j_τ are, in general, far from being injective. This shows existence of a non-faithful weak Markov flow. Of course, a trivial example is $j_t = 0$ for all t.

Now we are going to construct a univeral mapping $\mathfrak{T}$ very similar to the correlation kernels introduced in [AFL82]. We will see that $\mathfrak{T}$ and j_0 determine $(\mathcal{A}_\infty, j)$ completely. Moreover, $(\mathcal{A}_\infty, j)$ always admits a faithful representation on a suitable pre-Hilbert $j_0(\mathcal{B})$–module E^j (closely related to E as constructed in Theorem 5.8) as a minimal flow in the sense of Definition 3.1. This flow always extends to $\mathcal{B}^a(E^j)$ as a maximal flow on E^j. Both flows are determined by j_0 up to unitary equivalence and enjoy universal properties.

6.5 Lemma. *Denote by* $\mathbb{B} = \bigcup\limits_{n\in\mathbb{N}_0} (\mathbb{T}\times\mathcal{B})^n$ *the set of all finite tuples* $\big((t_1, b_1), \ldots, (t_n, b_n)\big)$ $(n \in \mathbb{N})$ *of pairs in* $\mathbb{T}\times\mathcal{B}$*. Let* V *be a vector space and* $\mathfrak{T}\colon \mathbb{B} \to V$ *a mapping, fulfilling*

$$\mathfrak{T}\big((t_1, b_1), \ldots, (s, a), (t, b), (s, c), \ldots, (t_n, b_n)\big) \\ = \mathfrak{T}\big((t_1, b_1), \ldots, (s, aT_{t-s}(b)c), \ldots, (t_n, b_n)\big), \quad (6.1)$$ T1

whenever $s \le t; a, b, c \in \mathcal{B}$*, and*

$$\mathfrak{T}\big((t_1, b_1), \ldots, (t_k, \mathbf{1}), \ldots, (t_n, b_n)\big) = \mathfrak{T}\big((t_1, b_1), \ldots, \widehat{(t_k, \mathbf{1})}, \ldots, (t_n, b_n)\big), \quad (6.2)$$ T2

whenever $t_{k-1} \le t_k$ $(1 < k)$*, or* $t_{k+1} \le t_k$ $(k < n)$*, or* $k = 1$*, or* $k = n$*.*

Then $\mathfrak{T}$ *is determined uniquely by the values* $\mathfrak{T}\big((0, b)\big)$ $(b \in \mathcal{B})$*. Moreover, the range of* $\mathfrak{T}$ *is contained in* $\mathsf{span}\,\mathfrak{T}\big((0, \mathcal{B})\big)$*.*

Proof. In a tuple $\big((t_1, b_1), \ldots, (t_n, b_n)\big) \in \mathbb{B}$ go to the position with maximal time t_m. By (6.1) we may reduce the length of this tuple by 2, possibly, after having inserted by (6.2) a $\mathbf{1}$ at a suitable time in the neighbourhood of (t_m, b_m). This procedure may be continued until the length is 1. If this is achieved, then we insert $(0, \mathbf{1})$ on both sides and, making again use of (6.1), we arrive at a tuple of the form $\big((0, b)\big)$. ■

6.6 Corollary. *Let* $(\mathcal{A}, j)$ *be a weak Markov flow of a conservative CP-semigroup* T*. Then the mapping* $\mathfrak{T}_j$*, defined by setting*

$$\mathfrak{T}_j\big((t_1, b_1), \ldots, (t_n, b_n)\big) = p_0 j_{t_1}(b_1) \ldots j_{t_n}(b_n) p_0,$$

is the unique mapping $\mathfrak{T}_j\colon \mathbb{B} \to j_0(\mathcal{B})$*, fullfilling (6.1), (6.2), and*

$$\mathfrak{T}_j\big((0, b)\big) = j_0(b). \quad (6.3)$$ T3

OCE **6.7 Corollary.** *The mapping* $\varphi_0\colon a \mapsto p_0 a p_0$ *defines a conditional expectation* $\mathcal{A}_\infty \to \mathcal{A}_0$*.*

tCE **6.8 Proposition.** *For all* $\tau \in \mathbb{T}$ *the mapping* $\varphi_\tau\colon a \mapsto p_\tau a p_\tau$ *defines a conditional expectation* $\mathcal{A}_\infty \to \mathcal{A}_\tau$*.*

Proof. Consider the time shifted weak Markov flow j^τ as in Remark 6.4. Since $j_0^\tau = j_\tau$, it follows by Corollary 6.7 that $p_\tau \bullet p_\tau$ defines a conditional expectation $\mathcal{A}_{[t} \to j_\tau(\mathcal{B})$.

Now consider a tuple in $\mathbb{B}$ and split it into subtuples which consist either totally of elements at times $\leq \tau$, or totally at times $> \tau$. At the ends of these tuples we may insert p_τ, so that the elements at times $> \tau$ are framed by p_τ. By the first part of the proof the product over such a subtuple (including the surrounding p_τ's) is an element of $j_\tau(\mathcal{B})$. The remaining assertions follow by the fact that p_τ is a unit for $\mathcal{A}_{t]}$. ■

6.9 Theorem. *There exists a unique mapping $\mathfrak{T}\colon \mathbb{B} \longrightarrow \mathcal{B}$, fullfilling (6.1), (6.2), and $\mathfrak{T}\big((0,b)\big) = b$ for all $b \in \mathcal{B}$. We call $\mathfrak{T}$ the* **correlation kernel** *of T.*

Proof. Suppose that j is a faithful weak Markov flow. Then the mapping $j_0^{-1} \circ \mathfrak{T}_j$ has the desired properties. Existence of a faithful weak Markov flow has been settled in Theorem 5.8. ■

6.10 Corollary. *Let j be a weak Markov flow. Then $\mathfrak{T}_j = j_0 \circ \mathfrak{T}$.*

6.11 Remark. In the sense of [Mur97] the module E from Theorem 5.8 may be considered as the *Kolmogorov decomposition* of the *positive definite $\mathcal{B}$–valued kernel* $\mathfrak{k}\colon \mathbb{B} \times \mathbb{B} \to \mathcal{B}$, defined by setting

$$\mathfrak{k}\Big(\big((t_n, b_n), \ldots, (t_1, b_1)\big)\,,\ \big((s_m, c_m), \ldots, (s_1, c_1)\big)\Big) \\ = \mathfrak{T}\big((t_1, b_1^*), \ldots, (t_n, b_n^*), (s_m, c_m), \ldots, (s_1, c_1)\big).$$

More generally, if $(\mathcal{A}, j)$ is a weak Markov flow, then the *GNS-module* E^j associated with j (see Definition 6.12 below) is the Kolmogorov decomposition for the positive definite kernel $j_0 \circ \mathfrak{k}$.

This interpretation throws a bridge to the reconstruction theorem in [AFL82], where $\mathfrak{k}$ is a usual $\mathbb{C}$–valued kernel, and the original construction of the minimal weak Markov flow in [BP94], which starts by writing down a positive definite kernel on $\mathbb{B} \times G$ (where G denotes a Hilbert space on which $\mathcal{B}$ is represented). Cf. also [Acc78] and [Bel85].

GNSj **6.12 Definition.** Let $(\mathcal{A}, j)$ be a weak Markov flow. Then by (E^j, ξ^j) we denote the GNS-representation of $\varphi_0\colon \mathcal{A}_\infty \to \mathcal{A}_0$. We call E^j the **GNS-module associated with** $(\mathcal{A}, j)$. Denote by $\alpha^j\colon \mathcal{A}_\infty \to \mathcal{B}^a(E^j)$ the canonical homomorphism. Obviously, $\alpha^j\colon (\mathcal{A}_\infty, j) \to (\alpha^j(\mathcal{A}_\infty), \alpha^j \circ j)$ is a morphism of weak Markov flows. We call $(\alpha^j(\mathcal{A}_\infty), \alpha^j \circ j)$ the **minimal** weak Markov flow associated with $(\mathcal{A}, j)$ and we call $(\mathcal{B}^a(E^j), \alpha^j \circ j)$ the **maximal** weak Markov flow associated with $(\mathcal{A}, j)$.

Observe that the minimal and the maximal weak Markov flow associated with a faithful flow $(\mathcal{A}, j)$ are essential flows in the sense of Definition 3.1. It is natural to ask under which conditions the representation of $\mathcal{A}_\infty$ on E^j is faithful or, more generally, extends to a faithful (isometric) representation of $\mathcal{A}$ on E^j. In other words, we ask under which conditions a weak Markov flow is isomorphic to an essential flow on a pre-Hilbert B–module. The following definition and proposition settle this problem. We leave a detailed analysis of similar questions for cyclic flows to future work. We mention, however, that satisfactory answers exist.

essdef **6.13 Definition.** A weak Markov flow $(\mathcal{A}, j)$ is called *essential*, if the ideal I_0 in $\mathcal{A}_\infty$ generated by p_0 is an ideal also in $\mathcal{A}$, and if I_0 is *essential* in the C^*–completion of $\mathcal{A}$ (i.e. for all $a \in \overline{\mathcal{A}}$ we have that $aI_0 = \{0\}$ implies $a = 0$).

Let us drop for a moment the condition in Definition 3.1 that j_0 be faithful.

essprop **6.14 Proposition.** *A weak Markov flow $(\mathcal{A}, j)$ is isomorphic to an essential weak Markov flow on E^j in the sense of Definition 3.1 (j_0 not necessarily faithful), if and only if it is essential in the sense of Definition 6.13. In this case also $\varphi(a) = p_0 a p_0$ defines a conditional expectation $\varphi\colon \mathcal{A} \to \mathcal{A}_0$.*

Proof. We have $\mathsf{span}\big(\mathcal{A}\mathcal{A}_\infty p_0\big) = \mathsf{span}\big(\mathcal{A}(\mathcal{A}_\infty p_0)p_0\big) \subset \mathsf{span}\big(\mathcal{A}I_0p_0\big) = \mathsf{span}\big(I_0p_0\big) = \mathcal{A}_\infty p_0$. Therefore, $\mathcal{A}_\infty p_0$ is a left ideal in $\mathcal{A}$ so that φ, indeed, takes values in $\mathcal{A}_0$. By construction φ is bounded, hence, extends to $\overline{\mathcal{A}}$. (Observe that $\mathcal{A}_0$ is the range of a C^*–algebra homomorphism and, therefore, complete.) Now our statement follows immediately by an application of Lemma B.1 to the extension of φ. ■

If j is essential, then we identify $\mathcal{A}$ as a pre–C^*–subalgebra of $\mathcal{B}^a(E^j)$. In this case, we write $(\mathcal{A}_\infty, j)$ and $(\mathcal{B}^a(E^j), j)$ for the minimal and the maximal weak Markov flow associated with $(\mathcal{A}, j)$, respectively. An essential weak Markov flow $(\mathcal{A}, j)$ lies in between the minimal and the maximal essential weak Markov flow associated with it, in the sense that $\mathcal{A}_\infty \subset \mathcal{A} \subset \mathcal{B}^a(E^j)$.

Observe that E^j is just $\overline{\mathcal{A}_\infty p_0}$ with cyclic vector p_0. If we weaken the cyclicity condition in Definition 3.1 to $E \subset \overline{\mathcal{A}_\infty p_0}$, then in order to have Proposition 6.14 it is sufficient to require that $\overline{I_0}$ is an essential ideal in $\overline{\mathcal{A}}$ (without requiring that I_0 is an ideal in $\mathcal{A}$).

Proposition 6.14 does not mean that φ_0 is faithful. In fact, as $\varphi_0(j_t(\mathbf{1}) - j_0(\mathbf{1})) = 0$, we see that φ_0 is faithful, if and only if $j_t(\mathbf{1}) = j_0(\mathbf{1})$ for all $t \in \mathbb{T}$. If j is also faithful, then we are precisely in the situation as described in Example 3.2.

The C^*–algebraic condition in Definition 6.13 seems to be out of place in our pre–C^*–algebraic framework for the algebra $\mathcal{A}$. In fact, we need it only in order to know that the GNS-representation of $\mathcal{A}$ is isometric. This is necessary, if we want that the E_0–semigroup ϑ in Theorem 5.8 extends to the completion of $\mathcal{B}^a(E)$; see Section 10. Example B.3 shows that the C^*–algebraic version is, indeed, indispensable.

Notice that there exist interesting non-essential weak Markov flows. For instance, tensor products of weak Markov flows with E_0–semigroups are rarely essential; see [Bha98a] for details.

By Observation B.2 I_0 may be identified with the compact operators $\mathcal{K}(E^j)$. The multiplier algebra of $\mathcal{K}(E^j)$ is $\mathcal{B}^a(\overline{E^j})$. In other words, $\mathcal{B}^a(\overline{E^j})$ is the biggest C^*–algebra, containing $\mathcal{K}(E^j)$ as an essential ideal. This justifies to say 'maximal essential weak Markov flow'.

6.15 Observation. Obviously, the minimal weak Markov flow $(\mathcal{A}_\infty, j)$ from Theorem 5.8 is essential, minimal, and faithful. For reasons which will become clear soon, and in accordance with Definition 3.1, we refer to $(\mathcal{A}_\infty, j)$ as *the* minimal weak Markov flow. Similarly, we refer to $(\mathcal{B}^a(E), j)$ as *the* maximal weak Markov flow.

6.16 Definition. For a (unital) C^*–algebra $\mathcal{B}$ we introduce the *homomorphism category* $\mathfrak{h}(\mathcal{B})$. The objects of $\mathfrak{h}(\mathcal{B})$ are pairs $(\mathcal{A}, j)$ consisting of a C^*–algebra $\mathcal{A}$ and a surjective homomorphism $j: \mathcal{B} \to \mathcal{A}$. A morphism $i: (\mathcal{A}, j) \to (\mathcal{C}, k)$ in $\mathfrak{h}(\mathcal{B})$ is a homomorphism $\mathcal{A} \to \mathcal{C}$, also denoted by i, such that $i \circ j = k$. Clearly, such a morphism exists, if and only if $\mathsf{ker}(j) \subset \mathsf{ker}(k)$. If there exists a morphism, then it is unique.

In the sequel, by (E, ξ) we always mean then GNS-module of the minimal weak Markov flow as constructed in Sections 4 and 5. Also j, the notions related to $\mathcal{A}_I$, and φ_τ refer to the minimal weak Markov flow. $(\mathcal{C}, k)$ stands for an essential weak Markov flow. $\mathcal{C}_I$ and related notions are defined similar to $\mathcal{A}_I$. (The flow k is not to be confused with the canonical mappings k_τ in Section 5.)

Mlem **6.17 Lemma.** *Let $(\mathcal{C}, k)$ be an essential weak Markov flow of T. Furthermore, denote by $(\mathcal{E}^k, \mathbf{1}^k)$ the GNS-construction of $k_0: \mathcal{B} \to \mathcal{C}_0 = k_0(\mathcal{B})$. Then $E^k = E \odot \mathcal{E}^k$ and $\xi^k = \xi \odot \mathbf{1}^k$. Moreover, in this identification we have*

$$k_t(b) = j_t(b) \odot \mathsf{id}\,. \tag{6.4}$$ Mhom

Proof. Clearly, $\mathcal{E}^k = k_0(\mathcal{B})$, when considered as a Hilbert $\mathcal{B}$–$k_0(\mathcal{B})$–module via $bk_0(b') := k_0(bb')$ and $\mathbf{1}^k = k_0(\mathbf{1})$. It follows that $E \odot \mathcal{E}^k$ is just E equipped with the new $\mathcal{C}_0$–valued inner product $\langle x, x' \rangle_k = k_0(\langle x, x' \rangle)$ divided by the kernel $\mathcal{N}$ of this inner product. $\xi \odot \mathbf{1}^k$ is just $\xi + \mathcal{N}$.

Let $x = j_{t_n}(b_n) \ldots j_{t_1}(b_1)\xi$ and $x' = j_{t'_m}(b'_m) \ldots j_{t'_1}(b'_1)\xi$ $(t_i, t'_j \in \mathbb{T}; b_i, b'_j \in \mathcal{B})$ be elements in E. Then

$$\langle x, x' \rangle = \mathcal{T}\big((t_1, b_1^*), \ldots, (t_n, b_n^*), (t'_m, b'_m) \ldots, (t'_1, b'_1)\big).$$

For $y = k_{t_n}(b_n) \ldots k_{t_1}(b_1)\xi^k$ and $y' = k_{t'_m}(b'_m) \ldots k_{t'_1}(b'_1)\xi^k$ in E^k we find

$$\langle y, y' \rangle = \mathcal{T}_k\big((t_1, b_1^*), \ldots, (t_n, b_n^*), (t'_m, b'_m) \ldots, (t'_1, b'_1)\big).$$

Therefore, by sending $x \odot \mathbf{1}^k$ to y we define a unitary mapping $u: E \odot \mathcal{E}^k \to E^k$. Essentially the same computations show that the isomorphism $\mathcal{B}^a(E \odot \mathcal{E}^k) \to \mathcal{B}^a(E^k), a \mapsto uau^{-1}$ respects (6.4). ■

6.18 Proposition. *Let $(\mathcal{C} = \mathcal{B}^a(E^k), k)$ and $(\mathcal{C}' = \mathcal{B}^a(E^{k'}), k')$ be two maximal weak Markov flows. Then there exists a morphism $\alpha: (\mathcal{C}, k) \to (\mathcal{C}', k')$, if and only if there exists a morphism $i: (\mathcal{C}_0, k_0) \to (\mathcal{C}'_0, k'_0)$. If i exists, then α is unique. In particular, $(\mathcal{C}, k)$ and $(\mathcal{C}', k')$ are isomorphic weak Markov flows, if and only if $(\mathcal{C}_0, k_0)$ and $(\mathcal{C}'_0, k'_0)$ are isomorphic objects in $\mathfrak{h}(\mathcal{B})$.*

Proof. If i does not exist, then there does not exist a morphism α. So let us assume that i exists. In this case we denote by $(\mathcal{E}^{kk'}, \mathbf{1}^{kk'})$ the GNS-construction of i. One easily checks that $\mathcal{E}^k \odot \mathcal{E}^{kk'} = \mathcal{E}^{k'}$ and $\mathbf{1}^k \odot \mathbf{1}^{kk'} = \mathbf{1}^{k'}$. Thus, $E^{k'} = E^k \odot \mathcal{E}^{kk'}$. By Observation 2.20 it follows that $\alpha: a \mapsto a \odot \mathsf{id}$ defines a contractive homomorphism $\mathcal{B}^a(E^k) \to \mathcal{B}^a(E^{k'})$. Clearly, we have $k'_t(b) = k_t(b) \odot \mathsf{id}$, so that α is a morphism of weak Markov flows.

If i is an isomorphism, then we may construct $\mathcal{E}^{k'k}$ as the GNS-module of i^{-1}. We find $E^{k'} \odot \mathcal{E}^{k'k} = E^k \odot \mathcal{E}^{kk'} \odot \mathcal{E}^{k'k} = E^k$. This enables us to reverse the whole construction, so that α is an isomorphism. The remaining statements are obvious. ■

6.19 Corollary. *Let $(\mathcal{C}, k)$ be an arbitrary weak Markov flow. Then the minimal and maximal weak Markov flows associated with $(\mathcal{C}, k)$ are determined up to isomorphism by the isomorphism class of $(\mathcal{C}_0, k_0)$ in $\mathfrak{h}(\mathcal{B})$.*

6.20 Corollary. *Let $(\mathcal{C}_0, k_0)$ be an object in $\mathfrak{h}(\mathcal{B})$. Then there exist a unique minimal and a unique maximal weak Markov flow extending k_0.*

Proof. Construct again the GNS-module $\mathcal{E}^k$ of k_0 and set $E^k = E \odot \mathcal{E}^k$. Then, obviously, (6.4) defines a maximal weak Markov flow $(\mathcal{B}^a(E^k), k)$ with a minimal weak Markov flow sitting inside. By the preceding corollary these weak Markov flows are unique. ■

The following theorem is proved by appropiate applications of the preceding results.

6.21 Theorem. *The maximal weak Markov flow $(\mathcal{B}^a(E), j)$ is the unique universal object in the category of maximal weak Markov flows. In other words, if $(\mathcal{C}, k)$ is another maximal weak Markov flow, then there exists a unique morphism $\alpha\colon (\mathcal{B}^a(E), j) \to (\mathcal{C}, k)$.*

The minimal weak Markov flow $(\mathcal{A}_\infty, j)$ is the unique universal object in the category of all essential weak Markov flows. In other words, if $(\mathcal{C}, k)$ is an essential weak Markov flow, then there exists a unique morphism $\alpha\colon (\mathcal{A}_\infty, j) \to (\mathcal{C}, k)$. Moreover, if $(\mathcal{C}, k)$ is minimal, then α is onto.

In this way we obtain a different proof of Corollary 5.5. (ϑ_τ is the morphism which sends j to j^τ.) Of course, also this proof is based on the factorization (5.1) so that there is nothing new in it.

Let $(\mathcal{C}, k)$ be an essential weak Markov flow. We could ask, whether the E_0–semigroup ϑ on $\mathcal{B}^a(E)$ gives rise to an E_0–semigroup on $\mathcal{B}^a(E^k)$ (or at least to an e_0–semigroup on $\mathcal{C}_\infty$). A necessary and sufficient condition is that the kernels of T_t should contain the kernel of k_0. (In this case, T_t gives rise to a completely positive mapping T^k_t on $k_0(\mathcal{B})$. Denote by $\mathcal{E}^k_t$ the GNS-module of T^k_t. It is not difficult to see that $E^k \odot \mathcal{E}^k_t$ carries a faithful representation of the time shifted weak Markov flow k^t, and that the mapping $a \mapsto a \odot \mathsf{id}$ sends the weak Markov flow k on E^k to the weak Markov flow k_t on $E^k \odot \mathcal{E}^k_t$. From this it follows that the time shift on $\mathcal{C}_\infty$ is contractive.) However, the following example shows that this condition need not be fufilled, even in the case, when $\mathcal{B}$ is commutative, and when T is uniformly continuous.

6.22 Example. Let $\mathcal{B} = \mathbb{C}^2$. By setting $T_t\binom{z_1}{z_2} = \frac{z_1+z_2}{2}\binom{1}{U} + e^{-t}\frac{z_1-z_2}{2}\binom{1}{-1}$ we define a conservative CP-semigroup T. We define a homomorphism $k\colon \mathbb{C}^2 \to \mathbb{C}$, by setting $k\binom{z_1}{z_2} = z_1$. Then $k\binom{0}{2} = 0$, but $k \circ T_t\binom{0}{2} = 1 - e^{-t} \neq 0$ (for $t \neq 0$).

For these reasons we dispense with an algebraic formulation of e_0–dilation and content ourselves with the module version in Section 3.

7 Units and cocycles

In Section 4 we started from a conservative CP-semigroup T on a unital C^*–algebra $\mathcal{B}$. We constructed a product system $E^\odot$ of pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules E_τ and a unit $\xi^\odot$ for this product system. This unit turned out to be unital and generating. In Section 5 we constructed an inductive limit E of the modules E_τ with the help of the unit. On E it was possible to realize a weak Markov flow $(\mathcal{B}^a(E), j)$ and an E_0–dilation ϑ of T.

In this section we reverse the proceeding and start with a product system and a unital unit. We construct an associated conservative CP-semigroup and investigate in how far the CP-semigroups are classified by such pairs. The results of Section 8 indicate how the constructions may be generalized to non-unital units bounded by 1, which correspond to general contractive CP-semigroups.

7.1 Definition. Let $\mathcal{B}$ be a unital C^*–algebra. Let $(E^\odot, \xi^\odot)$ be a pair consisting of a product system $E^\odot$ of pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules E_τ and a (unital) unit $\xi^\odot = \big(\xi_\tau\big)_{\tau\in\mathbb{T}}$. It is readily verified that $T = \big(T_\tau\big)_{\tau\in\mathbb{T}}$ with $T_\tau(b) = \langle\xi_\tau, b\xi_\tau\rangle$ defines a (conservative) CP-semigroup on $\mathcal{B}$. We call T the CP-semigroup *associated* with $(E^\odot, \xi^\odot)$.

Suppose the unit $\xi^\odot$ is unital. Then the family $\big(\gamma_{\tau\sigma}\big)_{\sigma\leq\tau}$ as defined in Section 5 provides an inductive limit over E_τ. We denote this inductive limit by $E^\xi = \limind_\xi E_\tau$ in order to indicate that it depends on the choice of the unit $\xi^\odot$. We say E^ξ is the inductive limit *associated* with $\xi^\odot$.

Again we find the factorization $E^\xi \odot E_\tau = E^\xi$ so that $\vartheta_\tau(a) = a \odot \mathsf{id}$ defines an E_0–semigroup on $\mathcal{B}^a(E^\xi)$, the E_0–semigroup *associated* with $\xi^\odot$. As $j_0 = |\xi\rangle b\langle\xi|$ acts faithfully on ξ, we find that $(\mathcal{B}^a(E^\xi), j)$ with $j_\tau = \vartheta_\tau \circ j_0$ defines a weak Markov flow of T, the weak Markov flow *associated* with $\xi^\odot$, and that ϑ is an E_0–dilation of T, the E_0–dilation *associated* with $\xi^\odot$.

7.2 Proposition. *Let T be the CP-semigroup associated with $(E^\odot, \xi^\odot)$. Furthermore, let $(E^0)^\odot$, $\xi_0^\odot$, and E^0 denote the product system, the unit, and the inductive limit, respectively, constructed from T as in Sections 4 and 5. Then for each $\tau \in \mathbb{T}$ the mapping*

$$u_\tau \colon b_n\xi_0^{t_n} \odot \ldots \odot b_1\xi_0^{t_1}b_0 \longmapsto b_n\xi_{t_n} \odot \ldots \odot b_1\xi_{t_1}b_0$$

$(t_n, \ldots, t_1) \in \mathbb{J}_\tau, b_i \in \mathcal{B})$ extends uniquely to a two-sided isometry $u_\tau \colon E_\tau^0 \to E_\tau$. In other words, the product system $(E^0)^\odot$ is isomorphic to a product subsystem of $E^\odot$. Of course, $u_\tau(\xi_0^\tau) = \xi_\tau$.

The mapping

$$u \colon k_\tau^0(x_\tau^0) \longmapsto k_\tau u_\tau(x_\tau^0)$$

$(\tau \in \mathbb{T}, x_\tau \in E_\tau^0)$ extends uniquely to an isometry $u \colon E^0 \to E$. In other words, E^0 is isomorphic to a submodule of E.

The product systems $(E^0)^\odot$ and $E^\odot$ are isomorphic, if and only if $\xi^\odot$ is generating for $E^\odot$. In this case also the weak Markov flows and the E_0–dilations constructed on E^0 and on E are the same (up to unitary isomorphism).

Proof. Clear. ■

We see that, given a certain (conservative) CP-semigroup T, then there is essentially one pair $(E^\odot, \xi^\odot)$ with a generating unit with which T is associated.

7.3 Theorem. *CP-semigroups are classified by pairs $(E^\odot, \xi^\odot)$ consisting of a product system $E^\odot$ and a generating unital unit $\xi^\odot$, up to isomorphism of the pairs.*

This is the classification of CP-semigroups by product systems *and* units. In [Arv89] Arveson classifies (normal, strongly continuous) E_0–semigroups on $\mathcal{B}(G)$ by product systems of Hilbert spaces up to *cocycle conjugacy.* In the sequel, we look in how far CP-semigroups are classified by their product sytems alone, and what *cocycle conjugacy* could mean in our context.

7.4 Definition. Let $\mathscr{B}$ be a unital C^*–algebra, let E be a pre-Hilbert $\mathscr{B}$–module, and let ϑ be an E_0–semigroup on $\mathcal{B}^a(E)$. A family $\mathfrak{u} = \big(\mathfrak{u}_\tau\big)_{\tau\in\mathbb{T}}$ of operators $\mathfrak{u}_\tau \in \mathcal{B}^a(E)$ is called a ***left (right) cocycle*** for ϑ, if for all $\sigma, \tau \in \mathbb{T}$

$$\mathfrak{u}_{\tau+\sigma} = \mathfrak{u}_\tau \vartheta_\tau(\mathfrak{u}_\sigma) \qquad \Big(\mathfrak{u}_{\tau+\sigma} = \vartheta_\tau(\mathfrak{u}_\sigma)\mathfrak{u}_\tau\Big).$$

A cocycle $\mathfrak{u}$ is called ***contractive, positive, unitary, isometric, partially isometric***, if $\mathfrak{u}_\tau$ is contractive, positive, unitary, isometric, partially isometric, respectively, for each $\tau \in \mathbb{T}$. A cocycle is called ***local***, if $\mathfrak{u}_\tau$ is in the relative commutant $\vartheta_\tau(\mathcal{B}^a(E))'$ of $\vartheta_\tau(\mathcal{B}^a(E))$ in $\mathcal{B}^a(E)$ for each $\tau \in \mathbb{T}$. (In this case $\mathfrak{u}$ is a left *and* a right cocycle.)

Let ϑ be the E_0–semigroup associated with a pair $(E^\odot, \xi^\odot)$. A cocycle $\mathfrak{u}$ for ϑ is called ***adapted***, if $p_\tau \mathfrak{u}_\tau p_\tau = \mathfrak{u}_\tau$ for each $\tau \in \mathbb{T}$ (cf. Proposition 5.9). In other words, $\mathfrak{u}_\tau$ is the image of the unique operator $u_\tau = k_\tau^* \mathfrak{u}_\tau k_\tau$ on E_τ under the embedding $k_\tau \bullet k_\tau^*\colon \mathcal{B}^a(E_\tau) \to \mathcal{B}^a(E)$.

Of course, $\mathfrak{u} = \big(\mathfrak{u}_\tau\big)_{\tau\in\mathbb{T}}$ is a left cocycle, if and only if $\mathfrak{u}^* = \big(\mathfrak{u}_\tau^*\big)_{\tau\in\mathbb{T}}$ is a right cocycle. Moreover, if $\mathfrak{u}$ is a unitary left cocycle for ϑ, then $\widehat{\vartheta}_\tau(a) = \mathfrak{u}_\tau \vartheta_\tau(a) \mathfrak{u}_\tau^*$ defines another E_0–semigroup on $\mathcal{B}^a(E)$. We provide the following lemma on local cocycles for later use in Section 14, where we establish an order isomorphism for partial orders defined on a certain set of local cocycles and and a certain set of CP-semigroups on a von Neumann algebra.

loccoc **7.5 Lemma.** *Let ϑ be the E_0–semigroup associated with a pair $(E^\odot, \xi^\odot)$. Let $\mathfrak{w}$ be a family of operators $\mathfrak{w}_\tau$ on E^ξ and define $w_\tau = k_\tau^* \mathfrak{w}_\tau k_\tau \in \mathcal{B}^a(E_\tau)$. Then $\mathfrak{w}$ is a local cocycle for ϑ, if and only if the following conditions are satisfied.*

1. *All w_τ are $\mathscr{B}$–$\mathscr{B}$–linear.*

2. *$\mathfrak{w}_\tau = \mathsf{id} \odot w_\tau$ in $\mathcal{B}^a(E^\xi \odot E_\tau) = \mathcal{B}^a(E^\xi)$.*

3. *$w_\sigma \odot w_\tau = w_{\sigma+\tau}$ for all $\sigma, \tau \in \mathbb{T}$.*

Of course, the w_τ are unique and $\mathfrak{w}$ is adapted.

Conversely, if w is a family of operators $w_\tau \in \mathcal{B}^a(E_\tau)$ fulfilling 1 and 3, then 2 defines a local cocycle $\mathfrak{w}$ for ϑ.

Proof. Recall that $k_\tau = \xi \odot \mathsf{id}_{E_\tau}$ and $k_\tau^* = \xi^* \odot \mathsf{id}_{E_\tau}$. Therefore, it is sufficient to consider the set-up where E is a pre-Hilbert $\mathscr{B}$–module with a unit vector ξ, where F is a pre-Hilbert $\mathscr{B}$–$\mathscr{B}$–module, and where $\mathfrak{w}$ is an operator on $E \odot F$ which commutes with all elements of $\mathcal{B}^a(E) \odot \mathsf{id} \subset \mathcal{B}^a(E \odot F)$.

Of course, for any $\mathscr{B}$–$\mathscr{B}$–linear mapping $w \in \mathcal{B}^a(F)$ the mapping $\mathsf{id} \odot w$ is well-defined and commutes with $a \odot \mathsf{id}$ for each $a \in \mathcal{B}^a(E)$.

Conversely, let $\mathfrak{w} \in \mathcal{B}^a(E \odot F)$ be in the relative commutant of $\mathcal{B}^a(E) \odot \mathsf{id}$. Set $w = (\xi^* \odot \mathsf{id})\mathfrak{w}(\xi \odot \mathsf{id}) \in \mathcal{B}^a(F)$. By $j(b) = |\xi\rangle b\langle\xi|$ we define a representation of $\mathcal{B}$ on E. Then $j(b) \odot \mathsf{id}$ is an element of $\mathcal{B}^a(E) \odot \mathsf{id}$ and, therefore, commutes with $\mathfrak{w}$. We find

$$bw = (\xi^* \odot \mathsf{id})(j(b) \odot \mathsf{id})\mathfrak{w}(\xi \odot \mathsf{id}) = (\xi^* \odot \mathsf{id})\mathfrak{w}(j(b) \odot \mathsf{id})(\xi \odot \mathsf{id}) = wb,$$

i.e. w is $\mathcal{B}$–$\mathcal{B}$–linear. In particular, $\mathsf{id} \odot w$ is a well-defined element of $\mathcal{B}^a(E \odot F)$. For arbitrary $x \in E$ and $y \in F$ we find

$$\begin{aligned}\mathfrak{w}(x \odot y) &= \mathfrak{w}(|x\rangle\langle\xi| \odot \mathsf{id})(\xi \odot y) = (|x\rangle\langle\xi| \odot \mathsf{id})\mathfrak{w}(\xi \odot y)\\ &= x \odot wy = (\mathsf{id} \odot w)(x \odot y),\end{aligned}$$

where $|x\rangle\langle\xi| \odot \mathsf{id}$ is an element of $\mathcal{B}^a(E) \odot \mathsf{id}$ and, therefore, commutes with $\mathfrak{w}$. In other words, $\mathfrak{w} = \mathsf{id} \odot w$.

Therefore, there is a one-to-one correspondence between operators $\mathfrak{w}$ in the commutant of $\mathcal{B}^a(E) \odot \mathsf{id}$ and $\mathcal{B}$–$\mathcal{B}$–linear operators in $\mathcal{B}^a(F)$. Applying this to $F = E_\tau$ we see that a family $\big(\mathfrak{w}_\tau\big)$ of mappings in the commutant of $\mathcal{B}^a(E) \odot \mathsf{id}_{E_\tau}$ is a cocycle, if and only if the corresponding family $\big(w_\tau\big)$ fulfills 3. ■

Let us return to the problem of finding the right notion of cocycle conjugacy. Notice that the members $\mathfrak{u}_\tau$ of an adapted right cocycle $\mathfrak{u}$ are necessarily of the form $\mathfrak{u}_\tau = |k_\tau\zeta_\tau\rangle\langle\xi|$ where ζ_τ are the unique elements $k_\tau^*\mathfrak{u}_\tau\xi \in E_\tau$. Indeed, by the cocycle property we have $\mathfrak{u}_\tau = \vartheta_0(\mathfrak{u}_\tau)\mathfrak{u}_0 = \mathfrak{u}_\tau\mathfrak{u}_0$. By adaptedness we have $\mathfrak{u}_0 = \mathfrak{u}_0 p_0$ and $\mathfrak{u}_\tau = p_\tau\mathfrak{u}_\tau$. Hence, $\mathfrak{u}_\tau = k_\tau k_\tau^*\mathfrak{u}_\tau\mathfrak{u}_0|\xi\rangle\langle\xi| = |k_\tau\zeta_\tau\rangle\langle\xi|$.

adacoc **7.6 Proposition.** *Let ϑ be the E_0–semigroup associated with a pair $(E^\odot, \xi^\odot)$. Then by setting $\zeta_\tau = k_\tau^*\mathfrak{u}_\tau\xi$ we establish a one-to-one correspondence between adapted right cocycles $\mathfrak{u}$ for ϑ and units $\zeta^\odot = \big(\zeta_\tau\big)$.*

Proof. Let $\mathfrak{u}$ be an adapted right cocycle. Then

$$\zeta_{\sigma+\tau} = k_{\sigma+\tau}^*\mathfrak{u}_{\sigma+\tau}\xi = k_{\sigma+\tau}^*\vartheta_\tau(\mathfrak{u}_\sigma)\mathfrak{u}_\tau\xi = k_{\sigma+\tau}^*(\mathfrak{u}_\sigma\xi \odot k_\tau^*\mathfrak{u}_\tau\xi) = k_\sigma^*\mathfrak{u}_\sigma\xi \odot k_\tau^*\mathfrak{u}_\tau\xi = \zeta_\sigma \odot \zeta_\tau,$$

i.e. $\zeta^\odot$ is a unit.

Conversely, let $\zeta^\odot$ be a unit and set $\mathfrak{u}_\tau = |k_\tau\zeta_\tau\rangle\langle\xi|$. Then

$$\vartheta_\tau(\mathfrak{u}_\sigma)\mathfrak{u}_\tau\xi = (\mathfrak{u}_\sigma \odot \mathsf{id})(\xi \odot \zeta_\tau) = k_{\sigma+\tau}(\zeta_\sigma \odot \zeta_\tau) = \mathfrak{u}_{\sigma+\tau}\xi.$$

Moreover, $\mathfrak{u}_{\sigma+\tau}$ is 0 on the orthogonal complement $(\mathbf{1} - |\xi\rangle\langle\xi|)E$ of ξ. In other words, $\mathfrak{u}_{\sigma+\tau} = \vartheta_\tau(\mathfrak{u}_\sigma)\mathfrak{u}_\tau$ so that the $\mathfrak{u}$ is an adapted right cocycle for ϑ. ■

coccondef **7.7 Definition.** If in the situation of Proposition adacoc 7.6 the cocycle is contractive, we say the CP-semigroup S associated with $\zeta^\odot$ is *cocycle subconjugate* to the CP-semigroup T associated with $\xi^\odot$.

If both $\xi^\odot$ and $\zeta^\odot$ are generating and unital, then we say S is *cocycle conjugate* to T.

coco **7.8 Theorem.** *Cocycle conjugacy is an equivalence relation among conservative CP-semigroups on $\mathcal{B}$, and CP-semigroups are classified by their product systems up to cocycle conjugacy.*

Proof. Of course, classifying CP-semigroups by their product systems is an equivalence relation. If S is cocycle conjugate to T, then S and T have the same product system $E^{\odot}$, but possibly different unital generating units $\xi^{\odot}$ and $\zeta^{\odot}$. Hence, Proposition 7.6 tells us that two CP-semigroups with the same product system, indeed, are cocycle conjugate. ■

Notice that our cocycles appearing in the definition of cocycle conjugacy, in general, are partial isometries. The cocycle conjugacy used by Arveson is through unitary cocylces. We see, however, in the following theorem that in the case of E_0–semigroups our cocycles are unitaries, automatically. In other words, it is the additional structure of E_0–semigroups (compared with a conservative CP-semigroup) which leads to unitary cocycles (cf. also Example 3.2). In Section 13 we will see that in the case of normal E_0–semigroups on $\mathcal{B}(G)$ our product systems of Hilbert modules are in one-to-one correspondence with Arveson's product systems of Hilbert spaces so that the classification of Arveson and ours coincide. More precisely, we will see that Arveson's product system sits inside our product system and, conversely, determines the structure of our product system completely; see Corollary 13.14.

7.9 Theorem. *Let ϑ and ϑ' be E_0–semigroups on $\mathcal{B}$ which are cocycle conjugate (as CP-semigroups). Then the unique right cocycle $\mathfrak{u}$ with respect to ϑ providing this equivalence is a unitary cocycle.*

Conversely, if $\mathfrak{u}$ is a unitary right cocycle with respect to ϑ, then the E_0–semigroup $\big(\mathfrak{u}_\tau^*\vartheta_\tau\mathfrak{u}_\tau\big)$ is cocycle conjugate to ϑ (as CP-semigroup).

Proof. Recall from Example 3.2 that $p_\tau = \mathbf{1}$. So any cocycle is adapted. And by Proposition 7.6 a right cocycle $\mathfrak{u}$ providing cocycle conjugacy of two E_0–semigroups is unique.

We encourage the reader to check that for a given E_0–semigroup ϑ on $\mathcal{B}$ the Hilbert $\mathcal{B}$–modules $E_\tau = \mathcal{B}$ equipped with the left multiplication $b.b' = \vartheta_\tau(b)b'$, indeed, form a product system. Of course, the elements $\xi_\tau = \mathbf{1}$ form a generating unital unit for this product system. The inductive limit provided by this unit is again $E = \mathcal{B}$ with cyclic vector $\xi = \mathbf{1}$. If we construct the maximal dilation as described in Section 5 we recover nothing but the original E_0–semigroup acting on $\mathscr{B}^a(E) = \mathscr{B}^a(\mathcal{B}) = \mathcal{B}$. All these assertions follow from the fact that $\mathcal{B}$ as a right module has a module basis which consists of one element $\mathbf{1}$, and any right linear mapping on $\mathcal{B}$ is determined by its value at $\mathbf{1}$.

Let ϑ' be another E_0–semigroup on $\mathcal{B}$ with the same product system E_τ and unit ξ'_τ. Of course, $\xi'_\tau \neq \mathbf{1}$ in the above identification, unless $\vartheta_\tau = \vartheta'_\tau$. The mapping $|\xi'_\tau\rangle\langle\xi_\tau|$ is nothing but multiplication with $\xi'_\tau \in E_\tau = \mathcal{B}$ from the left. ($\langle\xi_\tau, \bullet\rangle\colon E_\tau = \mathcal{B} \to \mathcal{B}$ is just the identity mapping.) It is an isometry as ξ'_τ has length $\sqrt{\langle\xi'_\tau, \xi'_\tau\rangle} = \mathbf{1}$. It is surjective, because E_τ is generated as a right module by ξ'_τ. (Otherwise E_τ was not isomorphic to the corresponding member in the product system for ϑ'.) In other words, $\xi'_\tau \in \mathcal{B}$ is a unitary. Observing that k_τ is nothing but the identification mapping $E_\tau = \mathcal{B} \to \mathcal{B} = E$, we find that also the lifting

$$k_\tau|\xi'_\tau\rangle\langle\xi_\tau|k_\tau^* = |k_\tau\xi'_\tau\rangle\langle k_\tau\xi_\tau| = |k_\tau\xi'_\tau\rangle\langle\xi| = \mathfrak{u}_\tau$$

of $|\xi'_\tau\rangle\langle\xi_\tau|$ from $\mathscr{B}^a(E_\tau) = \mathcal{B}$ to $\mathscr{B}^a(E) = \mathcal{B}$ is a unitary.

Conversely, let $\mathfrak{u}$ be a unitary right cocycle with respect to ϑ, and set $\vartheta'_\tau = \mathfrak{u}^*_\tau \vartheta_\tau \mathfrak{u}_\tau$. Interpret $\mathfrak{u}_\tau$ as an element of $\mathcal{B} = \mathcal{B}^a(E_\tau)$. Then for $b \in \mathcal{B}, b' \in E_\tau = \mathcal{B}$ we find

$$b.(\mathfrak{u}_\tau b') = \vartheta_\tau(b)\mathfrak{u}_\tau b' = \mathfrak{u}_\tau \vartheta'_\tau(b)b'.$$

In other words, the mapping which sends $b' \in E'_\tau$ (where $E'^{\odot}$ is the product system of ϑ') to $\mathfrak{u}_\tau b' \in E_\tau$ is a two-sided isomorphism $E'_\tau \to E_\tau$. So ϑ and ϑ' have the same product system. ■

7.10 Remark. The innocuous looking identifications in the preceding proof, actually, require some comments to avoid confusion. In fact, all modules appearing there are isomorphic as right Hilbert $\mathcal{B}$–modules to $\mathcal{B}$. This isomorphism even includes the cyclic vector $\mathbf{1}$ contained in $\mathcal{B}$. Indeed, also E_τ with the cyclic vector ξ'_τ is isomorphic to $\mathcal{B}$ with the cyclic vector $\mathbf{1}$. In other words, even the mapping $\xi_\tau \mapsto \xi'_\sigma$ extends uniquely as an isomorphism of right Hilbert modules. It is the left multiplication which distinguishes the different modules.

The decisive assumption of cocycle conjugacy is that there exists a unitary on E_τ (i.e. a unitary in $\mathcal{B}$) which intertwines ϑ_τ and ϑ'_τ. (This is the meaning of isomorphism of product systems.) However, even in the case of automorphisms this assumption is not true in general. Let $E = \mathcal{B}$ be the Hilbert $\mathcal{B}$–$\mathcal{B}$–module $\mathcal{B}$ with the natural left multiplication by elements of $\mathcal{B}$, and let $E_\alpha = \mathcal{B}$ be the Hilbert $\mathcal{B}$–$\mathcal{B}$–module $\mathcal{B}$ where $\mathcal{B}$ acts via an automorphism α. If there exists an intertwining unitary for these two-sided Hilbert modules, this means that α is inner. Of course, in general not all automorphisms of a C^*–algebra are inner. In fact, classifying all E_α up to two-sided unitary isomorphism is nothing but classifying the automorphisms of $\mathcal{B}$ up to (inner) unitary equivalence. We can reformulate the contents of Theorem 7.9 as follows. Two E_0–semigroups on $\mathcal{B}$ are cocycle conjugate, if and only if there exists a family (α_τ) of inner automorphisms of $\mathcal{B}$ such that $\vartheta'_\tau = \alpha_\tau \circ \vartheta_\tau$.

We should emphasize that our classification starts from the assumption that there are two CP-semigroups. They belong to the same class, if they have the same product system (i.e. they are cocycle conjugate). In contrast with Arveson's result in [Arv90] that (under certain measurability and separability assumptions) each Arveson product system can be obtained from an E_0–semigroup, in our case the analogue statement for conservative CP-semigroups is not true already in the case $\mathcal{B} = \mathbb{C}$. Indeed, a unital unit in a product system of Hilbert spaces is generating, if and only if this product system consists of one-dimensional Hilbert spaces.

If a product system of pre-Hilbert modules arises from our construction in Section 4, then it has a unital generating unit. In this context, and also in order to find a satisfactory definition of *type* and *index* of a product system of pre-Hilbert modules, we consider it as an interesting problem, to determine all units of a given product system. In particular, we ask whether for each $\mathcal{B}$ there are product systems without any unit. At least for $\mathcal{B} = \mathbb{C}$ Powers has shown in [Pow87] that there exists a type III E_0–semigroup on $\mathcal{B}(G)$ which means precisely that the associated product system of Hilbert spaces has no units.

In Section 11 we will see that (after suitable completion) any conservative CP-semigroup with bounded generator is cocycle subconjugate to the trivial semigroup. In Section 14 we will see that (also after suitable completion) any normal CP-semigroup S on a von

Neumann algebra which is *dominated* by a conservative normal CP-semigroup T is cocycle subconjugate to T through a local cocycle.

8 The non-conservative case

non1

In this section we study the procedures from Sections 4 and 5 in the case, when $\mathcal{B}$ still is unital, however, T may be non-conservative. We still assume that all T_t are contractive and, of course, that $T_0 = \mathsf{id}$.

There are two essentially different ways to proceed. The first way as done in [Bha96] uses only the possibly non-conservative CP-semigroup T. Although the first inductive limit still is possible, the second inductive limit breaks down, and the inner product must be defined *a priori*. The second way to proceed uses the unitization $\widetilde{T}$ on $\widetilde{\mathcal{B}}$ as indicated in Paragraph 2.4.

Here we mainly follow the second approach. In other words, we do the constructions of Sections 4 and 5 for the conservative CP-semigroup $\widetilde{T}$. As a result we obtain a pre-Hilbert $\widetilde{\mathcal{B}}$–$\widetilde{\mathcal{B}}$–module $\widetilde{E}$, a cyclic vector $\widetilde{\xi}$, a weak Markov flow $\widetilde{j}$ acting on $\widetilde{E}$, and an E_0–semigroup $\widetilde{\vartheta}$ on $\mathcal{B}^a(\widetilde{E})$. The restriction of $\widetilde{\vartheta}$ to the submodule E which is generated by $\widetilde{\xi}$ and $\widetilde{\vartheta}_{\mathbb{T}} \circ \widetilde{j}_0(\mathcal{B})$ is *cum grano salis* a dilation of T. We will see that the (linear) codimension of E in $\widetilde{E}$ is 1.

Recall that $\widetilde{\mathcal{B}} = \mathcal{B} \oplus \mathbb{C}\widetilde{\mathbf{1}}$, and that ($\mathcal{B}$ is unital)

$$\mathcal{B} \oplus \mathbb{C} \longrightarrow \widetilde{\mathcal{B}}, \quad (b, \mu) \longmapsto (b - \mu\mathbf{1}) \oplus \mu\widetilde{\mathbf{1}}$$

is an isomorphism of C^*–algebras, where $\mathcal{B} \oplus \mathbb{C}$ is the usual C^*–algebraic direct sum. In [BP94] the unitization has been introduced in the picture $\mathcal{B} \oplus \mathbb{C}$. In the sequel, we will switch between the pictures $\widetilde{\mathcal{B}}$ and $\mathcal{B} \oplus \mathbb{C}$ according to our needs.

We start by reducing the GNS-construction $(\widetilde{\mathcal{E}}_t, \widetilde{\xi}_t)$ for $\widetilde{T}_t$ to the GNS-construction $(\mathcal{E}_t, \xi_t)$ for T_t. Since $\mathcal{B}$ is an ideal in $\widetilde{\mathcal{B}}$, we may consider $\mathcal{E}_t$ also as a pre-Hilbert $\widetilde{\mathcal{B}}$–$\widetilde{\mathcal{B}}$–module. Since T_t is not necessarily conservative, ξ_t is not necessarily a unit vector. However, $\langle \xi_t, \xi_t \rangle \leq \mathbf{1}$ as T_t is contractive. Denote by $\widehat{\xi}_t$ the positive square root of $\widetilde{\mathbf{1}} - \langle \xi_t, \xi_t \rangle$ in $\widetilde{\mathcal{B}}$. Denote by $\widehat{\mathcal{E}}_t = \widehat{\xi}_t \widetilde{\mathcal{B}}$ the right ideal in $\widetilde{\mathcal{B}}$ generated by $\widehat{\xi}_t$ considered as a right pre-Hilbert $\widetilde{\mathcal{B}}$–module (see Example 2.8). By defining the left multiplication $b\widehat{\xi}_t = 0$ for $b \in \mathcal{B}$ and $\widetilde{\mathbf{1}}\widehat{\xi}_t = \widehat{\xi}_t$, we turn $\widehat{\mathcal{E}}_t$ into a pre-Hilbert $\widetilde{\mathcal{B}}$–$\widetilde{\mathcal{B}}$–module. We set $\widetilde{\mathcal{E}}_t = \mathcal{E}_t \oplus \widehat{\mathcal{E}}_t$ and $\widetilde{\xi}_t = \xi_t \oplus \widehat{\xi}_t$. One easily checks that $(\widetilde{\mathcal{E}}_t, \widetilde{\xi}_t)$ is the GNS-construction for $\widetilde{T}_t$.

xirules **8.1 Observation.** Among many other simple relations connecting ξ_t, $\widetilde{\xi}_t$, and $\widehat{\xi}_t$ with the central projections $\mathbf{1}$, and $\widetilde{\mathbf{1}} - \mathbf{1}$ like e.g. $\mathbf{1}\widetilde{\xi}_t = \xi_t$, $(\widetilde{\mathbf{1}} - \mathbf{1})\widetilde{\xi}_t = \widehat{\xi}_t$, or $\widetilde{\xi}_t(\widetilde{\mathbf{1}} - \mathbf{1}) = (\widetilde{\mathbf{1}} - \mathbf{1})\widehat{\xi}_t(\widetilde{\mathbf{1}} - \mathbf{1})$, the relation

$$\widehat{\xi}_t\mathbf{1} \;=\; (\widetilde{\mathbf{1}} - \mathbf{1})\widetilde{\xi}_t\mathbf{1} \;=\; \widetilde{\mathbf{1}}\widetilde{\xi}_t\mathbf{1} - \mathbf{1}\widetilde{\xi}_t\mathbf{1} \;=\; \widetilde{\xi}_t\mathbf{1} - \mathbf{1}\widetilde{\xi}_t$$

is particularly crucial for the proof of Theorem 8.4.

Notice that (like for any pre-Hilbert $\widetilde{\mathcal{B}}$–module; cf. Observation 9.1) the mapping $\omega_t \colon x \mapsto x(\widetilde{\mathbf{1}} - \mathbf{1})$ defines a projection on $\widetilde{\mathcal{E}}_t$. We denote $\Omega_t = \widetilde{\xi}_t(\widetilde{\mathbf{1}} - \mathbf{1})$ and $\widetilde{b} = (b, \mu)$ in the picture $\mathcal{B} \oplus \mathbb{C}$. The following proposition is verified easily by looking at the definition of $\widehat{\xi}_t$ and by the rules in Observation 8.1.

8.2 Proposition. *Ω_t may be identified with the element $\widetilde{\mathbf{1}} - \mathbf{1}$ in the right ideal $\widehat{\mathcal{E}}_t$ in $\widetilde{\mathcal{B}}$. We have*

$$\widetilde{b}\,\widetilde{\xi}_t\,\widetilde{b}'(\widetilde{\mathbf{1}} - \mathbf{1}) = \Omega_t\mu\mu' = (\widetilde{\mathbf{1}} - \mathbf{1})\Omega_t\mu\mu'.$$

In particular, ω_t is a projection onto $\mathbb{C}\Omega_t$. The orthogonal complement of Ω_t may be considered as a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module.

Doing the constructions of Sections 1st 4 and 2nd 5 for $\widetilde{T}$, we refer to $\widetilde{E}_t$, $\widetilde{E}_\tau = \liminf_{\mathfrak{t}\in\mathbb{J}_\tau} \widetilde{E}_\mathfrak{t}$, and $\widetilde{E} = \liminf_\tau \widetilde{E}_\tau$. Also other ingredients of these constructions are indicated by the *dweedle*. Letters without *dweedle* refer to analogue quantities coming from T_t. For instance, we already remarked that the first inductive limit may be performed also for non-conservative CP-semigroups. We obtain a family of pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules E_τ as inductive limits of pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules $E_\mathfrak{t}$ ($\mathfrak{t} \in \mathbb{J}_\tau$). These modules form a product system and the vectors $\xi^\tau \in E^\tau$ still form a generating unit. This unit is, however, not necessarily unital.

By sending $i_\mathfrak{t}(b_n\xi_{t_n} \odot \ldots \odot b_1\xi_{t_1}b_0)$ to $\widetilde{i}_\mathfrak{t}(b_n\xi_{t_n} \odot \ldots \odot b_1\xi_{t_1}b_0)$ ($\mathfrak{t} = (t_n, \ldots, t_1) \in \mathbb{J}_\tau; b_n, \ldots, b_0 \in \mathcal{B}$) we establish a $\mathcal{B}$–$\mathcal{B}$–linear isometric embedding $E_\tau \to \widetilde{E}_\tau$. In this identification we conclude from

$$\mathbf{1}\widetilde{b}_n\widetilde{\xi}_{t_n} \odot \ldots \odot \widetilde{b}_1\widetilde{\xi}_{t_1}\widetilde{b}_0 = b_n\xi_{t_n} \odot \ldots \odot b_1\xi_{t_1}b_0$$

that $\mathbf{1}\widetilde{E}_\tau = E_\tau$. We remark that here and in the remainder of this section it does not matter, whether we consider the tensor products as tensor products over $\mathcal{B}$ or over $\widetilde{\mathcal{B}}$. By definition of the tensor product the inner products coincide, so that the resulting pre-Hilbert modules are isometrically isomorphic. As long as the inner product takes values in $\mathcal{B}$ we are free to consider them as $\mathcal{B}$–modules or as $\widetilde{\mathcal{B}}$–modules.

8.3 Proposition. *Let $\tau \in \mathbb{T}$ and set $\Omega^\tau = \widetilde{\xi}^\tau(\widetilde{\mathbf{1}} - \mathbf{1}) \in \widetilde{E}_\tau$. Then $\widetilde{i}_\mathfrak{t}(\Omega_{t_n} \odot \ldots \odot \Omega_{t_1}) = \Omega^\tau$ for all $\mathfrak{t} \in \mathbb{J}_\tau$. Moreover, the Ω^τ form a unit for $\widetilde{E}_\tau$.*

Set $\widehat{\xi}^\tau = (\widetilde{\mathbf{1}} - \mathbf{1})\widetilde{\xi}^\tau \in \widetilde{E}_\tau$. Then $\widetilde{i}_{(\tau)}\widehat{\xi}_\tau = \widehat{\xi}^\tau$ for all $\tau \in \mathbb{T}$.

Set $\Omega = \widetilde{\xi}(\widetilde{\mathbf{1}} - \mathbf{1}) \in \widetilde{E}$. Then $\widetilde{k}_\tau\Omega^\tau = \Omega$ for all $\tau \in \mathbb{T}$.

PROOF. From Observation xirules 8.1 we find

$$\Omega^\tau = \widetilde{\xi}^\tau(\widetilde{\mathbf{1}} - \mathbf{1}) = \widetilde{i}_\mathfrak{t}(\widetilde{\xi}_{t_n} \odot \ldots \odot \widetilde{\xi}_{t_1})(\widetilde{\mathbf{1}} - \mathbf{1}) = \widetilde{i}_\mathfrak{t}(\Omega_{t_n} \odot \ldots \odot \Omega_{t_1})$$

from which all assertions of the first part follow. The second and third part are proved in an analogue manner. ■

Clearly, we have $\widetilde{E}(\widetilde{\mathbf{1}} - \mathbf{1}) = \mathbb{C}\Omega$. Denote by $E = \widetilde{E}\mathbf{1}$ the orthogonal complement of this submodule and denote by $\xi = \widetilde{\xi}\mathbf{1}$ the component of $\widetilde{\xi}$ in E. Denote by $\omega\colon x \mapsto x\mathbf{1}$ the projection in $\mathcal{B}^a(\widetilde{E})$ onto E. We may consider E as a pre-Hilbert $\mathcal{B}$–module.

ortho **8.4 Theorem.** *The operators in $\widetilde{j}_\mathbb{T}(\mathcal{B})$ leave invariant E, i.e. $\widetilde{j}_\tau(b)$ and ω commute for all $\tau \in \mathbb{T}$ and $b \in \mathcal{B}$. For the restrictions $j_\tau(b) = \widetilde{j}_\tau(b) \restriction E$ the following holds.*

1. *E is generated by $j_\mathbb{T}(\mathcal{B})$ and ξ.*
2. *The j_τ fulfill the Markov property (Markov 3.1) and, of course, j_0 is faithful.*

3. *The restriction of* $\widetilde{\vartheta}$ *to* $\mathcal{B}^a(E)$ *defines an* E_0*–semigroup* ϑ *on* $\mathcal{B}^a(E)$*, which fulfills* $\vartheta_\tau \circ j_\sigma = j_{\sigma+\tau}$*. Clearly,* ϑ *leaves invariant* $\mathcal{A}_\infty = \operatorname{\mathsf{span}} j_{\mathbb{T}}(\mathcal{B})$*.*

8.5 Remark. Extending our definitions to the non-conservative (contractive) case in an obvious way, we say $(\mathcal{A}_\infty, j)$ is the (unique) minimal weak Markov flow of T. Also the definition of essential generalizes, so that $(\mathcal{B}^a(E), j)$ is the (unique) maximal weak Markov flow of T. Extending the result in [Bha96] where an e_0–dilation on $\mathcal{A}_\infty$ was constructed, we say that ϑ is an E_0–dilation of T.

Proof of Theorem 8.4. Observe that $\widetilde{j}_\tau(\mathbf{1})\widetilde{E} = \widetilde{k}_\tau \mathbf{1} \widetilde{E}_\tau = \widetilde{\xi} \odot E_\tau$. By $\widehat{E} \subset E$ we denote the linear span of all these spaces. Clearly, $\widehat{E}$ is a pre-Hilbert $\mathcal{B}$–module. Moreover, all $\widetilde{j}_\tau(b)$ leave invariant $\widehat{E}$. We will show that $\widehat{E} = E$, which implies that also E is left invariant by $\widetilde{j}_\tau(b)$.

$\widetilde{E}$ is spanned by the subspaces $\widetilde{\xi} \odot \widetilde{E}_\tau$, so that E is spanned by the subspaces $\widetilde{\xi} \odot \widetilde{E}_\tau \mathbf{1}$. The space $\widetilde{E}_\tau \mathbf{1}$ is spanned by elements of the form $x_\tau = \widetilde{i}_{\mathfrak{t}}(\widetilde{b}_n \widetilde{\xi}_{t_n} \odot \ldots \odot \widetilde{b}_1 \widetilde{\xi}_{t_1} b_0)$. For each $1 \le k \le n$ we may assume that either $\widetilde{b}_k = b_k \in \mathcal{B}$ or $\widetilde{b}_k = \mu_k(\widetilde{\mathbf{1}} - \mathbf{1})$. If $\widetilde{b}_k = \mu_k(\widetilde{\mathbf{1}} - \mathbf{1})$ for some k, then we may assume that $\widetilde{b}_\ell = \mu_\ell(\widetilde{\mathbf{1}} - \mathbf{1})$ for all $\ell \ge k$. (Otherwise, the expression is 0.) We have to distinguish two cases. Firstly, all $\widetilde{b}_k$ are in $\mathcal{B}$. Then x_τ is in E_τ so that $\widetilde{\xi} \odot x_\tau \in \widehat{E}$. Secondly, there is a unique smallest number $1 \le k \le n$, such that $\widetilde{b}_\ell = \mu_\ell(\widetilde{\mathbf{1}} - \mathbf{1})$ for all $\ell \ge k$. Then it is easy to see that

$$x_\tau = \widetilde{i}_{(\sigma_3,\sigma_2,\sigma_1)}(\Omega_{\sigma_3} \odot \widehat{\xi}_{\sigma_2} \odot x_{\sigma_1}), \quad \text{i.e.} \quad \widetilde{\xi} \odot x_\tau = \widetilde{\xi} \odot \widehat{\xi}^{\sigma_2} \odot x_{\sigma_1}$$

where $\sigma_1 + \sigma_2 + \sigma_3 = \tau$ and $x_{\sigma_1} \in E_{\sigma_1}$. By Observation 8.1, we obtain

$$\begin{aligned} \widetilde{\xi} \odot \widehat{\xi}^{\sigma_2} \odot x_{\sigma_1} &= \widetilde{\xi} \odot (\widetilde{\xi}^{\sigma_2}\mathbf{1} - \mathbf{1}\widetilde{\xi}^{\sigma_2}) \odot x_{\sigma_1} = \widetilde{\xi} \odot (\widetilde{\xi}^{\sigma_2}\mathbf{1} - \mathbf{1}\xi^{\sigma_2}) \odot x_{\sigma_1} \\ &= (\widetilde{j}_{\sigma_1}(\mathbf{1}) - \widetilde{j}_{\sigma_2+\sigma_1}(\mathbf{1}))\widetilde{\xi} \odot x_{\sigma_1}, \end{aligned}$$

so that also in this case $\widetilde{\xi} \odot x_\tau \in \widehat{E}$. Therefore, $E \subset \widehat{E}$.

1. It remains to show that $\widetilde{\xi} \odot x_\tau$ for $x_\tau \in E_\tau$ can be expressed by applying a suitable collection of operators $j_t(b)$ to ξ and building linear combinations. But this follows inductively by the observation that $j_t(b)(\widetilde{\xi} \odot x_s) = \widetilde{\xi} \odot b\xi_{t-s} \odot x_s$ for $t > s$.
2. This assertion follows by applying ω to the Markov property of $\widetilde{j}$.
3. Clear. ■

8.6 Remark. Considering $\mathcal{B}$ as a C^*–subalgebra of $\mathcal{B}(G)$ for some Hilbert space G and doing the Stinespring construction for E as described in Example 2.16, we obtain the results from [Bha96]. It is quite easy to see that the inner products of elements in E_τ (that is for fixed τ) coincide, when tensorized with elements in the *initial space* G, with the inner products given in [Bha96]. We owe the reader to compute the inner products of elements in $\widetilde{k}_\tau E_\tau \subset E$ and $\widetilde{k}_\sigma E_\sigma \subset E$ for $\tau \neq \sigma$. Let $x_\tau \in E_\tau$ and $y_\sigma \in E_\sigma$ and assume without loss of generality that $\sigma < \tau$. We find

$$\langle \widetilde{\xi} \odot x_\tau, \widetilde{\xi} \odot y_\sigma \rangle = \langle \widetilde{\xi} \odot x_\tau, \widetilde{\xi} \odot \widetilde{\xi}^{\tau-\sigma} \odot y_\sigma \rangle = \langle x_\tau, \widetilde{\xi}^{\tau-\sigma} \odot y_\sigma \rangle = \langle x_\tau, \xi^{\tau-\sigma} \odot y_\sigma \rangle.$$

(In the last step we made use of $\mathbf{1}x_\tau = x_\tau$ and $\mathbf{1}\widetilde{\xi}^{\tau-\sigma} = \xi^{\tau-\sigma}$.) This shows in full correspondence with [Bha96] that an element in E_σ has to be lifted to E_τ by "inserting a **1** at time τ", before we can compare it with an element in E_τ. This lifting is done by tensorizing $\xi_{\tau-\sigma}$. As this operation is no longer an isometry, the second inductive limit breaks down in the non-conservative case. Cf. also Remark A.8.

nonex **8.7 Example.** Now we study in detail the most simple non-trivial example. We start with the non-conservative CP-semigroup $T_t \colon z \mapsto e^{-t}z$ on $\mathbb{C}$. Here the product system $E_\tau = \mathbb{C}$ consists of one-dimensional Hilbert spaces and the unit consists of the vectors $\xi^\tau = e^{-\frac{\tau}{2}} \in E_\tau$.

For the unitization we find it more convenient to consider $\mathbb{C}^2$ rather than $\widetilde{\mathbb{C}}$. The mappings $\widetilde{T}_t \colon \mathbb{C}^2 \to \mathbb{C}^2$ are given by $\widetilde{T}_t \binom{a}{b} = b \binom{1}{U} + (a-b)\binom{e^{-t}}{0}$. The first component corresponds to the original copy of $\mathbb{C}$, whereas the second component corresponds to $\mathbb{C}(\widetilde{\mathbf{1}} - \mathbf{1})$.

We continue by writing down $\widetilde{E}$ and $\widetilde{E}_\tau$, showing afterwards that these spaces are the right ones. (To be precise we are dealing rather with their completions. But, by Section 10 this difference is not too important.) We define the Hilbert $\mathbb{C}^2$–module $\widetilde{E}$ and its inner product by

$$\widetilde{E} = L^2(\mathbb{R}^+) \oplus \mathbb{C}\Omega \quad \text{and} \quad \left\langle \binom{f}{\mu}, \binom{g}{\nu} \right\rangle = \binom{\langle f,g\rangle}{\overline{\mu}\nu}.$$

The inner product already determines completely the right multiplication by elements of $\mathbb{C}^2$ to be the obvious one.

Let us define $e_\tau \in L^2(\mathbb{R}^+)$ by setting $e_\tau(t) = \chi_{[\tau,\infty)}(t)e^{-\frac{t}{2}}$. Observe that $\langle e_\tau, e_\tau\rangle = e^{-\tau}$. We define the Hilbert $\mathbb{C}^2$–submodule $\widetilde{E}_\tau$ of $\widetilde{E}$ by $\widetilde{E}_\tau = L^2(0,\tau) \oplus \mathbb{C}e_\tau \oplus \mathbb{C}\Omega$. (Observe that, indeed, $\langle L^2(0,\tau), e_\tau\rangle = \{0\}$.) We turn $\widetilde{E}_\tau$ into a Hilbert $\mathbb{C}^2$–$\mathbb{C}^2$–module by defining the left multiplication

$$\binom{a}{b} \begin{pmatrix} g \\ \beta \\ \nu \end{pmatrix} = b \begin{pmatrix} g \\ 0 \\ \nu \end{pmatrix} + a \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix}.$$

We define the homomorphism $\widetilde{j}_\tau \colon \mathbb{C}^2 \to \mathcal{B}^a(\widetilde{E})$ by, first, projecting down to the submodule $\widetilde{E}_\tau$, and then, applying the left multiplication of $\mathbb{C}^2$ on $\widetilde{E}_\tau \subset \widetilde{E}$. Clearly, the $\widetilde{j}_\tau$ form a weak Markov flow of $\widetilde{T}$.

Let us define the *shift* $\mathcal{S}_\tau$ on $L^2(\mathbb{R}^+)$ by setting $\mathcal{S}_\tau f(t) = f(t-\tau)$, if $t \geq \tau$, and $\mathcal{S}_\tau f(t) = 0$, otherwise. (Observe that also here $\langle L^2(0,\tau), \mathcal{S}_\tau L^2(\mathbb{R}^+)\rangle = \{0\}$.) One easily checks that the mappings

$$\binom{f}{\mu} \odot \begin{pmatrix} g \\ \beta \\ \nu \end{pmatrix} \longmapsto \binom{\mu g + e^{-\frac{\tau}{2}}\beta \mathcal{S}_\tau f}{\mu\nu} \quad \text{and} \quad \begin{pmatrix} f \\ \alpha \\ \mu \end{pmatrix} \odot \begin{pmatrix} g \\ \beta \\ \nu \end{pmatrix} \longmapsto \begin{pmatrix} \mu g + e^{-\frac{\tau}{2}}\beta \mathcal{S}_\tau f \\ \alpha\beta \\ \mu\nu \end{pmatrix} \tag{8.1}$$ C2fac

define isomorphisms $\widetilde{E} \odot \widetilde{E}_\tau \to \widetilde{E}$ and $\widetilde{E}_\sigma \odot \widetilde{E}_\tau \to \widetilde{E}_{\sigma+\tau}$, respectively. Remarkably enough, no completion is necessary here.

It remains to show that $\widetilde{E}$ (and, similarly, also $\widetilde{E}_\tau$) is generated by $\widetilde{\xi} = \binom{e_0}{1}$ and $\widetilde{j}_{\mathbb{T}}(\mathbb{C}^2)$. But this is simple, as we have $\widetilde{j}_0\binom{0}{1}\widetilde{\xi} = \Omega$ and $\left(\widetilde{j}_\tau\binom{1}{0} - \widetilde{j}_\sigma\binom{1}{0}\right)\widetilde{\xi} = \binom{\chi_{[\sigma,\tau]}e_0}{0}$ for $\sigma < \tau$. Therefore, we obtain all functions which consist piecewise of arcs of the form $e^{-\frac{t}{2}}$. Clearly, these functions form a dense subspace of $L^2(\mathbb{R}^+)$. Until now we, tacitly, have assumed to speak about Hilbert modules. It is, however, clear that the arcwise exponentials form an algebraically invariant subset.

In this example we see in an extreme case that the product system of a non-conservative CP-semigroup T may be *blown up* considerably, when changing to its unitization $\widetilde{T}$. Notice that the original one-dimensional product system of T is present in the middle component $\begin{pmatrix}0\\ \alpha\\ 0\end{pmatrix}\odot\begin{pmatrix}0\\ \beta\\ 0\end{pmatrix}=\begin{pmatrix}0\\ \alpha\beta\\ 0\end{pmatrix}$ in the factorization by (8.1). (Recall that $\left\|\begin{pmatrix}0\\ \beta\\ 0\end{pmatrix}\right\|^2=|\beta|^2\,e^{-\tau}$ depends on τ.) Responsible for the blow up is the part $\widehat{\xi}_t\mathbf{1}$ of $\widehat{\xi}_t$ which lies in $\mathcal{B}$. If T was already conservative, then this part is 0.

9 A classical process of operators on E

Our construction of a weak Markov flow is essentially non-commutative. The reason for this is that by definition $j_\tau(\mathbf{1})$ is a projection (at least in non-trivial examples) which "levels out" whatever happened before "in the future of τ". As a consequence, $j_\tau(b)$ and $j_\sigma(b)$ have no chance to commute in general. Indeed, for $\sigma<\tau$ we find

$$j_\tau(b)j_\sigma(b)x\odot x_{\tau-\sigma}\odot x_\sigma=\xi\odot b\xi^{\tau-\sigma}\odot\langle\xi\odot\xi^{\tau-\sigma}b^*,x\odot x_{\tau-\sigma}\rangle x_\sigma, \tag{9.1a}$$

whereas

$$j_\sigma(b)j_\tau(b)x\odot x_{\tau-\sigma}\odot x_\sigma=\xi\odot\xi^{\tau-\sigma}b\odot\langle\xi\odot b^*\xi^{\tau-\sigma},x\odot x_{\tau-\sigma}\rangle x_\sigma. \tag{9.1b}$$

Since b and $\xi^{\tau-\sigma}$ do not commute, unless T is the trivial semigroup, Equations (9.1a,9.1b) describe different elements of E.

If we restrict ourselves to the center of $\mathcal{B}$, then the weak Markov flow j can be modified as shown in [Bha93] to a commutative flow k called the *central flow*. If the initial algebra $\mathcal{B}$ is commutative to begin with, then the flow k can be interpreted as the classical Markov process obtained by the *Daniell-Kolmogorov construction*. Central flows play a crucial role in [AP96]. In this section we recover k as a process of operators on E. This example, almost a triviality now, illustrates once again the power of the module approach. (The central flow k appears only in this section and should not be confused with the canonical mappings $k_\tau\colon E_\tau\to E$.)

The approach is based on the following simple observation, which we already made use of in Section 8.

9.1 Observation. Let E be a pre-Hilbert $\mathcal{B}$–module and b in the *center* $C_{\mathcal{B}}(\mathcal{B})$ of $\mathcal{B}$, i.e. b commutes with all elements of $\mathcal{B}$. Then by setting $\omega_b x=xb$ $(x\in E)$, we define an element of $\mathcal{B}^a(E)$.

Now let T be a conservative CP-semigroup on $\mathcal{B}$. Let E be the pre-Hilbert $\mathcal{B}$–module as constructed in Sections 4 and 5. We define $k_0(b)=\omega_b$ $(b\in C_{\mathcal{B}}(\mathcal{B}))$ and $k_\tau=\vartheta_\tau\circ k_0$.

9.2 Proposition. *k_0 is an isomorphism onto the center of* $\mathcal{B}^a(E)$.

Proof. (Cf. also Lemma 7.5.) Clearly, k_0 maps into the center. So let c be in the center of $\mathcal{B}^a(E)$. Then

$$\langle\xi,c\xi\rangle b=\langle\xi,c\xi\rangle\langle\xi,j_0(b)\xi\rangle=\langle\xi,\xi\rangle\langle\xi,cj_0(b)\xi\rangle=\langle\xi,j_0(b)c\xi\rangle=b\langle\xi,c\xi\rangle$$

for all $b\in\mathcal{B}$, i.e. $\langle\xi,c\xi\rangle\in C_{\mathcal{B}}(\mathcal{B})$. Now let $x\in E$. Then

$$cx=cx\langle\xi,\xi\rangle=x\langle\xi,c\xi\rangle=k_0(\langle\xi,c\xi\rangle)x,$$

i.e. $c=k_0(\langle\xi,c\xi\rangle)$ so that k_0, indeed, is onto. ∎

9.3 Theorem. *The process $k = \big(k_\tau\big)_{\tau\in\mathbb{T}}$ is commutative (i.e. $[k_\mathbb{T}(C_\mathcal{B}(\mathcal{B})), k_\mathbb{T}(C_\mathcal{B}(\mathcal{B}))] = \{0\}$) and $\langle\xi, k_\tau(b)\xi\rangle = \langle\xi, j_\tau(b)\xi\rangle = T_\tau(b)$ for all $\tau \in \mathbb{T}, b \in C_\mathcal{B}(\mathcal{B})$. In particular, if $T_\mathbb{T}(C_\mathcal{B}(\mathcal{B})) \subset C_\mathcal{B}(\mathcal{B})$, then k is a classical Markov process.*

Proof. Clearly, $k_0(C_\mathcal{B}(\mathcal{B}))$ commutes with $k_\tau(C_\mathcal{B}(\mathcal{B})) \subset \mathcal{B}^a(E)$. The remaining statements follow by time shift. ■

The explicit action of k_τ is

$$k_\tau(b)x \odot x_\tau = xb \odot x_\tau. \tag{9.2}$$

Let us have a closer look at the difference between j and k. Both $j_\tau(b)$ and $k_\tau(b)$ let act the algebra element b at time τ. This can be seen explicitly by the observation that the actions of $j_\tau(b)$ and $k_\tau(b)$ restricted to the submodule $\xi \odot E_\tau$ coincide. In other words, both $j_\tau(b)$ and $k_\tau(b)$ can be thought of as the left multiplication of E_τ, first, lifted to $\xi \odot E_\tau \subset E$ and, then, extended to the whole of E. It is this extension which makes the difference. $j_\tau(b)$ is extended just by 0 to the orthogonal complement of $\xi \odot E_\tau$ in E. Correspondingly, $j_\tau(\mathbf{1})$ projects down the future $t > \tau$ to the presence $t = \tau$. Whereas $k_\tau(b)$ inserts b at time τ without changing the future part x of $x \odot x_\tau$. Therefore, all k_τ are unital homomorphisms.

A look at Equation (9.2) reminds us of the *ampliation* $\mathsf{id} \odot l_b$ of the operator of left multiplication $l_b \colon x_\tau \mapsto bx_\tau$ on E_τ by b to the tensor product $E \odot E_\tau$. We emphasize, however, that in contrast to $a \odot \mathsf{id}$ (see Observation 2.20), a mapping $\mathsf{id} \odot a$ on a tensor product of pre-Hilbert modules, in general, only exists, if a is $\mathcal{B}$–$\mathcal{B}$–linear. (This is the case, for instance, for ω_b, if $b \in C_\mathcal{B}(\mathcal{B})$. Cf. also Lemma 7.5.) The problem of how to find dilations to unital homomorphisms is also in the background of Section 11.

10 The C^*–case

Until now we considered product systems of pre-Hilbert modules. All definitions were understood algebraically. This was possible, essentially, because we were able to write down the mappings $\gamma_{\sigma\tau}$ and their adjoints explicitly on the algebraic domain. Unlike on Hilbert spaces, where existence of adjoints of bounded operators (or, equivalently, projections onto closed subspaces) is always guaranteed, the approach by Hilbert modules, forced us to find the adjoint in a different way. Retrospectively, this way turned out to be more effective. In principle, by Stinespring construction it is also possible to interpret the whole construction in terms of pre-Hilbert spaces. However, it seems impossible to see the contents of the crucial Observation 2.18 directly. In particular, in the Hilbert space approach there is no natural way to distinguish the algebra $\mathcal{B}^a(E)$, where the E_0–dilation ϑ lives, as a subalgebra of $\mathcal{B}(E \odot G)$.

In the following section we are going to consider quantum stochastic calculus. Since in calculus we are concerned with limits of operators, the spaces on which the operators act should be complete. Here we need completions for the first time essentially.

Let us repeat the facts which assure that we may complete all pre-Hilbert modules. Thanks to the fact that we are dealing with CP-semigroups on a C^*–algebra $\mathcal{B}$, all pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules are contractive. We may complete them to Hilbert $\mathcal{B}$–$\mathcal{B}$–modules. All j_t are representations of $\mathcal{B}$, therefore, they are contractions. Therefore, by Observation

2.6, $\mathcal{A}_\infty$ acts boundedly on E, i.e. we may complete also E. The C^*–norm of the pre–C^*–algebra $\mathcal{A}_\infty$ is just the operator norm. By Observation 2.20, the time shift ϑ_τ is contractive. In other words, we may complete $\mathcal{A}_\infty$ to a C^*–subalgebra of $\mathcal{B}^a(\overline{E})$. The E_0–semigroup ϑ extends to $\mathcal{B}^a(\overline{E})$ and leaves invariant $\overline{\mathcal{A}}_\infty$.

Modifiying the definitions to Hilbert modules in an obvious manner and taking into account Proposition A.10, we obtain "completed" versions of the results in Sections 4 – 7. We collect the most important.

C*thm **10.1 Theorem.** *Let T be a conservative CP-semigroup on a unital C^*–algebra $\mathcal{B}$.*

1. *The family $E^\odot = \big(\overline{E}_\tau\big)_{\tau\in\mathbb{T}}$ forms a product system of Hilbert modules, i.e. $\overline{E}_\sigma \bar{\odot} \overline{E}_\tau = \overline{E}_{\sigma+\tau}$.*
2. *The family $\xi^\odot = \big(\xi^\tau\big)_{\tau\in\mathbb{T}}$ forms a unital generating unit for this product system and $\langle\xi^\tau, b\xi^\tau\rangle = T_\tau(b)$.*
3. *The inductive limit $\overline{E}$ over $\overline{E}_\tau$ fulfills $\overline{E} \bar{\odot} \overline{E}_\tau = \overline{E}$ and $\xi \odot \xi^\tau = \xi$.*
4. *By setting $\vartheta_\tau(a) = a \bar{\odot} \mathsf{id}$ we define an E_0–semigroup $\vartheta = \big(\vartheta_\tau\big)_{\tau\in\mathbb{T}}$ of strict endomorphisms of $\mathcal{B}^a(\overline{E})$. Setting $j_0(b) = |\xi\rangle b\langle\xi| \in \mathcal{B}^a(\overline{E})$, we find that ϑ is* the *maximal dilation of T. In other words, by setting $j_\tau = \vartheta_\tau \restriction \mathcal{B}$, we find* the *maximal weak Markov flow $(j, \mathcal{B}^a(\overline{E}))$ of T.*
5. *We have $\vartheta_\tau \circ j_\sigma = j_{\sigma+\tau}$ so that ϑ leaves invariant $\overline{\mathcal{A}}_\infty$. In other words, $\vartheta \restriction \overline{\mathcal{A}}_\infty$ is* the *minimal dilation of T and $(j, \overline{\mathcal{A}}_\infty)$ is* the *minimal weak Markov flow of T.*

contsg **10.2 Theorem.** *Let T be a completely positive, conservative C_0–semigroup on $\mathcal{B}$ (i.e. $\mathbb{T} = \mathbb{R}^+$ and for each $b \in \mathcal{B}$ the mapping $t \mapsto T_t(b)$ is continuous). Then ϑ is strictly continuous (i.e. $\tau \mapsto \vartheta_\tau(a)x$ is continuous for all $a \in \mathcal{B}^a(\overline{E})$ and $x \in \overline{E}$).*

Proof. First, observe that the mapping $\mathfrak{s}^r_\tau \colon x \mapsto x \odot \xi^\tau$ is a contraction $E \to E$. The family $x \odot \xi^\tau$ depends continuously on τ. (On the dense subset E this follows from the fact that the correlation kernel $\mathfrak{T}$ in Section 6 depends jointly continuously on all time arguments. By contractivity of $\mathfrak{s}^r_\tau$ this extends to the whole of $\overline{E}$.) Now we easily see that for each $a \in \mathcal{B}^a(\overline{E})$ and for each $x \in \overline{E}$

$$
\begin{aligned}
ax - \vartheta_\tau(a)x = ax - ax \odot \xi^\tau + ax \odot \xi^\tau - \vartheta_\tau(a)x &= (ax) - (ax) \odot \xi^\tau + \vartheta_\tau(a)(x \odot \xi^\tau - x) \\
&= (\mathsf{id} - \mathfrak{s}^r_\tau)(ax) + \vartheta_\tau(a)(\mathfrak{s}^r_\tau - \mathsf{id})(x)
\end{aligned}
$$

is small for τ sufficiently small. Replacing a by $\vartheta_t(a)$ we obtain continuity at all times t. ■

10.3 Remark. Since $\vartheta_t \circ j_0(\mathbf{1}) = j_t(\mathbf{1})$ is an increasing family of projections, ϑ is in general not a C_0–semigroup.

Belrem **10.4 Remark.** Of course, $\mathfrak{s}^r_\tau$ is not an element of $\mathcal{B}^r(E)$, therefore, certainly neither adjointable, nor isometric (unless T is trivial). In particular, passing to the Stinespring construction (Example 2.16), $\mathfrak{s}^r_\tau$ will never be implemented by an operator in $\mathcal{B}(\overline{H})$. It follows that the j_τ (or better, the images of j_τ in $\mathcal{B}(\overline{H})$) do not form a stationary process in the sense of [Bel85]. In the Hilbert space picture obtained by Stinespring construction, in general, there is no time shift like $\mathfrak{s}^r_\tau$, acting directly on the Hilbert space.

11 The time ordered Fock module and dilations on the full Fock module

In this section we discuss to some extent, in how far it is possible to find *unital* dilations to E_0–semigroups, in the sense that the homomorphisms j_t are unital. This problem has been settled completely in [Sau86]. The solution does, however, not preserve continuity properties. Most recently, the problem of dilation to a strongly continuous E_0–semigroup has been solved in [GS99] in the case, when T has a bounded generator, with the help of a quantum stochastic calculus. This calculus is constructed on a symmetric Fock module as defined in [Ske98] and generalizes the calculus on the symmetric Fock space [HP84] in the notations of [Par92]. The goal of this section is to construct explicitly the product system of T. Our construction makes use of the calculus on the full Fock module developed in [Ske99]. This calculus is a direct generalization of the calculus on the full Fock space developed in [KS92].

Throughout this section we speak of product systems of Hilbert modules as explained in Section 10. Tensor products and direct sums are assumed to be completed in the norm topology. T is a conservative CP-semigroup on $\mathcal{B}$ with a bounded generator.

Let H be a Hilbert space. The best-known example of a product systems of Hilbert spaces is the family $\Gamma^{\otimes} = \big(\Gamma_\tau\big)_{\tau\in\mathbb{R}^+}$ of symmetric Fock spaces $\Gamma_\tau = \Gamma(L^2([0,\tau],H))$. Of course, the vacuum vectors $\Omega_\tau \in \Gamma_\tau$ form a unital unit for this product system, and $\Gamma = \Gamma(L^2(\mathbb{R}^+,H))$ may be thought of as the inductive limit of Γ_τ provided by the unit $\big(\Omega_\tau\big)_{\tau\in\mathbb{R}^+}$.

Notice that, following our approach and contrary to the usual conventions, in the factorization $\Gamma = \Gamma\otimes\Gamma_\tau$ we have to write the future on the left. Since this order is forced by the module approach, it seems appropiate to rethink the usual conventions. Thanks to the particularly simple $\mathbb{C}$–$\mathbb{C}$–module structure of Hilbert spaces we have two additional properties. Firstly, unlike on E, on Γ we also have a left action of the C^*–algebra $\mathbb{C}$ which is faithful and unital. Secondly, in a tensor product of such $\mathbb{C}$–$\mathbb{C}$–modules the order of the factors may be exchanged. Hencforth, in this particularly simple case it is possible to extend an operator a on Γ_τ to $\Gamma\otimes\Gamma_\tau$ via ampliation $\mathsf{id}\otimes a$. Whereas our extension $j_\tau(z)$ of the left multipliction by an algebra element $z\in\mathbb{C}$ corresponds to $|\Omega\rangle\langle\Omega|\otimes z\,\mathsf{id}$. Of course, this dilation of the conservative CP-semigroup $T_t\colon z\mapsto z$ is not minimal, because out of Ω, the j_t cannot create more than $\mathbb{C}\Omega$.

It is well-known that the symmetric Fock space Γ may be identified with a subspace of the full Fock space $\mathcal{F}(L^2(\mathbb{R}^+,H))$, the *time ordered Fock space* $\mathcal{F}^0(L^2(\mathbb{R}^+,H))$. The n–particle sector of $\mathcal{F}^0(L^2(\mathbb{R}^+,H))$ consists of those functions $F\colon(\mathbb{R}^+)^n\to H^{\otimes n}$ which are 0, unless the argument $(t_n,\ldots,t_1)$ is ordered decreasingly. See, for instance, [Sch93] for a proof based on exponential vectors, or [Bha98b] for a proof based on number vectors. We will see that also the time ordered Fock modules *until time* τ form a product system. Unlike the symmetric Fock module [Ske98] whose construction is based on the requirement that the one-particle sector is a *centered* Hilbert module (see Section 13), the time ordered Fock module may be constructed for modules of functions with values in an arbitrary two-sided Hilbert module as one-particle sector.

11.1 Definition [Pim97, Spe98]. Let $\mathcal{B}$ be a unital C^*–algebra and let F be a Hilbert

$\mathcal{B}$–$\mathcal{B}$–module. The *full Fock module* is defined as $\mathcal{F}(F) = \bigoplus_{n\in\mathbb{N}_0} F^{\odot n}$. By the *vacuum* ξ we mean the element $\mathbf{1} \in \mathcal{B} = F^{\odot 0}$.

For $x \in F$, we define the *creator* $\ell^*(x)$ and the *annihilator* $\ell(x)$ in $\mathcal{B}^a(\mathcal{F}(F))$, by setting

$$\ell^*(x)x_n \odot \ldots \odot x_1 = x \odot x_n \odot \ldots \odot x_1 \qquad\qquad \ell^*(x)b = xb$$
$$\ell(x)x_n \odot \ldots \odot x_1 = \langle x, x_n\rangle x_{n-1} \odot \ldots \odot x_1 \qquad\qquad \ell(x)b = 0.$$

11.2 Remark. Clearly, $\ell^*(x)$ and $\ell(x)$ are a pair of adjoint operators. Pimsner [Pim97] shows that, like the Cuntz algebras, the C^*–algebra generated by all $\ell^*(x)$ is *cum grano salis* determined by the relations $\ell(x)\ell^*(y) = \langle x, y\rangle$ where the algebra element acts on $\mathcal{F}(F)$ via left multiplication.

Let I be a measurable subset of $\mathbb{R}^+$ (or any other polish measure space). By $L^2(I, F)$ (or L^2_I) we mean the completion of the exterior tensor product $L^2(I) \otimes F$ with inner product $\langle f \otimes x, g \otimes y\rangle = \langle x, y\rangle \int_I \overline{f(t)}g(t)\,dt$ and the obvious module operations. We think of elements of $L^2(I, F)$ as functions on I with values in F. Of course, $L^2(I, F)^{\odot n} = L^2(I^n, F^{\odot n})$. We use the notations $L^2_{[s,t)} = L^2([s, t), F)$ $(0 \le s \le t \le \infty)$, $L^2_{t]} = L^2_{[0,t)}$, $L^2_{[t} = L^2_{[t,\infty)}$, and $L^2 = L^2_{[0,\infty)}$. Furthermore, we set $\mathcal{F}_I = \mathcal{F}(L^2(I, F))$ and use the same notations as for L^2. $\mathcal{F}$ and $\mathcal{F}_{[t}$ are isomorphic by the *time shift* $\mathfrak{s}_t \colon \mathcal{F} \to \mathcal{F}_{[t}$ in an obvious way.

The family $\big(\mathcal{F}_{t]}\big)_{t\in\mathbb{R}^+}$ does not form a product system. We have, however, factorizations like $\mathcal{F} = \mathcal{F}_{[t} \odot \big(\xi \oplus L^2_{t]} \odot \mathcal{F}\big)$. Defining the *time shift endomorphism* $\mathcal{S}_t$ on $\mathcal{B}^a\big(\mathcal{F}(L^2(\mathbb{R}^+, F))\big)$ by setting $\mathcal{S}_t(a) = \mathfrak{s}_t a \mathfrak{s}_t^{-1} \odot \mathsf{id}$ in the above identification, it is not difficult to check that the family $\mathcal{S} = \big(\mathcal{S}_t\big)_{t\in\mathbb{R}^+}$ forms an E_0–semigroup.

11.3 Definition. Let $\Delta_n \in L^2((\mathbb{R}^+)^n)$ $(n \in \mathbb{N})$ denote the indicator function of the (unbounded) n–simplex $\big\{(t_n, \ldots, t_1) \in (\mathbb{R}^+)^n \colon t_n \ge \ldots \ge t_1\big\}$ and set $\Delta_0 = 1$. Letting act Δ_n as a multiplication operator on $(L^2_I)^{\odot n}$ $(I \subset \mathbb{R}^+)$, we define a projection.

The *time ordered Fock module* $\mathcal{F}^0 \subset \mathcal{F}$ is defined, by setting $\mathcal{F}^0 = \big(\bigoplus_{n\in\mathbb{N}_0} \Delta_n\big)\mathcal{F}$. We use similar notations as for $\mathcal{F}_I$.

tofm **11.4 Theorem.** *The family $(\mathcal{F}^0)^\odot = \big(\mathcal{F}^0_{t]}\big)_{t\in\mathbb{R}^+}$ of time ordered Fock modules forms a product system of Hilbert modules. The family $\xi^\odot = \big(\xi_t\big)_{t\in\mathbb{R}^+}$ with $\xi_t = \xi$ forms a unital unit for this product system. The inductive limit for this unit is $\mathcal{F}^0$. In particular, we have $\mathcal{F}^0 = \mathcal{F}^0 \odot \mathcal{F}^0_{t]}$, and the associated E_0–semigroup is just $\mathcal{S} \restriction \mathcal{B}^a(\mathcal{F}^0)$. Moreover, $\mathcal{S} \restriction \mathcal{B}^a(\mathcal{F}^0)$ is a dilation of the trivial CP-semigroup on $\mathcal{B}$.*

Proof. Once established the first assertion, the remaining ones are obvious. So let us show that the $\mathcal{F}^0_{t]}$ form a product system.

First, observe that the time shift $\mathfrak{s}_t$ sends $\mathcal{F}^0_{s]}$ onto $\mathcal{F}^0_{[t,t+s]}$. In particular, the projection onto $\mathcal{F}^0$ commutes with all $\mathfrak{s}_t$. Let f be an element of the m–particle sector of $\mathcal{F}^0_{s]}$ and let g be an element of the n–particle sector of $\mathcal{F}^0_{t]}$. Let us define the function $[f \odot g] \in (L^2)^{\odot(m+n)}$ by setting

$$[f \odot g](s_m, \ldots, s_1, t_n, \ldots, t_1) = (\mathfrak{s}_t f)(s_m, \ldots, s_1) \odot g(t_n, \ldots, t_1). \tag{11.1}$$ F0fac

By the first observation $\mathcal{s}_t f$ is an element of $(L^2_{[t,t+s]})^{\odot m}$. However, we may identify $(L^2_{[t,t+s]})^{\odot m}$ as a subspace of $(L^2_{t+s]})^{\odot m}$. In this identification $\mathcal{s}_t f$ is a function which vanishes, unless $s_1 \geq t$. Therefore, the function $[f \odot g]$ is in $\mathcal{F}^0_{s+t]}$ so that $f \odot g \mapsto [f \odot g]$ extends to a two-sided isometric mapping $u_{st} \colon \mathcal{F}^0_s \odot \mathcal{F}^0_{t]} \to \mathcal{F}^0_{s+t]}$. It is clear that the u_{st} fulfill the associativity condition of Definition 4.7. It remains to show that u_{st} is surjective.

$(L^2_{s+t]})^{\odot n}$ is spanned by functions of the form $f = \chi_{[s_n,t_n]}x_n \odot \ldots \odot \chi_{[s_1,t_1]}x_1$. Since we are interested in $\Delta_n(L^2)^{\odot n}$ only, and since Δ_n is continuous, (splitting an interval into two, if necessary) we may restrict to the case where for each $i = 1, \ldots, n-1$ either $s_{i+1} \geq t_i$ or $s_{i+1} = s_i, t_{i+1} = t_i$. Furthermore, (by the same argument) we may assume, that $s_1 \geq t$, or that there exists m $(1 \leq m < n)$ such that $t_m \leq t$ and $s_{m+1} \geq t$, or that $t_n \leq t$. In the first case we have $f \in \mathcal{F}_{[t,t+s]}$ so that $\Delta_n f = [\mathcal{s}_t^{-1}\Delta_n f \odot \xi]$ is in the range of u_{st}. Similarly, in the third case $f \in \mathcal{F}_{t]}$ so that $\Delta_n f = [\xi \odot \Delta_n f]$ is in the range of u_{st}. In the second case we set $g_2 = \chi_{[s_n,t_n]}x_n \odot \ldots \odot \chi_{[s_{m+1},t_{m+1}]}x_{m+1} \in \mathcal{F}_{[t,t+s]}$ and $g_1 = \chi_{[s_m,t_m]}x_m \odot \ldots \odot \chi_{[s_1,t_1]}x_1 \in \mathcal{F}_{t]}$. Again, we see that $\Delta_n f = [\mathcal{s}_t^{-1}\Delta_{n-m}g_2 \odot \Delta_m g_1]$ is in the range of u_{st}. ■

Let $T = \big(T_t\big)_{t \in \mathbb{R}^+}$ be a conservative CP-semigroup on $\mathcal{B}$ with bounded generator $\mathcal{L} \colon \mathcal{B} \to \mathcal{B}$. We know from [CE79] that (if necessary, after passing to the CP-semigroup T^{**} on the bidual $\mathcal{B}^{**}$) the generator has the form $\mathcal{L}(b) = \mathcal{L}^0(b) - \frac{b\mathcal{L}^0(\mathbf{1}) + \mathcal{L}^0(\mathbf{1})b}{2} + i[h,b]$ where $\mathcal{L}^0$ is a completely positive mapping (usually, neither unital nor contractive) and h being a self-adjoint element of $\mathcal{B}$. Doing the GNS-construction for $\mathcal{L}^0$, we find a Hilbert module F and an element $\zeta \in F$ such that $\mathcal{L}(b) = \langle \zeta, b\zeta \rangle - \frac{b\langle\zeta,\zeta\rangle + \langle\zeta,\zeta\rangle b}{2} + i[h,b]$.

We summarize the necessary results from [Ske99]. Denoting $d\ell^*_t(\zeta) = \ell^*(\chi_{[t,t+dt]}\zeta)$, the quantum stochastic differential equation

$$du_t \;=\; u_t\big\{d\ell^*_t(\zeta) - d\ell_t(\zeta) + \big(ih - \tfrac{1}{2}\langle\zeta,\zeta\rangle\big)\,dt\big\}, \qquad u_0 = \mathbf{1} \tag{11.2}$$

has a unique unitary solution in the continuous $\mathcal{B}^a(\mathcal{F})$–valued functions on $\mathbb{R}^+$. This solution is adapted in the sense that for each $t \in \mathbb{R}^+$ there is a (unique) operator $u^0_t \in \mathcal{B}^a(\mathcal{F}_{t]})$ such that $u_t = u^0_t \odot \mathsf{id}$ in the identification $\mathcal{F} \;=\; \mathcal{F}_{t]} \odot \big(\xi \oplus L^2_{[t} \odot \mathcal{F}\big)$ and u_t is a left cocycle with respect to the time shift, i.e. $u_{t+s} = u_t \mathcal{S}_t(u_s)$. Consequently, $\widehat{\vartheta} = \big(\widehat{\vartheta}_t\big)_{t \in \mathbb{R}^+}$ with $\widehat{\vartheta}_t(a) = u_t \mathcal{S}_t(a) u^*_t$ is an E_0–semigroup on $\mathcal{B}^a(\mathcal{F})$. Moreover, $\widehat{\vartheta}$ is a dilation of T in the sense that $\langle \xi, \widehat{\vartheta}_t(b)\xi \rangle = T_t(b)$ for all $b \in \mathcal{B}$, where we identify $\mathcal{B}$ (faithfully) as a subalgebra of $\mathcal{B}^a(\mathcal{F})$ by left multiplication on $\mathcal{F}$.

11.5 Lemma. *We have $u^*_t \xi \in \mathcal{F}^0_{t]}$ for all $t \in \mathbb{R}^+$.*

PROOF. By adaptedness we have $u^*_t \xi \in \mathcal{F}_{t]}$. So let us show that it is time ordered. First, notice that u^*_t fulfills the adjoint of (11.2), i.e.

$$du^*_t \;=\; \big\{d\ell_t(\zeta) - d\ell^*_t(\zeta) - \big(ih + \tfrac{1}{2}\langle\zeta,\zeta\rangle\big)\,dt\big\}u^*_t \qquad u^*_0 = \mathbf{1}.$$

Recall that the solution of this differential equation is given by $u^* = \lim\limits_{n\to\infty}(u_n)^*$ where the processes $(u_n)^*$ are constructed inductively by setting $(u_0)^*_t = \mathbf{1}$ and $(u_{n+1})^*_t = \mathbf{1} + \int_0^t \big\{d\ell_t(\zeta) - d\ell^*_t(\zeta) - \big(ih + \frac{1}{2}\langle\zeta,\zeta\rangle\big)\,dt\big\}(u_n)^*_t$ $(n \in \mathbb{N}_0)$, and the integral is approximated in norm by Riemann-Stieltjes sums; see [Ske99]. Clearly, $(u_0)^*_t \xi = \xi \in \mathcal{F}^0$. Now let us assume that $(u_n)^*_t\xi \in \mathcal{F}^0$. Then $\bigl(ih + \frac{1}{2}\langle\zeta,\zeta\rangle\bigr)(u_n)^*_t\xi\,dt \in \mathcal{F}^0$, $d\ell_t(\zeta)\,(u_n)^*_t\xi = 0 \in \mathcal{F}^0$, and $d\ell^*_t(\zeta)\,(u_n)^*_t\xi \in \mathcal{F}^0$. As $(u_{n+1})^*_t\xi$ is approximated by sums over such elements, we find $(u_{n+1})^*_t\xi \in \mathcal{F}^0_{t]}$. ■

11.6 Theorem. *The $u^*_t\xi$ form a unital unit for $\mathcal{F}^0_{t]}$. Consequently, the family $v = \bigl(v_t\bigr)_{t\in\mathbb{R}^+}$ of mappings $v_t = |u^*_t\xi\rangle\langle\xi|$ forms an adapted right cocycle with respect to $\mathcal{S} \upharpoonright \mathcal{B}^a(\mathcal{F}^0)$ which takes values in the partial isometries. In other words, the CP-semigroup T is cocycle subconjugate to the trivial semigroup on $\mathcal{B}$. Moreover, the tensor product system of T is isomorphic to the subsystem of $\mathcal{F}^0_{t]}$ which is generated by the unit $u^*_t\xi$.*

Proof. By Section uco 7 it is sufficient only to show the first assertion. We find

$$u^*_s\xi \odot u^*_t\xi = \mathcal{S}_t(u^*_s)u^*_t\xi = u^*_{s+t}\xi. \blacksquare$$

11.7 Remark. The form of the unit $u^*_t\xi$ can be given explicitly; see LiSk00p1 [LS00].

12 The von Neumann case

vN

This section is the analogue of Section C* 10 for a normal CP-semigroup T on a von Neumann algebra $\mathcal{B}$ acting on a Hilbert space G. By Proposition normal1 C.4, the strong completions $\overline{\mathcal{E}_t}^s$ of the GNS-modules $\mathcal{E}_t$ are von Neumann $\mathcal{B}$–$\mathcal{B}$–modules. By Corollary normal3 C.6, also the tensor products $\overline{\mathcal{E}_s}^s \mathbin{\bar\odot}^s \overline{\mathcal{E}_t}^s$ are von Neumann $\mathcal{B}$–$\mathcal{B}$–modules. By Proposition topindlim A.10, the inductive limits $\overline{E_\tau}^s \subset \mathcal{B}(G, H_\tau)$ with $H_\tau = E_\tau \mathbin{\bar\odot} G$ are von Neumann $\mathcal{B}$–$\mathcal{B}$–modules.

Of course, the inductive limit $\overline{E}^s \subset \mathcal{B}(G, H)$ with $H = E \mathbin{\bar\odot} G$ is a von Neumann $\mathcal{B}$–module. Therefore, the algebra $\mathcal{B}^a(\overline{E}^s)$ is a von Neumann subalgebra of $\mathcal{B}(H)$; see Ske97p2 [Ske97]. By Corollary normal3 C.6, $\overline{E}^s \mathbin{\bar\odot}^s \overline{E_t}^s$ is a von Neumann $\mathcal{B}^a(\overline{E}^s)$–$\mathcal{B}$–module. In other words, the mapping $\vartheta_\tau \colon a \mapsto a \odot \mathsf{id}$ is a normal endomorphism of $\mathcal{B}^a(\overline{E}^s)$. This answers the question raised in Bha96p [Bha96], whether the e_0–semigroup $\vartheta \upharpoonright \mathcal{A}_\infty$ consists of normal mappings, in the affirmative sense.

After these preparations, it is clear that Theorem C*thm 10.1 remains true, replacing C^*–algebras by von Neumann algebras, Hilbert modules by von Neumann modules, and adding the word "normal" to all mappings between von Neumann algebras. We also find the analogue of Theorem contsg 10.2.

scont **12.1 Theorem.** *Let $T = \bigl(T_t\bigr)_{t\in\mathbb{R}^+}$ be a weakly continuous normal CP-semigroup on a von Neumann algebra $\mathcal{B}$ on a Hilbert space G. Then ϑ is a $*$–strongly continuous normal E_0–semigroup (i.e. $\vartheta_\tau(a)x \odot g$ is continuous for all $a \in \mathcal{B}^a(\overline{E}^s)$, $x \in \overline{E}^s$, and $g \in G$).*

Proof. Very much like the proof of Theroem contsg 10.2, but starting from the observation that the family $x \odot \xi^\tau \odot g$ of vectors in $H = \overline{E}^s \mathbin{\bar\odot} G$ depends continuously on τ. ■

13 Centered modules: The case $\mathcal{B} = \mathcal{B}(G)$

B(G)

13.1 Definition Ske98 [Ske98]. A pre-Hilbert (Hilbert, von Neumann) $\mathcal{B}$–$\mathcal{B}$–module E is called a *centered pre-Hilbert (Hilbert, von Neumann) $\mathcal{B}$–module*, if it is generated by its *center* $C_{\mathcal{B}}(E) = \{x \in E \colon xb = bx \ (b \in \mathcal{B})\}$ as a pre-Hilbert (Hilbert, von Neumann) $\mathcal{B}$–module.

13.2 Remark. The requirement that a pre-Hilbert module is centered is, in general, a rather serious restriction. Nevertheless, we will see in Theorem 13.11 that von Neumann $\mathcal{B}(G)$–$\mathcal{B}(G)$–modules are centered, automatically. Since the best understood examples are normal CP-semigroups and normal E_0–semigroups on $\mathcal{B}(G)$, there is a vast area for applications of centered modules. As examples, in Corollaries 13.10, 13.14, and 13.15 we recover some results of [Bha98a] as consequences of Theorems 5.8 and 13.11.

13.3 Example. Let $\mathfrak{H}$ be a pre-Hilbert space. Then $\mathcal{B} \otimes \mathfrak{H}$ is a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module with inner product $\langle b \otimes h, b' \otimes h' \rangle = b^* b' \langle h, h' \rangle$ and the obvious $\mathcal{B}$–$\mathcal{B}$–module structure. Moreover, $\mathcal{B} \otimes \mathfrak{H}$ is generated by its subset $\mathbf{1} \otimes \mathfrak{H}$ which, clearly, is contained in the center.

Assume that $\mathcal{B}$ is a pre–C^*–algebra of operators acting non-degenerately on a pre-Hilbert space G. Then $(\mathcal{B} \otimes \mathfrak{H}) \odot G = G \otimes \mathfrak{H}$ so that by Stinespring construction (Example 2.16) $\mathcal{B} \otimes \mathfrak{H}$ may be considered as a subset of $\mathcal{B}^a(G, G \otimes \mathfrak{H})$ via $b \otimes h \colon g \mapsto bg \otimes h$. In particular, the elements $\mathbf{1} \otimes h \in \mathbf{1} \otimes \mathfrak{H}$ are identified with mappings $g \mapsto g \otimes h$.

If G is a Hilbert space and $\mathcal{B} = \mathcal{B}(G)$ then the strong closure of $\mathcal{B}(G) \otimes \mathfrak{H}$ in $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$ is all of $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$ (cf. Proposition 13.9 below). In particular, $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$ is a centered von Neumann $\mathcal{B}(G)$–module (cf. Proposition 13.8 below). It is easy to see that the center coincides with $\mathbf{1} \otimes \mathfrak{H}$.

13.4 Remark. We mention that in the preceding example we changed the order of the factors in the tensor products $\mathcal{B} \otimes \mathfrak{H}$ and $G \otimes \mathfrak{H}$ compared with the conventions in [Ske98, Ske97]. We did this in order to avoid in Corollary 13.14 anti-product systems.

13.5 Proposition. *In a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module E we have $\langle C_{\mathcal{B}}(E), C_{\mathcal{B}}(E) \rangle \subset C_{\mathcal{B}}(\mathcal{B})$.*

Proof. Direct verification. ■

13.6 Corollary. *Let E and F be centered pre-Hilbert $\mathcal{B}$–modules. Then the mapping*

$$x \odot y \longmapsto y \odot x, \quad (x \in C_{\mathcal{B}}(E), y \in C_{\mathcal{B}}(F))$$

extends to a (unique) two-sided unitary $E \odot F \to F \odot E$.

13.7 Remark. If E is centered, then Corollary 13.6 allows for a symmetrization on $E^{\odot n}$. That is the basis for the construction of the *symmetric Fock module* over a centered one-particle sector; see [Ske98]. One can show that, like for Hilbert spaces, also the symmetric Fock module is isomorphic to the time ordered Fock module. For non-centered one-particle sectors, the symmetric Fock module cannot be constructed without additional effort. This suggests that the time ordered Fock module is a proper generalization of the symmetric Fock space.

13.8 Proposition [Ske97]. *A centered von Neumann module is a two-sided von Neumann module (i.e. the algebra is represented normally by multiplication from the left).*

13.9 Proposition. *Let E be a von Neumann $\mathcal{B}(G)$–module. Then $E = \mathcal{B}(G, H)$ (where $H = E \bar{\odot} G$). Moreover, $\rho \colon \mathcal{B}^a(E) \to \mathcal{B}(H)$ is a normal isomorphism.*

Proof. By definition ρ is normal. And it is easy to see that $\mathcal{B}^a(\mathcal{B}(G,H))$ is isomorphic to $\mathcal{B}(H)$; see the appendix in [Ske98]. So let us show that $E = \mathcal{B}(G,H)$.

$\mathcal{B}(G,H)$ is generated by rank-one operators as a von Neumann module. Since $\mathcal{B}(G)$ contains all rank-one operators and elements of the form $x \odot g$ form a total subset of $H = E \bar{\odot} G$, we can approximate (even in norm) arbitrary rank-one operators in $\mathcal{B}(G,H)$. ■

BHdil **13.10 Corollary.** *The maximal E_0–dilation ϑ of a normal conservative CP-semigroup on $\mathcal{B}(G)$ is isomorphic to a normal E_0–dilation on $\mathcal{B}(H)$ in the sense of [Bha98a].*

Now we are going to show that any von Neumann $\mathcal{B}(G)$–$\mathcal{B}(G)$–module E ($= \mathcal{B}(G,H)$ by Proposition 13.9) is centered. As E is a von Neumann $\mathcal{B}(G)$–$\mathcal{B}(G)$–module, we have a normal (unital) representation ρ of $\mathcal{B}(G)$ on H such that the left multiplication is $bx = \rho(b)x$. In the language of Arveson the center is the space of *intertwiners* between the representations id on G and ρ on H (i.e. mappings $x\colon G \to H$ such that $xb = \rho(b)x$ for all $b \in \mathcal{B}(G)$). By Proposition 13.5 the inner product of elements in the center takes values in the center of $\mathcal{B}$. In the case $\mathcal{B} = \mathcal{B}(G)$ the center of $\mathcal{B}$ is trivial so that, as observed by Arveson [Arv89], there is a $\mathbb{C}$–valued scalar product $\langle \bullet, \bullet \rangle_c$ on $C_{\mathcal{B}(G)}(E)$ fulfilling $\langle x,y \rangle = \langle x,y \rangle_c \mathbf{1}$. Obviously, $C_{\mathcal{B}(G)}(E)$ with this scalar product is a Hilbert space, which we denote by $\mathfrak{H}$.

BGcen **13.11 Theorem.** *Let E be a von Neumann $\mathcal{B}(G)$–$\mathcal{B}(G)$–module. Then E is isomorphic to $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$ as von Neumann $\mathcal{B}(G)$–$\mathcal{B}(G)$–module. In particular, E is a centered von Neumann $\mathcal{B}(G)$–module.*

Proof. The representation ρ of $\mathcal{B}(G)$ on H is normal. Therefore, there exists a Hilbert space $\mathfrak{H}$ such that ρ is unitarily equivalent to the representation $\mathsf{id} \otimes \mathbf{1}$ on $G \bar{\otimes} \mathfrak{H}$. Making use of Proposition 13.9 and identifying $G \bar{\otimes} \mathfrak{H}$ with H we find $E = \mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$. By a straightforward generalization of Proposition C.3 any homomorphism between von Neumann modules is strongly continuous so that identifying $\mathcal{B}(G,H)$ and $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$, indeed, respects the strong topology. By Example 13.3 the center of E is $\mathbf{1} \otimes \mathfrak{H}$ so that $\mathfrak{H}$, indeed, is the Hilbert space obtained from the center. ■

Henceforth, we will speak of centered von Neumann $\mathcal{B}(G)$–modules and von Neumann $\mathcal{B}(G)$–$\mathcal{B}(G)$–modules interchangeably.

centiso **13.12 Proposition.** *Let $E_1 = \mathcal{B}(G, G \bar{\otimes} \mathfrak{H}_1)$ and $E_2 = \mathcal{B}(G, G \bar{\otimes} \mathfrak{H}_2)$ be two arbitrary centered von Neumann $\mathcal{B}(G)$–modules. Then (identifying $\mathfrak{H}_i$ with $\mathbf{1} \otimes \mathfrak{H}_i$) $a \mapsto a \upharpoonright \mathfrak{H}_1$ establishes a canonical isomorphism from the $\mathcal{B}(G)$–$\mathcal{B}(G)$–linear mappings in $\mathcal{B}^a(E_1, E_2)$ to $\mathcal{B}(\mathfrak{H}_1, \mathfrak{H}_2)$. In particular, a $\mathcal{B}(G)$–$\mathcal{B}(G)$–linear mapping $E_1 \to E_2$ is a unitary, an isometry, a projection (for $\mathfrak{H}_1 = \mathfrak{H}_2$), etc., if and only if the corresponding mapping in $\mathcal{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ is.*

Proof. This follows from the fact that bilinear mappings respect the center (i.e. the range of $a \upharpoonright \mathfrak{H}_1$ is, indeed, contained in $\mathfrak{H}_2$). ■

13.13 Proposition. *Let $\mathcal{B}(G,H)$ be an arbitrary von Neumann $\mathcal{B}(G)$–module and let $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$ be an arbitrary centered von Neumann $\mathcal{B}(G)$–module. Then*

$$\mathcal{B}(G,H) \bar{\odot}^s \mathcal{B}(G, G \bar{\otimes} \mathfrak{H}) = \mathcal{B}(G, H \bar{\otimes} \mathfrak{H})$$

via $x \odot y \mapsto (x \otimes \mathsf{id}) \circ y$.

Let $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H}_1)$ *and* $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H}_2)$ *be two arbitrary centered von Neumann* $\mathcal{B}(G)$*–modules. Then the isomorphism*

$$\mathcal{B}(G, G \bar{\otimes} \mathfrak{H}_1) \mathbin{\bar{\odot}^s} \mathcal{B}(G, G \bar{\otimes} \mathfrak{H}_2) = \mathcal{B}(G, G \bar{\otimes} \mathfrak{H}_1 \bar{\otimes} \mathfrak{H}_2)$$

is two-sided. In particular, the restriction of this isomorphism to the centers is the tensor product of Hilbert spaces.

Proof. Simple verification. ■

13.14 Corollary. *Let* $E^\odot$ *be a product system of centered von Neumann* $\mathcal{B}(G)$*–modules. Denote by* $\mathfrak{H}_\tau$ *the center of* E_τ*. Then* $\mathfrak{H}^\otimes = \big(\mathfrak{H}_\tau\big)_{\tau \in \mathbb{T}}$ *is a product systems of Hilbert spaces.*

Moreover, two product systems $E^\odot$ *and* $E'^\odot$ *of centered von Neumann* $\mathcal{B}(G)$*–modules are isomorphic, if and only if the corresponding product systems* $\mathfrak{H}^\otimes$ *and* $\mathfrak{H}'^\otimes$ *of Hilbert spaces are isomorphic, where the isomorphisms* $E_\tau \longrightarrow E'_\tau$ *and* $\mathfrak{H}_\tau \longrightarrow \mathfrak{H}'_\tau$ *are identified via Proposition 13.12.*

13.15 Corollary. *Let* $\xi^\odot$ *be a unital unit for* $E^\odot$ *and denote by* E *the inductive limit associated with this unit. Let* $H = E \mathbin{\bar{\odot}} G$*. Then* $H \bar{\otimes} \mathfrak{H}_\tau = H$*.*

Proof. We have $E \mathbin{\bar{\odot}^s} E_\tau = E$, hence, $H \bar{\otimes} \mathfrak{H}_\tau = E \mathbin{\bar{\odot}} E_\tau \mathbin{\bar{\odot}} G = E \mathbin{\bar{\odot}} G = H$. ■

If $E^\odot$ is the product system of a normal E_0–semigroup ϑ on $\mathcal{B}(G)$, then $\mathfrak{H}^\otimes$ as given in Corollary 13.14 is the associated Arveson product system of Hilbert spaces. More precisely, if G is infinite-dimensional and separable, and if ϑ is strongly continuous and indexed by $\mathbb{R}^+$, then the associated Arveson system is

$$\Big\{(\tau, a) \in (0, \infty) \times \mathcal{B}(G) \colon a \in C_{\mathcal{B}(G)}(E_\tau)\Big\}$$

as a toplogical subspace of $(0,\infty) \times \mathcal{B}(G)$; see [Arv89]. Recall from the proof of Theorem 7.9 that all E_τ can be identified with $\mathcal{B}(G)$. Henceforth, $C_{\mathcal{B}(G)}(E_\tau)$ can, indeeed, be considered as a subset of $\mathcal{B}(G)$. Clearly, if G is separable and $G = G \bar{\otimes} \mathfrak{H}_\tau$, then also $\mathfrak{H}_\tau$ must be separable. Consequently, as operator norm and Hilbert space norm on $C_{\mathcal{B}(G)}(E_\tau)$ coincide, Arveson's product system is isomorphic to $(0,\infty) \times G$ as a topological space.

In Theorem 7.8 we have classified conservative CP-semigroups up to cocycle conjugacy or, what is the same, by product systems. In Theorem 7.9 we have shown that in the case of E_0–semigroups the cocyle providing the equivalence is unitary. In Corollary 13.14 we have seen that in the case of CP-semigroups on $\mathcal{B}(G)$ classification by product systems of Hilbert modules is the same as classification by product systems of Hilbert spaces. Altogether, we have shown that normal E_0–semigroups on $\mathcal{B}(G)$ are classified by product systems of Hilbert spaces up to unitary cocycle conjugacy. This generalizes Arveson's result [Arv89] to the case where G is not necessarily separable and where ϑ is not necessarily strongly continuous.

Let us repeat, however, that we do not have a one-to-one correspondence as in [Arv90]. Besides the question in how far this result depends on the assumption that ϑ is strongly continuous, droping the separibilty condition on G has changed the situation completely. Assuming G infinite-dimensional and separable means that there is essentially one C^*–algebra $\mathcal{B}(G)$ under consideration. Allowing arbitrary dimension for G raises the question, whether each product system of Hilbert spaces arises from an E_0–semigroup, once for each dimension. We remarked already that in the case of $G = \mathbb{C}$ the answer is negative, and it is very well possible that the answer generally depends on the dimension of G.

14 Domination and cocycles

Let $T\colon \mathcal{A} \to \mathcal{B}$ be a (bounded) completely positive mapping with GNS-construction (E, ξ). Let w be an operator in $\mathcal{A}'$, the *relative commutant* of (the image of) $\mathcal{A}$ in $\mathcal{B}^a(E)$, with $0 \leq w \leq \mathbf{1}$. Then $S(a) = \langle \xi, wa\xi \rangle = \langle \sqrt{w}\xi, a\sqrt{w}\xi \rangle$ defines a (bounded) completely positive mapping. Moreover, S is *dominated* by T, i.e. also $T - S$ is completely positive. Clearly, domination defines a partial order on the set of (bounded) completely positive mappings $\mathcal{A} \to \mathcal{B}$. The mapping $\mathfrak{O}\colon w \mapsto S$ is one-to-one (as $S_{w_1} = S_{w_2}$ implies $0 = b^*\langle \xi, (w_1 - w_2)a^*a'\xi \rangle b' = \langle a\xi b, (w_1 - w_2)a'\xi b' \rangle$ for all $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$) and, obviously, order preserving.

In the case when $\mathcal{B} = \mathcal{B}(G)$, Arveson [Arv69] has shown, based on the usual Stinespring construction, that $\mathfrak{O}$, actually, is surjective, hence, an order isomorphism. Paschke has generalized this to arbitrary von Neumann algebras $\mathcal{B} \subset \mathcal{B}(G)$. The proof of the following (slightly weaker) lemma shows that we need self-duality. Therefore, it is not clear, whether the result can be generalized to arbitrary C^*–algebras.

14.1 Lemma [Pas73]. *Let $\mathcal{A}$ be a C^*–algebra, let $\mathcal{B}$ be a von Neumann algebra on a Hilbert space G, and let $T \geq S$ be a completely positive mapping $\mathcal{A} \to \mathcal{B}$. Let (E, ξ) denote the GNS-construction for T. Then there exists an operator in $w \in \mathcal{A}' \subset \mathcal{B}^a(\overline{E}^s)$ such that $S(a) = \langle \xi, wa\xi \rangle$.*

Proof. Let (F, ζ) denote the GNS-construction for S. As $T - S$ is completely positive, the mapping $v\colon \xi \mapsto \zeta$ extends to an $\mathcal{A}$–$\mathcal{B}$–linear contraction $E \to F$ and further (similar to Proposition C.3) to a contraction $\overline{E}^s \to \overline{F}^s$. By Remark C.2, v has an adjoint $v^* \in \mathcal{B}^a(\overline{F}^s, \overline{E}^s)$. Since adjoints of bilinear mappings and compositions among them are bilinear, again, it follows that $w = v^*v$ commutes with all $a \in \mathcal{A}$. Of course, $\langle \xi, wa\xi \rangle = \langle \xi, v^*va\xi \rangle = \langle \zeta, a\zeta \rangle = S(a)$. ■

Of course, we can equip also the set of CP-semigroups on a unital C^*–algebra $\mathcal{B}$ with a partial order, by saying that $T \geq S$, if $T_t \geq S_t$ for all $t \in \mathbb{T}$. In [Bha98a] the order structure of the set of normal strongly continuous CP-semigroups on $\mathcal{B}(G)$ which are dominated by a fixed normal E_0–semigroup (with $\mathbb{T} = \mathbb{R}^+$) is studied. In the remainder of this section we generalize the results in [Bha98a] to arbitrary von Neumann algebras $\mathcal{B} \subset \mathcal{B}(G)$, to normal CP-semigroups (not necessarily strongly continuous) dominated by a fixed conservative normal CP-semigroup, and arbitrary $\mathbb{T}$.

To begin with, let T be an arbitrary normal CP-semigroup on a von Neumann algebra $\mathcal{B} \subset \mathcal{B}(G)$ and let S be a normal CP-semigroup dominated by T. Denote by $\mathcal{E}_t, \mathcal{E}_{\mathfrak{t}}, E_\tau$ and $\mathcal{F}_t, \mathcal{F}_{\mathfrak{t}}, F_\tau$ $(t, \tau \in \mathbb{T}, \mathfrak{t} \in \mathbb{J}_\tau)$ the strong closures of the modules related to the first inductive limit in Section 4 for the CP-semigroups T and S, respectively. ($\mathcal{F}_t$ should not be confused with the Fock modules in Section 11.) Denote by $\zeta_t \in \mathcal{F}_t, \zeta_{\mathfrak{t}} \in \mathcal{F}_{\mathfrak{t}}, \zeta^\tau \in F_\tau$ the anlogues for S of the elements $\xi_t, \xi_{\mathfrak{t}}, \xi^\tau$ for T. Denote by $\beta^T_{\mathfrak{t}\mathfrak{s}}, i^T_{\mathfrak{t}}$ and $\beta^S_{\mathfrak{t}\mathfrak{s}}, i^S_{\mathfrak{t}}$ the embeddings related to the constructions for T and for S, respectively.

For each $t \in \mathbb{T}$ denote by $v_t \in \mathcal{B}^a(\mathcal{E}_t, \mathcal{F}_t)$ the $\mathcal{B}$–$\mathcal{B}$–linear contraction extending $\xi_t \mapsto \zeta_t$ as constructed in the proof of Lemma 14.1. For $\mathfrak{t} \in \mathbb{J}_\tau$ define the $\mathcal{B}$–$\mathcal{B}$–linear contraction

$$v_{\mathfrak{t}} = v_{t_n} \odot \ldots \odot v_{t_1} \in \mathcal{B}^a(\mathcal{E}_{\mathfrak{t}}, \mathcal{F}_{\mathfrak{t}}).$$

Obviously,

$$v_{\mathfrak{s}} \odot v_{\mathfrak{t}} = v_{\mathfrak{s} \smile \mathfrak{t}} \tag{14.1}$$ vtensor

for $\mathfrak{s} \in \mathbb{J}_\sigma$ and $\mathfrak{t} \in \mathbb{J}_\tau$. Moreover, $v_{\mathfrak{t}} \beta^T_{\mathfrak{t}\mathfrak{s}} = \beta^S_{\mathfrak{t}\mathfrak{s}} v_{\mathfrak{s}}$ for all $\mathfrak{s} \leq \mathfrak{t} \in \mathbb{J}_\tau$. Applying $i^S_{\mathfrak{t}}$ to both sides, we find $i^S_{\mathfrak{t}} v_{\mathfrak{t}} \beta^T_{\mathfrak{t}\mathfrak{s}} = i^S_{\mathfrak{s}} v_{\mathfrak{s}}$. Therefore, by Proposition A.3 (extended to strong completions via Proposition C.3), for each $\tau \in \mathbb{T}$ there exists a unique $\mathcal{B}$–$\mathcal{B}$–linear contraction $v^\tau \in \mathcal{B}^a(E_\tau, F_\tau)$ fulfilling $v^\tau i^T_{\mathfrak{t}} = i^S_{\mathfrak{t}} v_{\mathfrak{t}}$ for all $\mathfrak{t} \in \mathbb{L}_\tau$. By (14.1) we find $v^\sigma \odot v^\tau = v^{\sigma+\tau}$. By Remark C.2 these operators have adjoints. By $\mathcal{B}$–$\mathcal{B}$–linearity one easily checks that also the adjoint equation $(v^\sigma)^* \odot (v^\tau)^* = (v^{\sigma+\tau})^*$ is valid. Therefore, by setting $w^\tau = (v^\tau)^* v^\tau$ ($\tau \in \mathbb{T}$), we define $\mathcal{B}$–$\mathcal{B}$–linear positive contraction on E_τ. This family of contractions fulfills

$$w^\sigma \odot w^\tau = w^{\sigma+\tau}. \tag{14.2}$$ precoc

wcor **14.2 Corollary.** *Also the family* $\left(\sqrt{w^\tau}\right)_{\tau \in \mathbb{T}}$ *fulfills* (14.2). *The family* $\xi^\odot_S = \left(\xi^\tau_S\right)_{\tau \in \mathbb{T}}$ *with* $\xi^\tau_S = \sqrt{w^\tau} \xi^\tau$ *is a unit for the product system* $E^\odot$. *Moreover, the CP-semigroup* $\langle \xi^\tau_S, \bullet \xi^\tau_S \rangle$ *associated with* $\xi^\odot_S$ *is* S.

In the sequel, we assume that T is conservative. (Of course, S is not conservative, unless $S = T$.) Then we may construct the strong closure E of the second inductive limit from Section 5 and a normal E_0–semigroup ϑ on $\mathcal{B}^a(E)$.

By Lemma 7.5 the family $\mathfrak{w} = \left(\mathfrak{w}_\tau\right)_{\tau \in \mathbb{T}}$ of operators $\mathfrak{w}_\tau = \mathsf{id} \odot w^\tau \in \mathcal{B}^a(E) = \mathcal{B}^a(E \mathbin{\bar{\odot}^s} E_\tau)$ is a positive contractive local cocycle with respect to ϑ.

On the set of positive local cocycles we define a partial order by saying $\mathfrak{w} \geq \mathfrak{v}$, if $\mathfrak{w}_\tau \geq \mathfrak{v}_\tau$, pointwise.

CPdoco **14.3 Theorem.** *Let T be a conservative normal CP-semigroup on a von Neumann algebra $\mathcal{B}$ and let ϑ be the maximal dilation of T. Then the mapping $\mathfrak{w} \mapsto S$ defined, by setting*

$$S_t = \langle \sqrt{\mathfrak{w}_t} \xi, \bullet \sqrt{\mathfrak{w}_t} \xi \rangle \ \ (= \langle \xi, \bullet \mathfrak{w}_t \xi \rangle),$$

establishes an order isomorphism from the partially ordered set of positive contractive local cocycles $\mathfrak{w}$ with respect to ϑ to the partially ordered set of all normal CP-semigroups S dominated by T.

Proof. Of course, the mapping $\mathfrak{w} \mapsto S$ maps into the normal CP-semigroups dominated by T, and we have just shown that it is surjective. By Lemma 14.1 the restriction of w^τ to $\mathcal{B}\xi^\tau\mathcal{B} = i_{(\tau)}(\mathcal{B}\xi_\tau\mathcal{B})$ is determined uniquely by S_τ. By (14.2) this determines w^τ completely. By Lemma 7.5 the correspondence w^τ and $\mathfrak{w}_\tau$ is one-to-one. Therefore, the mapping is also injective.

Certainly, the mapping respects the order, i.e. $\mathfrak{w} \geq \mathfrak{w}' \Rightarrow S \geq S'$. Conversely, if $S \geq S'$, then (cf. the discussion before Corollary 14.2) there exists a family $u^\tau \in \mathcal{B}^a(F_\tau, F'_\tau)$ of $\mathcal{B}$–$\mathcal{B}$–linear contractions such that $w'^\tau = (v'^\tau)^* v'^\tau = (v^\tau)^*(u^\tau)^* u^\tau v^\tau \leq (v^\tau)^* v^\tau = w^\tau$. This implies $\mathfrak{w}' \leq \mathfrak{w}$. ∎

14.4 Remark. Notice that also the embedding of w^τ via $\mathfrak{w}_\tau^0 = (|\xi\rangle\langle\xi|) \odot w^\tau$ defines a cocycle. This cocycle is positive and *adapted*. Therefore, S is cocycle subconjugate to T. Here we have, at least, that $\mathfrak{w}_\tau^0$ is in the relative commutant of $j_\tau(\mathcal{B})$, so that $\mathfrak{w}^0$ is *local* in the sense of [Bha98a]. Also in this sense we have a one-to-one correspondence.

Appendix

A Inductive limits

A.1 Definition. Let $\mathbb{L}$ be a partially ordered set which is directed increasingly. An *inductive system over* $\mathbb{L}$ is a family $\big(E_t\big)_{t\in\mathbb{L}}$ of vector spaces E_t with a family $\big(\beta_{ts}\big)_{t\geq s}$ of linear mappings $\beta_{ts}\colon E_s \to E_t$ fulfilling

$$\beta_{tr}\beta_{rs} = \beta_{ts}$$

for all $t \geq r \geq s$ and $\beta_{tt} = \mathsf{id}_{E_t}$.

The *inductive limit* $E = \limind_{t\in\mathbb{L}} E_t$ of the family $\big(E_t\big)$ is defined as $E = E^\oplus/\mathcal{N}$, where $E^\oplus = \bigoplus_{t\in\mathbb{L}} E_t$ and $\mathcal{N}$ denotes the subspace of $E^\oplus$ consisting of all those $x = \big(x_t\big)$ for which there exists $s \in \mathbb{L}$ (with $s \geq t$ for all t with $x_t \neq 0$) such that $\sum_{t\in\mathbb{L}} \beta_{st}x_t = 0 \in E_s$. (Clearly, if s fulfills this condition, then so does each $s' \geq s$.)

A.2 Proposition. *The family $\big(i_t\big)_{t\in\mathbb{L}}$ of canonical mappings $i_t\colon E_t \to E$ fulfills $i_t\beta_{ts} = i_s$ for all $t \geq s$. Clearly, $E = \bigcup_{t\in\mathbb{L}} i_tE_t$.*

PROOF. Let us identify $x_t \in E_t$ with its image in $E^\oplus$ under the canonical embedding. We have to check, whether $\beta_{ts}x_s - x_s \in \mathcal{N}(\subset E^\oplus)$ for all $x_s \in E_s$. But this is clear, because $\beta_{tt}(\beta_{ts}x_s) - \beta_{ts}(x_s) = 0$. ■

A.3 Proposition. *Let F be another vector space and suppose $f\colon E \to F$ is a linear mapping. Then the familiy $\big(f_t\big)_{t\in\mathbb{L}}$ of linear mappings, defined by setting*

$$f_t = fi_t, \tag{A.1}$$

fulfills

$$f_t\beta_{ts} = f_s \quad \text{for all} \quad t \geq s. \tag{A.2}$$

Conversely, if $\big(f_t\big)_{t\in\mathbb{L}}$ is a family of linear mappings $f_t\colon E_t \to F$ fulfilling (A.2), then there exists a unique linear mapping $f\colon E \to F$ fulfilling (A.1).

PROOF. Of course, $f = 0$, if and only if $f_t = 0$ for all $t \in \mathbb{L}$, because E is spanned by all i_tE_t. In other words, the correspondence is one-to-one.

Consider a linear mapping $f\colon E \to F$ and set $f_t = fi_t$. Then by Proposition A.2 we have $f_t\beta_{ts} = fi_t\beta_{ts} = fi_s = f_s$.

For the converse direction let $\big(f_t\big)$ be a family of linear mappings $f_t\colon E_t \to F$ which satisfies (A.2). Define $f^\oplus = \bigoplus_{t\in\mathbb{L}} f_t\colon E^\oplus \to F$ and let $x = \big(x_t\big) \in \mathcal{N}$ so that for some $s \in \mathbb{L}$

we have $\sum\limits_{t\in\mathbb{L}} \beta_{st}x_t = 0$. Then $f^{\oplus}(x) = \sum_{t\in\mathbb{L}} f_t x_t = \sum_{t\in\mathbb{L}} f_s\beta_{st}x_t = f_s\sum_{t\in\mathbb{L}}\beta_{st}x_t = 0$, so that $f^{\oplus}$ defines a mapping f on the quotient E fulfilling (A.1). ■

A.4 Remark. The inductive limit E together with the family $\bigl(i_t\bigr)$ is determined by the second part of Proposition A.3 up to vector space isomorphism. This is refered to as the *universal property* of E.

If the vector spaces E_t carry additional structures, and if the mediating mappings β_{ts} respect these structures, then simple applications of the universal property show that, usually, also the inductive limit carries the same structures.

A.5 Example. If all E_t are right (left) modules and all β_{ts} are right (left) module homomorphisms, then E inherits a right (left) module structure in such a way that all i_t also become right (left) module homomorphisms. A similar statement is true for two-sided modules.

Moreover, if F is another module (right, left, or two-sided) and $\bigl(f_t\bigr)$ is a family of homomorphisms of modules (right, left, or two-sided) fulfilling (A.2), then also f is homomorphism.

Sometimes it is necessary to work slightly more in order to see that E carries the same structure. Denote by $i\colon E^{\oplus}\to E$ the canonical mapping.

A.6 Proposition. *Let all E_t be pre-Hilbert modules and let all β_{ts} be isometries. Then*

$$\langle x, x'\rangle = \sum_{t,t'}\langle \beta_{st}x_t, \beta_{st'}x'_{t'}\rangle \tag{A.3}$$

($x = i((x_t))$, $x' = i((x'_t)) \in E$, and s such that $x_t = x'_t = 0$ whenever $t > s$) defines an inner product on E. Obviously, also the i_t are isometries.

Moreover, if $\bigl(f_t\bigr)_{t\in\mathbb{L}}$ with $f_t\beta_{ts} = f_s$ $(t \geq s)$ is a family of isometries from E_t into a pre-Hilbert module F, then so is f.

Proof. We have to show that (A.3) does not depend on the choice of s. So let s_1 and s_2 be different possible choices. Then choose s such that $s \geq s_1$ and $s \geq s_2$ and apply the isometries β_{ss_1} and β_{ss_2} to the elements of E_{s_1} and E_{s_2}, respectively, which appear in (A.3).

Since any element of E may be written in the form i_tx_t for suitable $t \in \mathbb{L}$ and $x_t \in E_t$, we see that that the inner product defined by (A.3) is, indeed, strictly positive.

The remaining statements are obvious. ■

A.7 Remark. Of course, the inductive limit over two-sided pre-Hilbert modules E_t with two-sided β_{ts} is also a two-sided pre-Hilbert module and the canonical mappings i_t respect left multiplication.

A.8 Remark. If the mappings β_{ts} are non-isometric, then Equation (A.3) does not make sense. However, if $\mathbb{L}$ is a lattice, then we may define an inner product of two elements i_tx_t and $i_{t'}x'_{t'}$ by $\langle\beta_{st}x_t, \beta_{st'}x'_{t'}\rangle$ where s is the unique maximum of t and t'. This idea is the basis for the construction in [Bha96] where also non-conservative CP-semigroups are considered. Cf. also Remark 8.6.

Sometimes, however, in topological contexts it will be necessary to enlarge the algebraic inductive limit in order to preserve the structure. For instance, the inductive limit of Hilbert modules will only be rarely complete. In this case, we refer to the limit in Definition A.1 as the *algebraic* inductive limit.

A.9 Definition. By the *inductive limit* of an inductive system of Hilbert modules we understand the norm completion of the algebraic inductive limit.

By the *inductive limit* of an inductive system of von Neumann modules we understand the strong completion of the algebraic inductive limit; see Appendix C.

A.10 Proposition. *1. Let $\mathcal{A}$ be a pre–C^*–algebra and let $\mathcal{B}$ be a unital C^*–algebra. Then the inductive limit of contractive Hilbert $\mathcal{A}$–$\mathcal{B}$–modules is a contractive Hilbert $\mathcal{A}$–$\mathcal{B}$–module.*

2. Let $\mathcal{A}$ be a von Neumann algebra and let $\mathcal{B}$ be a von Neumann algebra acting on a Hilbert space G. Then the inductive limit of von Neumann $\mathcal{A}$–$\mathcal{B}$–modules is a von Neumann $\mathcal{A}$–$\mathcal{B}$–module.

Proof. Any element in the algebraic inductive limit may be written as $i_t x_t$ for suitable $t \in \mathbb{L}$ and $t_t \in E_t$. Therefore, the action of $a \in \mathcal{A}$ is bounded by $\|a\|$ on a dense subset of the inductive limit of Hilbert modules. Moreover, if all E_t are von Neumann modules, then the functionals $\langle i_t x_t \odot g, \bullet i_t x_t \odot g \rangle$ on $\mathcal{A}$ all are normal. (Cf. Appendix C.) ■

B Conditional expectations generated by projections and essential ideals

B.1 Lemma. *Let $\mathcal{A}$ be a C^*–algebra with a unital C^*–subalgebra $\mathcal{B}$, for which $\varphi\colon a \mapsto \mathbf{1}_{\mathcal{B}} a \mathbf{1}_{\mathcal{B}}$ defines a conditional expectation. Denote by I the closed ideal in $\mathcal{A}$ generated by $\mathbf{1}_{\mathcal{B}}$.*

If I is an essential ideal, then the algebra $\mathcal{A}$ acts faithfully on the GNS-Hilbert module E for the conditional expectation φ. In particular, $\|a\| = \|a\|_E$ for all $a \in \mathcal{A}$, where $\|\bullet\|_E$ denotes the operator norm in $\mathcal{B}^a(E)$.

Proof. One easily, checks that the GNS-Hilbert module is precisely $E = \overline{\mathcal{A}\mathbf{1}_{\mathcal{B}}}$ and that the cyclic vector is $\xi = \mathbf{1}_{\mathcal{B}}$. We are done, if show that $\mathcal{A}$ acts faithfully on E, because faithful homomorphisms from one C^*–algebra into another are isometric, automatically. So let a be a non-zero element in $\mathcal{A}$. We know that there exists an element $i \in I$, such that $ai \neq 0$. Since $I = \overline{\operatorname{span}}\, \mathcal{A}\mathbf{1}_{\mathcal{B}}\mathcal{A}$, we may find (by polarization, if necessary) $a' \in \mathcal{A}$, such that $aa'\mathbf{1}_{\mathcal{B}}a'^* \neq 0$. Therefore, $aa'\mathbf{1}_{\mathcal{B}} \neq 0$, where $a'\mathbf{1}_{\mathcal{B}}$ is an element of E. ■

B.2 Observation. The preceding proof also shows that we may identify I with the compact operators $\mathcal{K}(E)$ on E.

B.3 Example. We show that an algebraic version of 'essential' is not sufficient. Consider the $*$–algebra $\mathcal{P} = \mathbb{C}\langle x \rangle$ of polynomials in one self-adjoint indeterminate. By $p \mapsto p(x)$ we define a homomorphism from $\mathcal{P}$ into the C^*–algebra of continuous functions on the subset $[0, 1] \cup \{2\}$ of $\mathbb{R}$. Denote by $\mathcal{A}$ the image of $\mathcal{P}$ under this homomorphism. Furthermore, choose the ideal I in $\mathcal{A}$ consisting of all functions which vanish at 2. Clearly, I is essential in $\mathcal{A}$. But, the completion of $\mathcal{A}$ contains just all continuous functions. These are no longer separated by I.

C Von Neumann modules

In this appendix we recall the definition of *von Neumann modules* and their basic properties. Like von Neumann algebras, which are strongly closed subalgebras of $\mathcal{B}(G)$, we think of von Neumann modules as strongly closed submodules of $\mathcal{B}(G, H)$ [Ske97]. Other authors (e.g. Schweitzer [Sch96]) follow an abstract approach paralleling Sakai's characterization of W^*–algebras. Consequently, they define W^*–modules as Hilbert modules with a pre-dual. Both approaches are more or less equivalent. The most important properties of von Neumann modules or W^*–modules already can be found in the first paper [Pas73] on Hilbert modules by Paschke.

For two reasons we decided to follow the concrete operator approach. Firstly, the access to topological questions seems to be more direct. For instance, using the embedding $\mathcal{B}(G, H) \subset \mathcal{B}(G \oplus H)$, it is almost a triviality to see that a von Neumann module can be *embedded* as a strongly closed subset into a von Neumann algebra. In the W^*–approach one needs to work slightly harder to see this. Secondly, starting from the usual Stinespring construction, the existing results on both CP-semigroups and E_0–semigroups are formulated exclusively, using the language of operators on or between Hilbert spaces. Therefore, in order to keep contact with earlier work, von Neumann modules are the more reasonable choice. Our general reference for von Neumann modules is [Ske97]. This is just, because we do not know another reference where the operator approach is used systematically.

We start by repeating some well-known facts on normal mappings which can be found in any text book like [Sak71, Tak79]. We also recommend the almost self-contained appendix in Meyer's book [Mey93]. First of all, recall that a von Neumann algebra is *order complete*, i.e. any bounded increasing net of positive elements in a von Neumann algebra converges in the strong topology to its unique least upper bound. A positive linear mapping T between von Neumann algebras is called *normal*, if it is *order continuous*. In other words, T is normal, if and only if $\limsup_\lambda T(a_\lambda) = T(\limsup_\lambda a_\lambda)$ for each bounded increasing net $\big(a_\lambda\big)$. Of particular interest is the set of *normal states* on a von Neumann algebra. An increasing net $\big(a_\lambda\big)$ converges to a in the strong topology, if and only if $\varphi(a_\lambda)$ converges to $\varphi(a)$ for any normal state φ. The linear span of the normal states is a Banach space, the *pre-dual*. As normality is a matter of bounded subsets, a positive mapping T is normal, if $\varphi \circ T(a_\lambda)$ converges to $\varphi \circ T(a)$ for all bounded increasing nets $\big(a_\lambda\big)$ and all φ in a subset of normal states which is total in the pre-dual. If a von Neumann algebra acts on a Hilbert space G, then the functionals of the form $\langle f, \bullet f\rangle$ form a total subset of the pre-dual, whenever f ranges over a dense subset of G. Moreover, using the technique of *cyclic decomposition* (see [Ske97]), one can show that also the set of functionals $\langle x \odot g, \bullet x \odot g\rangle$ is total in the pre-dual of $\mathcal{B}(E \bar{\odot} G)$, whenever x ranges over a dense subset of E and g ranges over a dense subset of G.

C.1 Definition. Let $\mathcal{B} \subset \mathcal{B}(G)$ be a von Neumann algebra on a Hilbert space G. A Hilbert $\mathcal{B}$–module E is called a *von Neumann $\mathcal{B}$–module*, if the set L_E constructed via

Stinespring construction (Example 2.16) is a strongly closed subset of $\mathcal{B}(G,H)$. In this case we assume that $H = E \bar{\odot} G$ is a part of the definition of E, and do no longer distinguish between $x \in E$ and $L_x \in \mathcal{B}(G,H)$.

Let $\mathcal{A}$ be another von Neumann algebra. A Hilbert $\mathcal{A}$–$\mathcal{B}$–module E is called a *von Neumann $\mathcal{A}$–$\mathcal{B}$–module*, if it is a von Neumann $\mathcal{B}$–module, and if the representation $\rho\colon \mathcal{A} \to \mathcal{B}(H)$ (see Example 2.16) is normal.

sd **C.2 Remark.** A von Neumann module is self-dual. More precisely, for any element $\Phi \in \mathcal{B}^r(E,\mathcal{B})$ there exists an element $x \in E$ such that $\Phi(y) = x^*y$ $(y \in E)$. Therefore, $\mathcal{B}^a(E) = \mathcal{B}^r(E)$. The algebra $\mathcal{B}^a(E)$ is a von Neumann algebra. A Hilbert module over a von Neumann algebra is self-dual, if and only if it is a von Neumann module. A bounded $\mathcal{B}$–sesquilinear form $(\bullet,\bullet)\colon E \times F \to \mathcal{B}$ on a pair of von Neumann $\mathcal{B}$–modules defines a unique operator $a \in \mathcal{B}^a(F,E)$ fulfilling $\langle x, ay\rangle = \langle a^*x, y\rangle = (x,y)$. For Hilbert modules, in general, this is not so. See [Pas73, Ske97].

smcont **C.3 Proposition.** *Let $\mathcal{A}$ be a C^*–algebra and let $\mathcal{B}$ be von Neumann algebra acting on a Hilbert space G. Let E be a Hilbert $\mathcal{A}$–$\mathcal{B}$–module. Then the operations $x \mapsto xb$, $x \mapsto \langle y,x\rangle$, and $x \mapsto ax$ are strongly continuous. Therefore, $\overline{E}^s$ is a Hilbert $\mathcal{A}$–$\mathcal{B}$–module and a von Neumann $\mathcal{B}$–module.*

Proof. This trivially follows from the embedding $E \subset \mathcal{B}(G,H) \subset \mathcal{B}(G \oplus H)$; see [Ske97]. ■

normal1 **C.4 Proposition.** *If E is the GNS-module of a normal completely positive mapping $T\colon \mathcal{A} \to \mathcal{B}$ between von Neumann algebras, then $\overline{E}^s$ is a von Neumann $\mathcal{A}$–$\mathcal{B}$–module.*

Proof. We have to show that the representation ρ for the GNS-module of T is normal. So let $\big(a_\lambda\big)$ be a bounded increasing net in $\mathcal{A}$. This net converges strongly to some $a \in \mathcal{A}$. Then for each $b \in \mathcal{A}$ also the net $\big(b^*a_\lambda b\big)$ is bounded and increasing, and it converges strongly to b^*ab, because multiplication in $\mathcal{A}$ is separately strongly continuous. Since T is normal, we have $\lim\limits_\lambda T(b^*a_\lambda b) = T(b^*ab)$. Let $g \in G$ be a unit vector and define the normal state $\langle g, \bullet g\rangle$ on $\mathcal{B}$. Then for $f = (b \otimes \mathbf{1} + \mathcal{N}_{\mathcal{A}\otimes\mathcal{B}}) \odot g \in E \odot G$ we have

$$\lim_\lambda \langle f, \rho(a_\lambda) f\rangle = \lim_\lambda \langle g, T(b^*a_\lambda b)g\rangle = \langle g, T(b^*ab)g\rangle) = \langle f, \rho(a)f\rangle$$

where f ranges over all vectors of the form $x \odot g$. ■

normal2 **C.5 Lemma.** *Let E be a von Neumann $\mathcal{A}$–$\mathcal{B}$–module. Let π be a normal representation of $\mathcal{B}$ on G. Then the representation ρ of $\mathcal{A}$ on $H = E \bar{\odot} G$ is normal.*

Proof. This is a small modification of the preceding proof. Let $\big(a_\lambda\big)$ be a bounded increasing net in $\mathcal{A}$. Then

$$\lim_\lambda \langle x, a_\lambda x\rangle = \langle x, ax\rangle \tag{C.1}$$ netconv

for all $x \in E$. This can be seen by choosing $\pi = \mathsf{id}$ to be the defining representation of $\mathcal{B}$ and then checking (C.1) with normal states $\langle g, \bullet g\rangle$ $(g \in G)$. Our assertion follows by the same check, however, turning to arbitrary normal π. ■

normal3 **C.6 Corollary.** *Let E be a von Neumann $\mathcal{A}$–$\mathcal{B}$–module and let F be a von Neumann $\mathcal{B}$–$\mathcal{C}$–module where $\mathcal{C}$ acts on a Hilbert space G. Then the strong closure $E \bar{\odot}^s F$ of $E \odot F$ in $\mathcal{B}(G, E \bar{\odot} F \bar{\odot} G)$ is a von Neumann $\mathcal{A}$–$\mathcal{C}$–module.*

PROOF. We have to show that the representation ρ of $\mathcal{A}$ on $E \bar{\odot} F \bar{\odot} G$ is normal. But this follows from Lemma C.5 (normal2) and the fact that the representation of $\mathcal{B}$ on $F \bar{\odot} G$ is normal. ■

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