The
Lévy-Khintchine Formula
for the
Quantum Group $SU_q(2)$

Inaugural Dissertation

zur

Erlangung der Doktorwürde

der

Naturwissenschaftlich–Mathematischen Gesamtfakultät

der

Ruprecht–Karls–Universität

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vorgelegt von

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Meinen Eltern

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Contents

Introduction

Convolution semi-groups of probability measures are among the most important objects of classical probability theory. On the one hand, they turn out to be classified by their infinitesimal generators. On the other hand, they classify themselves all stochastic processes with stationary and independent increments (or 'white noises') up to stochastic equivalence. In other words, it is possible to classify white noises by giving a formula for all infinitesimal generators. In the case of processes on the real line (or, more generally, on $\mathbf{R}^n$) this is the contents of the famous LÉVY–KHINTCHINE formula for the logarithm of the FOURIER transform of an infinitely divisible probability measure. This formula has been generalized to locally compact groups, basically in two ways. There is HUNT's formula [13] for the infinitesimal generators of convolution semi-groups of probability measures on an arbitrary LIE group (cf. also VON WALDENFELS [34]). In the other direction of generalization there is a formula for the logarithm of an infinitely divisible positive definite function on a (not necessarily commutative) locally compact group (see ARAKI [4], GUICHARDET [11], PARTHASARATHY, SCHMIDT [20], and STREATER [30]).

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We quickly recall the quantum stochastic generalizations of these notions. One *dualizes* the notion of a probability space $(\Omega, \mathcal{F}, \mu)$ by introducing a pair $(\mathcal{A}, \varphi_\mu)$ consisting of a (commutative) $*$–algebra $\mathcal{A}$ of certain integrable functions on Ω and a state φ_μ on $\mathcal{A}$ given by $\varphi_\mu(f) = \int f(p)\,\mu(dp)$. A quantum probability space is a pair $(\mathcal{A}, \varphi)$ where the algebra $\mathcal{A}$ is allowed to be non-commutative.

If the probability space Ω is also a compact group (with $\mathcal{F}$ being the BOREL sets) and $\mathcal{A}$ the coefficient algebra of Ω, then $\mathcal{A}$ has a natural HOPF $*$–algebra structure, where the comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is defined by $[\Delta(f)](x, y) = f(xy)$ for $x, y \in \Omega$, and the counit $\delta : \mathcal{A} \to \mathbf{C}$ is given by $\delta(f) = f(I)$ with I being the idenity of Ω and the antipode $S : \mathcal{A} \to \mathcal{A}$ is defined by $[S(f)](x) = f(x^{-1})$. (Cf. Section 5.4 of these notes where we make this explicit for $SU(2)$. See also the preliminaries in Chapter 1.) Consequently, a (not necessarily commutative) HOPF $*$–algebra is a candidate for a compact quantum group. We emphasize that in the literature, in general, more structure is required for a quantum group. However, in all cases a quantum group is a HOPF $*$–algebra. If Ω is only a semi-group, then the antipode is missing and one obtains only a $*$–bialgebra. If Ω is only locally compact (for instance the real line), we may choose the C^*–algebra of continuous bounded functions on Ω (or, even more generally, the $*$–algebra of bounded measurable functions). It is still possible to define a 'comultiplication' Δ in the stated way. However, we want to emphazise that this Δ is not the comultiplication of a HOPF algebra, because it does, in general, not map into the algebraic tensor product $\mathcal{A} \otimes \mathcal{A}$.

If μ and ν are probability measures on the (locally compact) group Ω then there is a convolution product $\mu \star \nu$ which is again a probability measure. This convolution structure turns over to the corresponding states. One has

$$\varphi_{\mu\star\nu} = \varphi_\mu \star \varphi_\nu = (\varphi_\mu \otimes \varphi_\nu) \circ \Delta.$$

A classical random variable is a function $J : \Omega \to E$ from Ω into a measurable space E. The probability measure $\mu \circ J^{-1}$ is called the distribution of J. Denoting by $\mathcal{A}_E$ the $*$–algebra of bounded measurable functions on E, we see that in the dualized language the mapping $j : \mathcal{A}_E \to \mathcal{A}$ defined by

$$j : f \longmapsto f \circ J$$

is a homomorphism between $*$–algebras and the state defined by the measure $\mu \circ J^{-1}$ is given by $\varphi_J = \varphi \circ j$. Consequently, a quantum random variable on a quantum probability space is a $*$–algebra homomorphism j from a $*$–algebra $\mathcal{B}$ into $\mathcal{A}$ and the state $\varphi \circ j$ is called its distribution. One says the quantum random variable is *over* $\mathcal{A}$ and *on* $\mathcal{B}$.

A pair of classical random variables j, k is called independent, if $\varphi \circ m \circ (j \otimes k) = \varphi \circ j \otimes \varphi \circ k$ with m denoting the multiplication map of $\mathcal{A}$. Since $\mathcal{A}$ is commutative, this property is sufficient to calculate all the momenta of the joined distribution. In the case of quantum random variables an additional assumption is required. Throughout these notes a pair j, k of quantum random variables is called independent if the above condition holds and if in addition $\Big[j(\mathcal{B}_j), k(\mathcal{B}_k)\Big] = \{0\}$ where $[\bullet, \bullet]$ is the usual commutator. For other notions of non-commutative independence see e.g. [15, 29, 33].

A classical stochastic process is a family of random variables $J_i, i \in \mathcal{I}$, mapping into the same measurable space and, consequently, a quantum stochastic process is a family of quantum random variables $j_i, i \in \mathcal{I}$, on the same $*$–algebra $\mathcal{B}$ (see Accardi, Frigerio, Lewis [2]). If the random variables of a classical stochastic process map into a compact group (semi-group) the dualized process is on a commutative Hopf $*$–algebra ($*$–bialgebra) $\mathcal{B}$. For the quantum analogue, $\mathcal{B}$ is allowed to be non-commutative. In this case one defines the usual convolution of mappings from coalgebra into an algebra by $j_k \star j_\ell = m \circ (j_k \otimes j_\ell) \circ \Delta$. If j_k and j_ℓ are independent then $j_k \star j_\ell$ is also a $*$–algebra homomorphism.

A stochastic process (dualized classical or quantum) on a $*$–bialgebra with independent and stationary increments (or white noise) is a stochastic process j_{st} indexed by $0 \leq s \leq t$, such that for all $s_1 \leq t_1 \leq \ldots \leq s_n \leq t_n$ the random variables $j_{s_i t_i}$ and $j_{s_\ell t_\ell}$ are independent for all $i, \ell \in \{1, \ldots, n\}$, the distribution $\varphi \circ j_{st}$ converges to $\varphi \circ j_{ss}$ weakly for $t \to s$,

$$j_{rs} \star j_{st} \;=\; j_{rt}$$

for $r \leq s \leq t$, and the distributions $\varphi \circ j_{st}$ depend only on the difference $t - s$. (See Accardi, Schürmann, von Waldenfels [3].)

In a series of papers [22, 23, 24, 25, 26] (which are summarized in [27]) Schürmann has shown as an extension of the classical results for stochastic processes with values in a compact Lie group that any quantum stochastic process j_{st} with stationary and independent increments on a $*$–bialgebra $\mathcal{B}$ gives rise to an infinitesimal generator ψ from which the process can be reconstructed up to quantum stochastic equivalence. All infinitesimal generators ψ are conditionally positive (linear, hermitian) functionals, i.e. ψ is positive on $\ker(\delta)$, vanishing at identity. Moreover, given any such ψ, there is a quantum stochastic process with stationary and independent increments associated with ψ. A representation of this process can be obtained on a Boson Fock space $\Gamma(H)$ over a Hilbert space H in the vacuum state by finding the unique solution of the system of quantum stochastic differential equations in the sense of Hudson and Parthasarathy [12]

$$dj_{st} \;=\; j_{st} \star (dA_t^* \circ \eta + d\Lambda_t \circ \pi \circ (Id - \delta\mathbf{1}) + dA_t \circ \tilde{\eta} + \psi\, dt), \qquad j_{ss} \;=\; \delta\mathbf{1},$$

where the $*$–representation π of $\mathcal{B}$ and the 1–cocycle η with repect to π can be constructed from ψ by Theorem 1.1 in Chapter 1 of these notes, and $\tilde{\eta} = \eta \circ *$.

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In these notes we investigate the structure of all conditionally positive functionals on the quantum group $SU_q(2)$. The $SU_q(2)$ introduced by Woronowicz [35] (for further references see the survey of Koornwinder [14]) is one of the standard examples of a quantum group and it seems, therefore, natural to investigate its behaviour as a state space for quantum white noise. Partly, our results (in particular those of the first three chapters) are already published in [28].

We find that there is, in some sense, a strong formal anology between our results and the classical ones. Let, for instance, $\mu_t, t \geq 0$, be a weakly continuous semi-group of probability measures on the real line. Then it is not difficult to see that it is possible to find a function $\widetilde{m}(k)$, such that the Fourier transform $\widetilde{\mu}_t(k) = \int e^{ikx}\, d\mu_t(x)$ is of the form

$$\widetilde{\mu}_t(k) \;=\; e^{t\widetilde{m}(k)}.$$

In 1934 Lévy [17] found that $\widetilde{m}(k)$ can always be chosen to be of the form

$$\widetilde{m}(k) = ir_1k - rk^2 + \int_{\mathbf{R}\setminus\{0\}} \left(e^{ikx} - 1 - \frac{ikx}{1+x^2} \right) d\mathcal{L}(x)$$

where $r_1 \in \mathbf{R}$, $r \geq 0$, and the Lévy measure $\mathcal{L}$ fulfills the condition $\int \frac{x^2}{1+x^2} d\mathcal{L}(x) < \infty$. By replacing in the integrand the function $\frac{ikx}{1+x^2}$ with $ik \sin x$, we obtain the equivalent formulation

$$\widetilde{m}(k) = ir_1k - rk^2 + \int_{\mathbf{R}\setminus\{0\}} (e^{ikx} - 1 - ik\sin x)\, d\mathcal{L}(x)$$

which is more convenient for our purposes.

For $q \in [-1,1]$ the quantum group $SU_q(2)$ can be considered as the $*$–bialgebra $\mathcal{A}_q$ having the matrix $\left(\begin{smallmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{smallmatrix}\right)$ as unitary corepresentation (see the preliminaries in Chapter 1). Our Theorem 4.16 in Chapter 4 says that in the case when $|q| < 1$ the infinitesimal generators of white noises on $\mathcal{A}_q$ have the form

$$\psi(a) = \psi_\delta(a) + \langle \eta_1 | \pi \circ \mathcal{P}(a) | \eta_1 \rangle. \qquad (*)$$

Here ψ_δ is a Gaussian part, $\mathcal{P}$ is a projection onto a certain ideal K_2 of $\mathcal{A}_q$, and π is a $*$–representation of $\mathcal{A}_q$ acting on a Hilbert space H where H contains no invariant subspace on which the representation is given by δ. The vector $\eta_1 \in \overline{H}$ is an element of a certain completion of H, and the brackets are to be understood as the continuation of the usual scalar product on H to elements of $\overline{H}$.

Notice that the mappings δ_φ, defined by $\alpha \mapsto e^{i\varphi}, \gamma \mapsto 0$ can be extended to a $*$–homomorphism of $\mathcal{A}_q$. Using the derivatives

$$\delta^{(n)}(a) = \frac{d^n \delta_\varphi(a)}{d\varphi^n}\Big|_{\varphi=0},$$

the precise form of ψ_δ and $\mathcal{P}$ is given by

$$\psi_\delta = r_1\delta' + r\delta''$$

and

$$\mathcal{P} = Id - \delta\mathbf{1} - \delta' \frac{\alpha - \alpha^*}{2i}.$$

$\overline{H}$ is the completion of H with respect to the scalar product

$$\langle \pi(\alpha^* - \mathbf{1}) \bullet | \pi(\alpha^* - \mathbf{1}) \bullet \rangle$$

where $\pi(\alpha^* - \mathbf{1})$ has to be an injective operator according to the condition on H. It is easy to check that $\mathcal{P}$ projects onto the ideal K_2 being the linear span of all products of elements of $\ker(\delta)$.

Now we want to explain why the formula for $\widetilde{m}$ is in analogy to $(*)$. It is well-known that the semi-group T_t of linear transformations on the algebra $C_b(\mathbf{R})$ of continuous bounded functions on the real line, defined by setting

$$T_t f(x) = \int f(x+y)\, d\mu_t(y),$$

has an infinitesimal generator L. If we define a linear functional on a suitable dense subset of $C_b(\mathbf{R})$ by setting

$$\psi(f) = Lf(0),$$

we obtain (see [13]) that the 'Fourier transform' of ψ is given by

$$\psi(e^{ikx}) = \widetilde{m}(k).$$

By defining the derivatives

$$\delta^{(n)}(f) \quad = \quad f^{(n)}(0)$$

and the projection

$$[\mathcal{P}(f)](x) \quad = \quad f(x) - \delta(f) - \delta'(f)\sin x,$$

we obtain

$$\psi(f) \quad = \quad r_1\delta' + r\delta'' + \int_{\mathbf{R}\setminus\{0\}} [\mathcal{P}(f)](x)\, d\mathcal{L}(x).$$

Notice that ψ is positive on $\ker(\delta)$, that the domain of $\mathcal{P}$ consists of all functions in $C_b(\mathbf{R})$ differentiable at 0 and that $\mathcal{P}$ projects onto the ideal K_2 in this domain consisting of all functions $f \in C_b(\mathbf{R})$ with $f(0) = f'(0) = 0$.

Now consider the positive functional φ_λ on $C_b(\mathbf{R})$, defined by setting

$$\varphi_\lambda(f) \quad = \quad \int f(x)\, d\lambda(x)$$

where λ is the finite measure

$$d\lambda(x) \quad = \quad \frac{x^2}{1+x^2}\, d\mathcal{L}(x).$$

It has a GNS representation π on $H = L^2(\mathbf{R}, \lambda)$ with the constant function 1 being the cyclic vector. If we introduce the function space

$$\overline{H} \quad = \quad \left\{ \sqrt{\frac{1+x^2}{x^2}}\eta(x) \,\middle|\, \eta \in H \right\}$$

and $\eta_1 \in \overline{H}$ by setting

$$\eta_1(x) \quad = \quad \sqrt{\frac{1+x^2}{x^2}},$$

we indeed obtain that the functional ψ on $C_b(\mathbf{R})$ can also be written in the form $(*)$.

Having a look at the results for the classical case ($q = 1$) in Section 5.4 (cf. also the general results of Hunt [13]), we see that the 'quantization' $SU_q(2)$ of the three-parameter group $SU(2)$ behaves much more like a one-parameter group. It turns out that also the classical case can be described in the form $(*)$; see Theorem 5.18. However, the Gaussian part ψ_δ and the projection $\mathcal{P}$ are considerably more complicated and the space $\overline{H}$ out of which η_1 can be chosen is no longer a completion of the representation space H. Notice that the mapping $a \mapsto f(\varphi) = \delta_\varphi(a)$ defines a homomorphism from $\mathcal{A}_q$ onto the coefficient algebra of the one-dimensional torus which, therefore, can be considered to be contained in $\mathcal{A}_q$ as a subgroup. In view of this 'embedding' we can say that in the case $|q| < 1$ the Gaussian part and the projection are those of the one-dimensional torus.

The case $q = -1$ which we call anti-classical is in some respects a 'mixture' of the foregoing cases (see Section 5.5). It can also be described in the form $(*)$ with the exception that we have to add a part $\psi_{\underline{\delta}}$ which we call anti-Gaussian; see Theorem 5.26. This part corresponds to the Gaussian part of the classical case and is similarly complicated. On the other hand, the Gaussian part and the projection of the anti-classical case are those of the case $|q| < 1$.

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The contents of these notes is organized as follows. By Theorem 1.1 in Chapter 1 the search for all possible white noises (i.e. for all conditionally positive functionals) becomes, as in classical probability theory, a cohomological problem; cf. [4, 11, 20, 30]. Throughout the first four chapters we solve the problem of finding all conditionally positive functionals for the cases when $0 < |q| < 1$ in which we have the crucial Lemma 1.6. This lemma shows to be the key to the whole theory.

In Chapter 2 we solve the cohomological problems. It turns out that the conditionally positive functionals are classified (more or less) by states.

The representation theory is placed in Chapter 3. By Lemma 1.6 we are able to give a new completely algebraic proof and recover the well-known irreducible representations. Using the comultiplication which induces a convolution of representations, we are able to decompose our results not only according to irreducible representations but into expressions which are built up in terms of one-dimensional representations and only one infinite dimensional irreducible representation.

In Chapter 4 we recover a faithful representation, introduced by WORONOWICZ, as the convolution square of an irreducible representation. We show that this representation is a C^*–algebra isomorphism. We find that all continuous infinitesimal generators are (more or less) given by states. By solving the problem of finding a topology in which all conditionally positive functionals are continuous, we find as a main result of these notes a new formulation of our results of Chapter 2. This new formula is in perfect anology to the classical LÉVY-KHINTCHINE formula.

Chapter 5 deals with the exceptional cases $q = 1$ (which is the classical case), $q = -1$ (which is a GRASSMANN analogue of the classical case), and $q = 0$. The cases $q = -1, 1$ show to be quite similar and much more complicated than the other ones. For $q \in (-1, 1)$ we find that all our results are, in a certain sense, equivalent. As a further main result we point out (Theorem 5.27) that in all cases the set consisting of all functionals $\varphi \circ \mathcal{Q}$, where φ runs over all positive functionals on $\mathcal{A}_q$ and $\mathcal{Q}$ runs over all possible projections onto K_2, is dense in the set of all conditionally positive functionals on $\mathcal{A}_q$ with respect to pointwise convergence.

In the final chapter we show that it is possible to approximate any given conditionally positive functional for the cases $q = -1, 1$ by conditionally positive functionals for $|q| < 1$. This is nothing but a *correspondence principle* for $SU_q(2)$ and shows that the quantum group $SU_q(2)$ may indeed serve as convenient quantization of $SU(2)$.

Chapter 1

Basic definitions and results

1.1 Preliminaries

Let $\mathcal{A}$ be a unital $*$–algebra and $\delta : \mathcal{A} \to \mathbf{C}$ a $*$–algebra homomorphism into the complex numbers. Let $K_1 = \ker(\delta)$ be the kernel of δ. We call a linear hermitian functional ψ on $\mathcal{A}$ *conditionally positive* (with respect to δ) if it is positive on the ideal K_1, i.e. if

$$\psi(a^*a) \geq 0 \quad \text{for all} \quad a \in K_1.$$

Let $\mathcal{D}$ be a pre–Hilbert space and π a $*$–representation of $\mathcal{A}$ acting on $\mathcal{D}$. By a *1–cocycle* with respect to π we mean a linear mapping $\eta : \mathcal{A} \to \mathcal{D}$ such that

$$\eta(ab) = \pi(a)\eta(b) + \eta(a)\delta(b) \quad \text{for all} \quad a, b \in \mathcal{A}. \tag{1.1}$$

Let $K_2 = \overline{\text{lin}}(K_1 \cdot K_1)$ be the ideal which is given by the linear span of all products of elements of K_1. In [26] the following has been proved.

Theorem 1.1 (Schürmann) *For an arbitrary conditionally positive functional ψ there is a triplet $(\mathcal{D},\pi,\eta)$ consisting of a pre–Hilbert space $\mathcal{D}$, a $*$–representation π on $\mathcal{D}$ and a 1–cocycle η with respect to this representation such that the values of ψ on K_2 are given by*

$$\psi(ab) = \langle \eta(a^*) | \eta(b) \rangle \quad \textit{for all} \quad a, b \in K_1. \tag{1.2}$$

The restriction of π to the invariant subspace $\eta(\mathcal{A})$ of $\mathcal{D}$ is determined up to unitary equivalence.

By Theorem 1.1 we are able to reduce the problem of finding all conditionally positive functionals to that of finding all $*$–representations, all 1–cocycles with respect to these representations, and checking for which of them we can find a conditionally positive functional satisfying (1.2).

We define for any $*$–representation π on a pre–Hilbert space $\mathcal{D}$ and any vector $\eta \in \mathcal{D}$ the mappings $\langle \eta | \pi | \eta \rangle : \mathcal{A} \to \mathbf{C}$ and $\pi\eta : \mathcal{A} \to \mathcal{D}$ by

$$\langle \eta | \pi | \eta \rangle(a) = \langle \eta | \pi(a) | \eta \rangle \text{ and } (\pi\eta)(a) = \pi(a)\eta \tag{1.3}$$

respectively. The mapping $Id - \delta\mathbf{1} : \mathcal{A} \to \mathcal{A}$ defined by

$$(Id - \delta\mathbf{1})(a) = a - \delta(a)\mathbf{1},$$

is a canonical projection onto K_1. We immediately see that

$$(\pi\eta) \circ (Id - \delta\mathbf{1}) \text{ and } \langle \eta | \pi | \eta \rangle \circ (Id - \delta\mathbf{1})$$

are a 1–cocycle and a conditionally positive functional respectively, satisfying (1.2). In cohomology theory such a 1–cocycle is called a *coboundary* and the functional is a positive multiple of $\varphi - \delta$, where φ is a *state* on $\mathcal{A}$.

From Leibniz rule of differentiation we obtain the following

Proposition 1.2 *Let δ_φ be a family of homomorphisms $\delta_\varphi : \mathcal{A} \to \mathbf{C}$ which are pointwise continuous in φ and such that $\delta_0 = \delta$. Then we have*

(i) If δ_φ is pointwise differentiable with respect to φ at $\varphi = 0$, then δ_0' is a 1–cocycle with respect to δ.

(ii) If δ_φ is pointwise twice differentiable with respect to φ at $\varphi = 0$, then δ_0' is a 1–cocycle with respect to δ and $\frac{\delta_0''}{2}$ is a conditionally positive functional satisfying (1.2).

A $*$–*coalgebra* $\mathcal{C}$ is a ($\mathbf{C}$–)linear space together with two linear mappings, the comultiplication $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and the counit $\delta : \mathcal{C} \to \mathbf{C}$, such that

$$(\Delta \otimes Id) \circ \Delta = (Id \otimes \Delta) \circ \Delta \qquad \text{(coassociativity)}$$
$$(\delta \otimes Id) \circ \Delta = Id = (Id \otimes \delta) \circ \Delta, \qquad \text{(counit property)}$$

and an involution $* : \mathcal{C} \to \mathcal{C}$ such that

$$\begin{aligned} \Delta \circ * &= (* \otimes *) \circ \Delta \text{ and} \\ \delta \circ * &= \overline{\delta}. \end{aligned}$$

A $*$–*bialgebra* $\mathcal{B}$ is a unital $*$–algebra and also a $*$–coalgebra such that Δ and δ are unital algebra homomorphisms, i.e.

$$\begin{aligned} \Delta(\mathbf{1}) &= \mathbf{1} \otimes \mathbf{1}, \\ \Delta(ab) &= \Delta(a)\Delta(b) \text{ and} \\ \delta(ab) &= \delta(a)\delta(b) \text{ for all } a, b \in \mathcal{B} \end{aligned}$$

where $\mathcal{B} \otimes \mathcal{B}$ is equipped with the natural multiplication

$$(a \otimes b)(a' \otimes b') = (aa' \otimes bb').$$

If we define an involution on $\mathcal{B} \otimes \mathcal{B}$ by $* \otimes *$, then Δ and δ are also $*$–algebra homomorphisms.

A Hopf $*$–*algebra* $\mathcal{H}$ is a $*$–bialgebra together with a linear mapping $\mathcal{S} : \mathcal{H} \to \mathcal{H}$, called antipode, such that

$$m \circ (Id \otimes \mathcal{S}) \circ \Delta = \delta \mathbf{1} = m \circ (\mathcal{S} \otimes Id) \circ \Delta$$

where $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ denotes the multiplication coming from the algebra structure of $\mathcal{H}$. It can be shown that $\mathcal{S}$ is an algebra anti-homomorphism, i.e.

$$\mathcal{S}(ab) = \mathcal{S}(b)\mathcal{S}(a).$$

An $n \times n$–matrix $U = (u_{ij})_{i,j=1,\ldots,n}$ with entries in a $*$–bialgebra $\mathcal{B}$ is said to be a *corepresentation* of $\mathcal{B}$ if $\mathcal{B}$, as a unital $*$–algebra, is generated by the matrix entries and if the coalgebra structure of $\mathcal{B}$ is determined by

$$\begin{aligned} \Delta(u_{ij}) &= \sum_{k=1}^{n} u_{ik} \otimes u_{kj} \text{ and} \\ \delta(u_{ij}) &= \delta_{ij}. \end{aligned}$$

U is said to be a corepresentation of a Hopf $*$–algebra $\mathcal{H}$ if it is a corepresentation of the $*$–bialgebra structure of $\mathcal{H}$ and if the antipode of $\mathcal{H}$ is determined by

$$\mathcal{S}(u_{ij}) = {u_{ji}}^*.$$

A corepresentation U of a $*$–bialgebra $\mathcal{B}$ and a Hopf $*$–algebra $\mathcal{H}$ respectively is said to be unitary if

$$U^*U = \mathbf{1} = UU^*.$$

For further details on co–, bi– and Hopf algebras see the textbooks of Abe [1] and Sweedler [31] on these subjects.

1.2 Conventions

Our conditionally positive functionals are always assumed to be hermitian. All representations are supposed to be non-degenerate $*$–representations. This yields immediately that all 1–cocycles vanish at $\mathbf{1}$. Cocycle always means 1–cocycle.

For a Hilbert space H by $\mathcal{B}(H)$ we mean the von Neumann algebra of bounded operators on H.

The natural, integer, real, and complex numbers are denoted by $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, and $\mathbf{C}$ respectively. Let a be an element of an involutive algebra (e.g. $\mathbf{C}$ together with complex conjugation). For $\ell \in \mathbf{Z}$ we define $a^{\check{\ell}}$ by putting

$$a^{\check{\ell}} \quad = \quad \begin{cases} a^{\ell} & \text{for } \ell \geq 0 \\ (a^*)^{-\ell} & \text{for } \ell < 0 \end{cases} .$$

If an equation concerning a and a^* is also valid if we replace a by a^* (and conversely), we indicate this by writing $a^{(*)}$ for a and a^*, and writing $a^{*(*)}$ for a^* and a, respectively.

1.3 Definition of the algebra structure

For the time being, we only need the algebra structure of the quantum group $SU_q(2)$ and a homomorphism δ for investigating its conditionally positive functionals. (Later this homomorphism will be the counit of the underlying coalgebra structure.)

Definition 1.1 *For a real number q with $|q| \in (0,1)$ we denote by $\mathcal{A}_q$ the unital $*$–algebra generated by α, γ, with the following relations:*

$$\begin{array}{lrcl} \text{(a)} & \alpha\gamma & = & q\gamma\alpha \\ \text{(b)} & \alpha\gamma^* & = & q\gamma^*\alpha \\ \text{(c)} & \gamma^*\gamma & = & \gamma\gamma^* \\ \text{(d)} & \alpha\alpha^* - \alpha^*\alpha & = & (1-q^2)\gamma^*\gamma \\ \text{(e)} & \gamma^*\gamma + \alpha^*\alpha & = & \mathbf{1}. \end{array} \tag{1.4}$$

These relations describe the algebra structure of $SU_q(2)$ in the case when $|q| \in (0,1)$ (cf. [35]). The remaining cases $q = -1, 0, 1$ are treated seperately in Chapter 5. Throughout the other chapters we always refer to the case $|q| \in (0,1)$ unless stated otherwise. The irreducible $*$–representations (cf. [32]) are given by the following two families:

Theorem 1.3 (Vaksman, Soibelman) *Let h_0 be a Hilbert space with an ONB $\{e_k\}_{k\in\mathbf{N}_0}$. The following equations*

(i)

$$\begin{array}{rcll} \rho_\varphi(\alpha)e_k & = & \sqrt{1-q^{2k}}\,e_{k-1}, & k \in \mathbf{N} \\ \rho_\varphi(\alpha)e_0 & = & 0, & k = 0 \\ \rho_\varphi(\gamma)e_k & = & e^{i\varphi}q^k e_k, & k \in \mathbf{N}_0 \end{array}$$

(ii)

$$\begin{array}{rcl} \delta_\varphi(\alpha) & = & e^{i\varphi} \\ \delta_\varphi(\gamma) & = & 0 \end{array}$$

define irreducible $*$–representations $\rho_\varphi : \mathcal{A}_q \to \mathcal{B}(h_0)$ and $\delta_\varphi : \mathcal{A}_q \to \mathbf{C}$ of $\mathcal{A}_q$ for any $\varphi \in [0, 2\pi)$.

Any irreducible $*$–representation must be unitarily equivalent to one of these representations.

We will obtain this result (i.e. the completeness and the well-definedness of ρ_φ and δ_φ) as Corollary 3.5 in the general representation theory in Chapter 3.

The homomorphism δ is just δ_0. Clearly, δ_φ evaluated at a fixed algebra element a is an analytic function of φ. We use the notation of Proposition 1.2 and ommit the subscript $\varphi = 0$.

1.4 The structures of K_1 and K_2

Now we investigate the sets $K_1 = \ker(\delta)$ and K_2 the latter being the linear span of all products of elements of K_1. Clearly, if we introduce

$$\beta = \alpha - \mathbf{1},$$

the set $\{\mathbf{1}, \beta, \beta^*, \gamma, \gamma^*\}$ generates the whole algebra. Henceforth, since we have $\beta, \beta^*, \gamma, \gamma^* \in K_1$ and $\mathbf{1} \notin K_1$, the set

$$G = \{\beta, \beta^*, \gamma, \gamma^*\} \tag{1.5}$$

generates K_1. Relations (1.4), expressed in terms of β and γ, transform into

$$\begin{array}{llcl}
(\widetilde{\mathrm{a}}) & \beta\gamma & = & q\gamma\beta - (1-q)\gamma \\
(\widetilde{\mathrm{b}}) & \beta\gamma^* & = & q\gamma^*\beta - (1-q)\gamma^* \\
(\widetilde{\mathrm{c}}) & \gamma^*\gamma & = & \gamma\gamma^* \\
(\widetilde{\mathrm{d}}) & \beta\beta^* - \beta^*\beta & = & (1-q^2)\gamma^*\gamma \\
(\widetilde{\mathrm{e}}) & \gamma^*\gamma + \beta^*\beta + \beta^* + \beta & = & 0.
\end{array} \tag{1.6}$$

An arbitrary element a of $\mathcal{A}_q$ can be written in the form

$$a \;=\; c_1\mathbf{1} \;+\; \sum_{g\in G} c_g g \;+\; c$$

where c_1, c_g are complex numbers and $c \in K_2$. From Relations $(\widetilde{\mathrm{a}})$, $(\widetilde{\mathrm{b}})$, and $(\widetilde{\mathrm{e}})$, we immediately see that the elements γ, γ^*, and $\beta + \beta^*$ can be expressed as sums of products of elements of K_1, hence are elements of K_2. In other words, we have that any $a \in \mathcal{A}_q$ can be written as

$$a \;=\; c_1\mathbf{1} + c_2\frac{\beta - \beta^*}{2i} + c \;=\; c_1\mathbf{1} + c_2\frac{\alpha - \alpha^*}{2i} + c \tag{1.7}$$

where c_1, c_2 are complex numbers and $c \in K_2$, in at least one way.

Proposition 1.4 *Decomposition (1.7) is unique for any $a \in \mathcal{A}_q$.*

PROOF We apply δ and δ' to (1.7). By definition δ is 0 on K_1 and K_2, hence $\delta(a) = c_1$.

Using the factorization property of δ, we obtain by an application of LEIBNIZ rule of differentiation that δ' vanishes on K_2 as well as it does on $\mathbf{1}$. Hence

$$\delta'(a) = c_2\delta'\left(\tfrac{\alpha-\alpha^*}{2i}\right) = c_2.$$

Therefore, the numbers c_1, c_2 are determined by a, and so is c by (1.7). ∎

We conclude this paragraph by writing down the canonical projection onto K_2.

Corollary 1.5 *The mapping $\mathcal{P}$*

$$\begin{array}{rcl}
\mathcal{P} : \mathcal{A}_q & \longrightarrow & \mathcal{A}_q \\
a & \longmapsto & a - \delta(a)\mathbf{1} - \delta'(a)\dfrac{\alpha - \alpha^*}{2i}
\end{array}$$

is a projection onto K_2. *I.e.* $\mathcal{P}(\mathcal{A}_q) = K_2$ *and* $\mathcal{P}^2 = \mathcal{P}$.

1.5 A fundamental lemma on the representations

As a preparation for the next chapters, we formulate a lemma concerning the representations of $\mathcal{A}_q$. It replaces the *spectral theorem* applied to the representing operator of γ (which is *normal* due to Relation (c)). We will use this lemma to prove the properties of the cocycles (Chapter 2) and to establish the representation theory (Chapter 3) *without* using the spectral theorem.

We mention that due to Relation (e) all representation operators are bounded. Therefore, we can assume the pre-HILBERT space $\mathcal{D}$ (on which the representation acts) to be a HILBERT space.

Lemma 1.6 *Let H be a* HILBERT *space and $\pi : \mathcal{A}_q \to \mathcal{B}(H)$ a $*$–representation of $\mathcal{A}_q$ acting on it. Then we have*

$$\pi(\gamma) \text{ injective } \Longrightarrow \lim_{k\to\infty} \pi(\alpha^k) = 0$$

in the strong operator topology.

PROOF It is easy to see, that the range of an injective normal operator is dense. Therefore, if $\pi(\gamma)$ is injective, we have that $\pi(\gamma)H$ is dense in H.

Let f be any element of H. We can find a sequence $\{f_n\}_{n\in\mathbf{N}}$ with $f_n \in \pi(\gamma)H$ which approximates f. In other words, for any $\epsilon > 0$ we can find $N \in \mathbf{N}$ such that for any bounded operator $B \in \mathcal{B}(H)$

$$\|Bf - Bf_n\| < \|B\|\frac{\epsilon}{2} \text{ for all } n > N$$

holds. From Relation (e) we obtain $\|\pi(\alpha)\| \le 1, \|\pi(\gamma)\| \le 1$ and immediately

$$\|\pi(\alpha^k)\| \le 1, \|\pi(\gamma^k)\| \le 1$$

for all $k \in \mathbf{N}$. Thus, we have

$$\|\pi(\alpha^k)f - \pi(\alpha^k)f_n\| < \frac{\epsilon}{2} \text{ for all } n > N \tag{1.8}$$

independent of $k \in \mathbf{N}$.

We associate with $\{f_n\}_{n\in\mathbf{N}}$ the sequence $\{\hat{f}_n\}_{n\in\mathbf{N}}$ defined by $\pi(\gamma)\hat{f}_n = f_n$. By Relation (a) we obtain

$$\|\pi(\alpha^k)f_n\| = \|\pi(\alpha^k\gamma)\hat{f}_n\| = q^k\|\pi(\gamma\alpha^k)\hat{f}_n\| \le q^k\|\hat{f}_n\|. \tag{1.9}$$

Thus, choosing $n > N$ and K such that $q^K\|\hat{f}_n\| < \frac{\epsilon}{2}$ we obtain, by combining estimates (1.8) and (1.9) that

$$\|\pi(\alpha^k)f\| \le \|\pi(\alpha^k)f - \pi(\alpha^k)f_n\| + \|\pi(\alpha^k)f_n\| < \epsilon \text{ for all } k > K.$$

This concludes the proof. ■

1.6 Topologies on $\mathcal{A}_q$

The algebra $\mathcal{A}_q$ is usually equipped with a C^*–norm $\| \bullet \|$ given by

$$\|a\| = \sup_\pi \|\pi(a)\|$$

where the supremum is taken over all $*$–representations. Clearly, this is a semi–C^*–norm. On the other hand, the existence of a faithful representation (see Section 4.1) yields that it is positive definite, hence indeed a norm.

Let $\mathcal{A}$ be the C^*–algebra which we obtain by completion of $\mathcal{A}_q$ with respect to this norm. Then by the above definition it is clear that any representation $\pi : \mathcal{A}_q \to \mathcal{B}(H)$ is continuous. Thus, it can be extended to $\mathcal{A}$ and, in this manner, we obtain all representations of $\mathcal{A}$ because representations of C^*–algebras are continuous.

The C^*–completion is already introduced here because we will be concerned with geometric series. However, we want to emphasize that all our results on $\mathcal{A}_q$ can be expressed and proved in terms of the algebra $\mathcal{A}_q$ without using the C^*–language; see our discussion in Section 2.4.

The range of any irreducible representation contains the subset of all compact operators on the underlying HILBERT space. ($\mathbf{1} - \alpha^*\alpha = \gamma^*\gamma$ is mapped to a compact operator. On the other hand, it is well-known that the range of any C^*–algebra under a $*$–representation contains all compact operators if it contains at least one.) Henceforth, $\mathcal{A}$ is a type I C^*–algebra.

Now let ω be a $*$–representation of $\mathcal{A}$ which is also an isomorphism between C^*–algebras (i.e. an isometry) such that $\omega(\gamma)$ is injective. (Such a representation exists; cf. WORONOWICZ [35] and Section 4.1 of these notes.) Then ω induces a notion of strong (weak) convergence in $\mathcal{A}$ by

$$\lim_{n\to\infty} a_n = a \text{ strongly (weakly)}$$

if and only if

$$\lim_{n\to\infty} \omega(a_n) = \omega(a) \text{ strongly (weakly)}$$

for $a_n, a \in \mathcal{A}$. (In general, these topologies will depend on the special choice of ω, but this does not affect our results.) Now Lemma 1.6 reads $\lim_{k\to\infty} \alpha^k = 0$ in the strong topology.

In Section 4.4 we will introduce two other norms, in order to make all cocycles and all conditionally positive functionals continuous.

Chapter 2

Cocycles and conditionally positive functionals

In this chapter we classify all cocycles and conditionally positive functionals on $\mathcal{A}_q$. This is done by establishing two linear mappings $\mathcal{O}, \mathcal{T} : \mathcal{A}_q \to \mathcal{A}$ which satisfy the following conditions analogous to Equations (1.1) and (1.2):

$$\mathcal{O}(ab) = a\mathcal{O}(b) + \mathcal{O}(a)\delta(b) \text{ for all } a, b \in \mathcal{A}_q \tag{2.1}$$

and

$$\mathcal{T}(ab) = \mathcal{O}(a^*)^*\mathcal{O}(b) \text{ for all } a, b \in K_1. \tag{2.2}$$

Assume for the moment that $\mathcal{O}$ and $\mathcal{T}$ have already been constructed. Then we obtain that

$$\psi \;=\; \langle\eta|\pi|\eta\rangle \circ \mathcal{T}$$

is a conditionally positive functional and

$$(\pi\eta) \circ \mathcal{O}$$

is a cocycle such that Equation (1.2) is fulfilled. Notice that the mappings $\langle\eta|\pi|\eta\rangle$ and $\pi\eta$ defined by (1.3) are continuous.

The continuous projection $Id - \delta\mathbf{1}$ onto K_1 is the simplest possible choice for $\mathcal{O}$ *and* $\mathcal{T}$. We return to this case in Section 4.3 where we are concerned with continuous conditionally positive functionals.

In the following, $\mathcal{O}$ will show to be fixed by the additional requirement that $\mathcal{O}(\alpha^*) = \mathbf{1}$.

2.1 The mappings $\mathcal{O}$ and $\mathcal{T}$

The key of establishing these two mappings is

Lemma 2.1 *We have*

$$\lim_{\substack{p\to 1\\ p\in[0,1)}} \frac{\mathbf{1} - \alpha^*}{\mathbf{1} - p\alpha^*} = \mathbf{1}$$

in the strong topology.

PROOF Notice that for $|p| < 1$ the left hand side is well-defined since $\|\alpha^*\| \leq 1$. The first step is to compute

$$\mathbf{1} - \frac{\mathbf{1} - \alpha^*}{\mathbf{1} - p\alpha^*} = (1-p)\frac{\alpha^*}{\mathbf{1} - p\alpha^*}.$$

We make the substitution

$$\mathcal{J}_p = \frac{\mathbf{1}}{\mathbf{1} - p\alpha^*}$$

and we will show that $(1-p)\mathcal{J}_p$ converges to 0. First we show that we have at least weak convergence .

Indeed, given any two unit vectors $e, e' \in H_\omega$ (the representation space of ω) we obtain by expanding into a VON NEUMANN series

$$(1-p)\left\langle e \left| \frac{\mathbf{1}}{\mathbf{1} - p\omega(\alpha^*)} \right| e' \right\rangle = (1-p)\sum_{k=0}^{\infty} p^k \overline{\langle e\,|\omega(\alpha)^k|\,e'\rangle}. \tag{2.3}$$

As can be seen by standard proof technique (cf. e.g. the proof of ABEL's limit theorem), this power series in p is already known to converge to 0 as p tends to 1, if its coefficients converge to 0. But this is true due to Lemma 1.6.

To show strong convergence of $(1-p)\mathcal{J}_p$ we have to show weak convergence of $(1-p)^2\mathcal{J}_p^*\mathcal{J}_p$. We have

$$(1-p)^2 = \frac{1-p}{1+p}(1-p^2)$$

and

$$\begin{aligned}
\mathcal{J}_p^*\mathcal{J}_p &= \left(\mathbf{1} + \frac{p\alpha}{\mathbf{1} - p\alpha}\right)\left(\mathbf{1} + \frac{p\alpha^*}{\mathbf{1} - p\alpha^*}\right) \\
&= \mathbf{1} + \frac{p\alpha}{\mathbf{1} - p\alpha} + \frac{p\alpha^*}{\mathbf{1} - p\alpha^*} + \frac{p\alpha}{\mathbf{1} - p\alpha}\,\frac{p\alpha^*}{\mathbf{1} - p\alpha^*}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
(1-p)^2\mathcal{J}_p^*\mathcal{J}_p &= \frac{1-p}{1+p}\Bigg(\mathbf{1} + \frac{p\alpha}{\mathbf{1} - p\alpha} + \frac{p\alpha^*}{\mathbf{1} - p\alpha^*} \\
&\qquad - \frac{p}{\mathbf{1} - p\alpha}(\mathbf{1} - \alpha\alpha^*)\frac{p}{\mathbf{1} - p\alpha^*}\Bigg) \\
&= \frac{1-p}{1+p}\Bigg(\mathbf{1} + \frac{p\alpha}{\mathbf{1} - p\alpha} + \frac{p\alpha^*}{\mathbf{1} - p\alpha^*} \\
&\qquad - q^2\frac{p}{\mathbf{1} - p\alpha}\,\frac{p}{\mathbf{1} - pq^2\alpha^*}\gamma^*\gamma\Bigg)
\end{aligned}$$

where we made use of $\mathbf{1} - \alpha\alpha^* = q^2\gamma^*\gamma$. From our foregoing considerations we see that the first three summands multiplied by the factor in front of the brackets converge to 0 weakly. In the last summand the factor

$$\frac{p}{\mathbf{1} - pq^2\alpha^*}$$

converges in norm to a bounded operator. The factor

$$\frac{1-p}{1+p}\,\frac{p}{\mathbf{1} - p\alpha}$$

converges to 0 weakly. Therefore, the last summand converges also to 0 at least weakly. ■

For $p \in [0,1)$ let $\mathcal{O}_p$ be the mapping $\mathcal{O}_p : \mathcal{A}_q \to \mathcal{A}$ defined by

$$\mathcal{O}_p(a) \quad = \quad (Id - \delta\mathbf{1})(a)(p\alpha^* - \mathbf{1})^{-1}.$$

Then we have the following

Theorem 2.2 *For all $a \in \mathcal{A}_q$, the elements $\mathcal{O}_p(a)$ of $\mathcal{A}$ converge strongly for $p \to 1$ to an element of $\mathcal{A}$. The mapping $\mathcal{O} : \mathcal{A}_q \to \mathcal{A}$ defined by*

$$\mathcal{O}(a) \quad = \quad \lim_{p \to 1} \mathcal{O}_p(a)$$

fulfills Equation (2.1) and the additional condition $\mathcal{O}(\alpha^*) = \mathbf{1}$.

PROOF First we show that the limit exists. This is obvious for $a = \mathbf{1}$ (since $\mathcal{O}_p(\mathbf{1}) = 0$). Therefore, it suffices to show it on the set G of generators. (Cf. our argument leading to Equation (1.7).)

For β^* our work is already done by Lemma 2.1. Moreover, we find that the condition $\mathcal{O}(\alpha^*) = \mathcal{O}(\beta^*) = \mathbf{1}$ is fulfilled.

For γ and γ^* we obtain, by using the VON NEUMANN series and Relations (b)* and (a)* respectively,

$$\gamma^{(*)}(p\alpha^* - \mathbf{1})^{-1} = (pq\alpha^* - \mathbf{1})^{-1}\gamma^{(*)}.$$

Thus, we find that $\mathcal{O}(\gamma^{(*)})$ exists and

$$\mathcal{O}(\gamma^{(*)}) = (q\alpha^* - \mathbf{1})^{-1}\gamma^{(*)}.$$

Using the relation

$$(\mathbf{1} - x)^{-1} = \mathbf{1} + x(\mathbf{1} - x)^{-1}, \tag{2.4}$$

which holds for any $x \in \mathcal{A}$ if $\|x\| < 1$, we obtain after straightforward calculations that

$$\begin{aligned} \mathcal{O}_p(\beta) &= (\alpha - \mathbf{1})(p\alpha^* - \mathbf{1})^{-1} \\ &= (1 - p)(\mathbf{1} - p\alpha^*)^{-1} + p(\mathbf{1} - \alpha\alpha^*)(\mathbf{1} - p\alpha^*)^{-1} - \alpha. \end{aligned}$$

From Relations (d) and (e) we see that $\mathbf{1} - \alpha\alpha^* = q^2\gamma^*\gamma$. We insert this and obtain by repeated use of Relations (a) and (b):

$$\mathcal{O}_p(\beta) = (1 - p)(\mathbf{1} - p\alpha^*)^{-1} + pq^2(\mathbf{1} - pq^2\alpha^*)^{-1}\gamma^*\gamma - \alpha.$$

In this expression the limit p to 1 can be performed without any problems. The first summand disappears (see the proof of Lemma 2.1). In the second summand p is just replaced by 1. The third one does not depend on p at all. Substituting $\gamma^*\gamma = \mathbf{1} - \alpha^*\alpha$, we obtain

$$\begin{aligned} \mathcal{O}(\beta) &= q^2(\mathbf{1} - q^2\alpha^*)^{-1}(\mathbf{1} - \alpha^*\alpha) - \alpha \\ &= q^2(q^2\alpha^* - \mathbf{1})^{-1}(q^{-2}\alpha - \mathbf{1}) \end{aligned}$$

where we again made use of (2.4).

Notice that we did not only show the existence of $\mathcal{O}$, but also listed its values on the generators. Clearly, the limit fulfills Equation (2.1), because $\mathcal{O}_p$ does for any p. ∎

N.B.: In Section 2.3 we will see (cf. Corollary 2.7) that $\mathcal{O}$ is already determined by the properties stated in the theorem.

EXAMPLE 2.1 *Consider the restriction of $\mathcal{O}$ to the subspace $\mathbf{C}\mathbf{1} \oplus \mathcal{A}_q\beta^*$. An element of this subspace is given by $c\mathbf{1} + a\beta^*$ with c and a being unique elements of $\mathbf{C}$ and $\mathcal{A}_q$, respectively. Obviously $\mathcal{O}$ maps such an element to a. In Section 4.4 we will see that the above subspace of $\mathcal{A}_q$ is dense in $\mathcal{A}$. Thus, it actually suffices to know the values of $\mathcal{O}$ on this subspace.*

Now we define for $p \in [0, 1)$ the mapping $\mathcal{T}_p : \mathcal{A}_q \to \mathcal{A}$ by

$$\mathcal{T}_p(a) \quad = \quad (p\alpha - \mathbf{1})^{-1}\mathcal{P}(a)(p\alpha^* - \mathbf{1})^{-1}.$$

Theorem 2.3 *For all $a \in \mathcal{A}_q$, the elements $\mathcal{T}_p(a)$ of $\mathcal{A}$ converge strongly for $p \to 1$ to an element of $\mathcal{A}$. The mapping $\mathcal{T} : \mathcal{A}_q \to \mathcal{A}$ defined by*

$$\mathcal{T}(a) \quad = \quad \lim_{p \to 1} \mathcal{T}_p(a)$$

fulfills Equation (2.2).

PROOF The projection $\mathcal{P}$ projects onto K_2 and a typical element of K_2 is given by ab with $a, b \in K_1$. Inserting this in $\mathcal{T}_p$ we obtain

$$\mathcal{T}_p(ab) = \mathcal{O}_p(a^*)^* \mathcal{O}_p(b).$$

From this it is immediate that the weak limit exists and fulfills Equation (2.2). Let us for the moment identify the elements of $\mathcal{A}$ with their images under the isomorphism ω. In order to see that the limit is indeed a strong limit, we observe that

$$\begin{aligned} &\|(\mathcal{O}(a^*)^* \mathcal{O}(b) - \mathcal{O}_p(a^*)^* \mathcal{O}_p(b))f\| \\ &\quad = \quad \|(\mathcal{O}(a^*)^* - \mathcal{O}_p(a^*)^*)\mathcal{O}(b)f + \mathcal{O}_p(a^*)^*(\mathcal{O}(b) - \mathcal{O}_p(b))f\| \\ &\quad \leq \quad \|(\mathcal{O}(a^*)^* - \mathcal{O}_p(a^*)^*)g\| + \|\mathcal{O}_p(a^*)^*\| \, \|(\mathcal{O}(b) - \mathcal{O}_p(b))f\| \end{aligned}$$

for any given but fixed vector $f \in H_\omega$ and $g = \mathcal{O}(b)f$. Thus, the proof is complete if we show that $\|\mathcal{O}_p(a^*)^*\|$ or equivalently $\|\mathcal{O}_p(a)\|$ is bounded uniformly in p for any fixed a.

Consider $\mathcal{J}_p$ as introduced in the proof of Lemma 2.1. One easily checks that

$$\mathcal{O}_p(a) \quad = \quad \mathcal{O}(a)(\mathbf{1} - (1-p)\alpha^* \mathcal{J}_p).$$

This yields

$$\|\mathcal{O}_p(a)\| \quad \leq \quad \|\mathcal{O}(a)\|(1 + \|(1-p)\mathcal{J}_p\|).$$

From Equation (2.3) it is clear that the norm of $(1-p)\mathcal{J}_p$ cannot be greater than 1. ■

EXAMPLE 2.2 *Consider the restriction of $\mathcal{T}$ to the subspace $\mathbf{C1} \oplus \mathbf{C}\frac{\alpha - \alpha^*}{2i} \oplus \beta \mathcal{A}_q \beta^*$. An element of this subspace is given by $c_1 \mathbf{1} + c_2 \frac{\alpha - \alpha^*}{2i} + \beta a \beta^*$ with c_i and a being unique elements of $\mathbf{C}$ and $\mathcal{A}_q$, respectively. Obviously $\mathcal{T}$ maps such an element to a. In Section 4.4 we will see that the above subspace of $\mathcal{A}_q$ is dense in $\mathcal{A}$. Thus, it actually suffices to know the values of $\mathcal{T}$ on this subspace.*

2.2 General remarks on the $*$–representations

In view of Lemma 1.6 we are interested in separating the injective part from an arbitrary representation.

Proposition 2.4 *Let π be a $*$–representations of $\mathcal{A}_q$ on a* HILBERT *space H. Then there is a unique decomposition into a direct sum*

$$\pi = \pi_1 \oplus \pi_2 \quad on \quad H = H_1 \oplus H_2 \tag{2.5}$$

where $\pi_i, i = 1, 2$ are representations on $H_i, i = 1, 2$ respectively with

- *π_1 maps γ to 0 and α to a unitary operator on H_1.*
- *π_2 maps γ to an injective operator on H_2.*

PROOF We show that $\ker(\pi(\gamma))$ is an invariant subspace of H. Indeed, given any $f \in H$ with $\pi(\gamma)f = 0$ we immediately see from Relations (a), (b)* and (c) that $\pi(\alpha)f$, $\pi(\alpha^*)f$ and $\pi(\gamma^*)f$ respectively are in $\ker(\pi(\gamma))$. (E.g. $\pi(\gamma)\pi(\alpha)f = q^{-1}\pi(\alpha)\pi(\gamma)f = 0$.) Henceforth, $\ker(\pi(\gamma))$ is invariant under the action of $\pi(a)$ for any $a \in \mathcal{A}_q$.

We denote $\ker(\pi(\gamma))$ by H_1 and its orthogonal complement by H_2. Since we are concerned with $*$–representations, H_2 is an invariant subspace as well. Clearly, if we denote by π_i the restriction of π to H_i respectively, the operator $\pi_1(\gamma)$ is 0 and the operator $\pi_2(\gamma)$ is injective. ■

N.B.: Since the subspaces H_i remain invariant under π the components η_i of a given cocycle $\eta = \eta_1 \oplus \eta_2$ with respect to π are cocycles with respect to π_i respectively.

In order to indicate that the representations of type π_1 are already determined by fixing the unitary operator u on H_1 to which α is mapped, we denote π_1 also by ρ_u. Another type of representations to be separated are those behaving like δ which lead to the so-called GAUSS*ian parts* of the functionals (see [23]).

Proposition 2.5 *The subspace $H_\delta = \ker(\pi(\beta))$ of H is an invariant subspace, and the restriction of π to H_δ is given by*

$$\pi_\delta(\bullet) = \delta(\bullet)\mathbf{1}_{H_\delta}.$$

PROOF α is mapped to $\mathbf{1}$ on H_δ. According to Relation (e), we have $\pi(\gamma)f = 0$ for $f \in H_\delta$. On the other hand, we see from Relation (d) that $\pi(\alpha\alpha^*)f = \pi(\alpha^*\alpha)f = \pi(\alpha^*)f$ for any $f \in H_\delta$, hence H_δ is invariant under $\pi(\alpha^*)$. ■

2.3 Classification of cocycles and conditionally positive functionals

Now we describe all 1–cocycles and all conditionally positive functionals associated with a cocycle via (1.2). We classify them by all pairs (π, η) consisting of a representation and a vector in the representation space H (actually the functional is determined in this way only up to two real constants). It turns out that for any given cocycle there is a conditionally positive functional such that (1.2) holds. Let us give a simple counter example to make clear that this is not always the case, but depends on the algebra under consideration (cf. also the cases $q = 1$ and $q = -1$ in Sections 5.4 and 5.5).

EXAMPLE 2.3 *Consider the free unital commutative $*$–algebra generated by the symbol x. (This $*$–algebra can be turned into a* HOPF *$*$–algebra; cf.* SWEEDLER *[31].) We define a $*$–algebra homomorphism δ_x by $\delta_x(\mathbf{1}) = 1$ and $\delta_x(x) = 0$, with respect to which a functional can be conditionally positive or not. On the other hand, δ_x is a representation. Given any two numbers $\eta_x, \eta_{x^*} \in \mathbf{C}$, the mapping η defined by*

$$\begin{aligned} \eta(x) &= \eta_x \\ \eta(x^*) &= \eta_{x^*} \\ \eta(y) &= 0 \text{ , } y \text{ any monomial with degree not equal to 1} \end{aligned}$$

and linear extension is a cocycle with respect to δ_x. However, if we have $|\eta_x| \neq |\eta_{x^*}|$ and try to define a conditionally positive functional ψ by (1.2), we obtain

$$\psi(x^*x) = \overline{\eta_x}\eta_x \quad \text{and} \quad \psi(xx^*) = \overline{\eta_{x^*}}\eta_{x^*}$$

which contradicts the commutativity of the underlying algebra.

Let us proceed in our main stream. We remind the reader of the fact that for any cocycle η we must have $\eta(\mathbf{1}) = 0$, because our representations are non–degenerate by convention. Thus, we have $\eta(\alpha^{(*)}) = \eta(\beta^{(*)})$. Clearly, the cocycle property (1.1) reads on K_1

$$\eta(ab) = \pi(a)\eta(b) \text{ for all } a, b \in K_1. \tag{2.6}$$

Thus, η is determined by its values on the generators. We will see that any cocycle is already determined by its value on α^*.

Lemma 2.6 *Let η and $\widetilde{\eta}$ be two 1–cocycles with respect to π. They coincide if and only if they coincide on α^*, i.e.*

$$\eta(\alpha^*) = \widetilde{\eta}(\alpha^*) \quad \Longleftrightarrow \quad \eta = \widetilde{\eta}.$$

Proof Clearly, two cocycles do not coincide if they assume different values on α^*. Thus, we have to show the other direction. We split the proof into the two cases $\pi(\gamma) = 0$ (i.e. $H_2 = \{0\}$) and $\pi(\gamma)$ injective (i.e. $H_1 = \{0\}$) which can be treated separately.

Let η be a cocycle with respect to $\pi = \rho_u$ and η_{α^*} a vector in $H = H_1$ with $\eta(\alpha^*) = \eta_{\alpha^*}$. We apply η to Relation $(\widetilde{\text{a}})$, use the cocycle property (2.6) and obtain, taking also into account that $\pi(\gamma) = 0$,

$$\pi(\beta)\eta(\gamma) = -(1-q)\eta(\gamma) \quad \text{or} \quad \pi(\alpha)\eta(\gamma) = u\eta(\gamma) = q\eta(\gamma). \tag{2.7}$$

We take the norm of both sides and arrive at

$$\eta(\gamma) = 0.$$

(Otherwise, we would have $|q| = 1$.) Similarly, starting from Relation $(\widetilde{\text{b}})$ we obtain

$$\eta(\gamma^*) = 0.$$

Using Relation $(\widetilde{\text{e}})$ in the same manner, we obtain

$$0 \;=\; \pi(\beta^*)\eta(\beta) + \eta(\beta) + \eta(\beta^*) \;=\; u^*\eta(\beta) + \eta(\beta^*)$$

or

$$\eta(\beta) \;=\; -u\eta(\beta^*) \;=\; u\eta_{\alpha^*}.$$

Now let η be a cocycle with respect to $\pi = \pi_2$ and η_{α^*} a vector in $H = H_2$ with $\eta(\alpha^*) = \eta_{\alpha^*}$. We write

$$\epsilon = \eta(\gamma^*\gamma).$$

Relation $(\widetilde{\text{c}})$ yields

$$\epsilon = \pi(\gamma^*)\eta(\gamma) = \pi(\gamma)\eta(\gamma^*). \tag{2.8}$$

$\pi(\gamma)$ is injective and so is $\pi(\gamma^*)$ (notice that $\|\pi(\gamma)f\| = \|\pi(\gamma^*)f\|$ for all f due to Relation $(\widetilde{\text{c}})$). Therefore, $\eta(\gamma)$ and $\eta(\gamma^*)$ are determined by ϵ. Applying η and (2.6), Relation $(\widetilde{\text{a}})$ reads

$$\pi(\beta)\eta(\gamma) = q\pi(\gamma)\eta(\beta) - (1-q)\eta(\gamma).$$

Now we multiply by $\pi(\gamma^*)$. Using Relation $(\widetilde{\text{b}})$ in order to eliminate $\gamma^*\beta$, we obtain after some short calculations

$$\pi(\gamma^*\gamma)\eta(\beta) = (q^{-2}\pi(\alpha) - \mathbf{1})\epsilon. \tag{2.9}$$

Notice that $\pi(\gamma^*\gamma)$ is injective. Therefore, $\eta(\beta)$ is determined by ϵ as well.

The same procedure, now starting from Relation $(\widetilde{\text{b}})^*$ and using $(\widetilde{\text{a}})^*$, yields

$$\pi(\gamma^*\gamma)\eta(\beta^*) = (q^2\pi(\alpha^*) - \mathbf{1})\epsilon. \tag{2.10}$$

Since $\|\pi(\alpha^*)\| \leq 1$ the operator $q^2\pi(\alpha^*) - \mathbf{1}$ is invertible on H. Thus, ϵ is expressible in terms of $\eta(\beta^*) = \eta_{\alpha^*}$

We see that in both cases the values of η on the generators are determined by η_{α^*}, and clearly this extends to the whole algebra. ■

Now we can prove

Corollary 2.7 *$\mathcal{O}$ is the unique linear mapping satisfying Equation (2.1) and $\mathcal{O}(\alpha^*) = \mathbf{1}$.*

PROOF By the foregoing lemma we see that any cocycle is already determined by its value η_{α^*} on α^*. If we had two mappings $\mathcal{O}, \widetilde{\mathcal{O}}$ satisfying the claimed conditions, their difference $\Delta\mathcal{O} = \mathcal{O} - \widetilde{\mathcal{O}}$ must vanish on α^*. Thus, the cocycle $(\pi\eta) \circ \Delta\mathcal{O}$ must vanish for any π and η. By the existence of a faithful representation we see that $\Delta\mathcal{O}$ must vanish itself. ■

Together with the introductory remarks of this section and Theorems 1.1, 2.2 and 2.3 we obtain the classification theorem as a simple corollary.

Theorem 2.8 *Let π be a $*$–representation of $\mathcal{A}_q$. For a vector η_{α^*} in the representation space H of π the mapping*

$$\eta \;=\; (\pi\eta_{\alpha^*}) \circ \mathcal{O}$$

is a 1–cocycle with respect to π fulfilling $\eta(\alpha^*) = \eta_{\alpha^*}$. Moreover, all 1–cocycles with respect to π are of this form, so that there is a one-to-one correspondence between elements of the representation space of π and 1–cocycles with respect to π.

For all numbers r_1, r_2 in $\mathbf{R}$ the mapping

$$\psi \;=\; r_1\delta + r_2\delta' + \langle \eta_{\alpha^*} | \pi | \eta_{\alpha^*} \rangle \circ \mathcal{T}$$

is a conditionally positive functional fulfilling Equation (1.2). Moreover, for any conditionally positive functional ψ satisfying (1.2) there are unique numbers r_1, r_2 in $\mathbf{R}$ such that ψ is of the above form.

PROOF By the equation $\eta = (\pi\eta_{\alpha^*}) \circ \mathcal{O}$ we assign to any pair (π, η_{α^*}) a cocycle with respect to π assuming the value η_{α^*} on α^*. By Lemma 2.6 these cocycles must be all.

By the equation $\psi' = \langle \eta_{\alpha^*} | \pi | \eta_{\alpha^*} \rangle \circ \mathcal{T}$ we define indeed a conditonally positive functional which fulfills Equation (1.2). An arbitrary functional, fulfilling Equation (1.2), can differ from ψ' only on the two basis vectors $\mathbf{1}$ and $\frac{\alpha - \alpha^*}{2i}$. In order to take this into account we have to add a linear combination $r_1\delta + r_2\delta'$. The constants r_1, r_2 have to be chosen real, because our functionals are supposed to be hermitian. By Theorem 1.1 the functionals of the stated form must indeed be all. ■

Corollary 2.9 *Let ψ be a conditionally positive functional on $\mathcal{A}_q$. Then there are unique numbers r_1, r_2 in $\mathbf{R}$ and a unique positive functional φ on $\mathcal{A}_q$ such that*

$$\psi \;=\; r_1\delta + r_2\delta' + \varphi \circ \mathcal{T}$$

EXAMPLE 2.4 *For the* GAUSS*ian part of a representation, i.e. representations proportional to δ, we obtain by Proposition 1.2*

Corollary 2.10 *Let $\eta_{\delta_{\alpha^*}}$ be a vector in H_δ. Then*

$$\eta_\delta = i\delta' \eta_{\delta_{\alpha^*}}$$

is the *cocycle η_δ which assumes the value $\eta_{\delta_{\alpha^*}}$ on α^*. Moreover,*

$$\psi_\delta = \frac{\delta''}{2} \|\eta_{\delta_{\alpha^*}}\|^2$$

defines a conditionally positive functional fulfilling Equation (1.2).

Corollary 2.11 *All* GAUSS*ian conditionally positive functionals are of the form*

$$\psi_\delta \;=\; r_1\delta + r_2\delta' + r\delta'',$$

where r_1, r_2 are in $\mathbf{R}$, and r is in $\mathbf{R}_+$.

EXAMPLE 2.5 *In Appendix A we introduce the q–exponential function* e_q^z. *By the properties stated there the following becomes obvious. Let* p *be in* $(0,1)$. *Firstly,*

$$\frac{\mathbf{1}}{\mathbf{1}-p\alpha^*} \quad = \quad e_{q^2}^{p\alpha^*}\left(e_{q^2}^{pq^2\alpha^*}\right)^{-1}.$$

Secondly, $\left(e_{q^2}^{pq^2\alpha^*}\right)^{-1}$ *converges in norm to* $\left(e_{q^2}^{q^2\alpha^*}\right)^{-1}$ *for* $p\to 1$.

In Appendix B we define the q–coherent states $e_{q^2}(\lambda)\in h_0, \lambda\in U_1(0)$, *where* h_0 *is the representation space of the irreducible representation* ρ_0. *Notice that* $e_{q^2}(\lambda)=e_{q^2}^{\lambda\rho_0(\alpha^*)}e_0$. *For the conditionally positive functional* $\langle e_{q^2}(q^2)|\rho_0|e_{q^2}(q^2)\rangle\circ\mathcal{T}$ *we, thus, obtain*

$$\begin{aligned}&\langle e_{q^2}(q^2)|\rho_0|e_{q^2}(q^2)\rangle\circ\mathcal{T}\\ &\quad = \lim_{p\to1}\left\langle e_{q^2}(q^2)\left|\frac{\mathbf{1}}{\mathbf{1}-p\rho_0(\alpha)}\rho_0\circ\mathcal{P}\frac{\mathbf{1}}{\mathbf{1}-p\rho_0(\alpha^*)}\right|e_{q^2}(q^2)\right\rangle \quad = \quad \lim_{p\to1}\langle e_{q^2}(p)|\rho_0\circ\mathcal{P}|e_{q^2}(p)\rangle.\end{aligned}$$

By Relations (1.4) it is sufficient to know a linear mapping on the vectors $\alpha^{\check{\ell}}\gamma^{*m}\gamma^n$, $\ell\in\mathbf{Z}, m,n\in\mathbf{N}_0$. *Since our functional is hermitian and* ρ_0 *does not distinguish between* γ *and* γ^*, *it is even sufficient to restrict ourselves to monomials* $\gamma^m\alpha^n$, $m,n\in\mathbf{N}_0$. *First let* $m>0$ *which makes the projection* $\mathcal{P}$ *dissapear. Using the formulae of Appendix B we obtain*

$$\lim_{p\to1}\langle e_{q^2}(p)|\rho_0(\gamma^m\alpha^n)|e_{q^2}(p)\rangle \quad = \quad \lim_{p\to1}p^n e_{q^2}^{p^2q^m} \quad = \quad e_{q^2}^{q^m}.$$

In the remaining cases the projection yields

$$\mathcal{P}(\alpha^n) \quad = \quad \alpha^n - n\frac{\alpha-\alpha^*}{2}-\mathbf{1}.$$

We obtain

$$\lim_{p\to1}\langle e_{q^2}(p)|\rho_0\circ\mathcal{P}(\alpha^n)|e_{q^2}(p)\rangle \quad = \quad \lim_{p\to1}(p^n-1)e_{q^2}^{p^2} \quad = \quad -\frac{n}{2}e_{q^2}^{q^2}.$$

It is not difficult to check the equality on simple monomials, for instance $\beta\beta^*$ *and* $\beta\gamma\beta^*$.

2.4 Remarks on motivation

If we apply a given cocycle η to Relations ($\widetilde{\text{a}}$) – ($\widetilde{\text{e}}$) and their adjoints we get a new set of relations for the values of η on the generators. This set of relations enables us to express these values in terms of $\eta(\alpha^*)$. Clearly, we could have tried to extend these four vectors by means of Property (2.6) to the whole algebra. (This is partly done in the course of the proof of Proposition 2.6.) The main problem was to examine if this procedure is well-defined.

To avoid this complication we followed another path. First we restrict ourselves to the orthogonal complement of H_δ. By construction the operator $\pi(\beta^*)$ is injective on H. Thus, if it was invertible, the function given by

$$a \quad\longmapsto\quad (\pi(a)-\delta(a)\mathbf{1})\pi(\beta^*)^{-1}\eta(\alpha^*)$$

would define a cocycle assuming the correct value on β^*. Thus, we had to find a way of *inverting* the operator $\pi(\beta^*)$. Our way to do this was to approximate '$\pi(\beta^*)^{-1}\eta(\alpha^*)$' (which does not always exist in H; cf. Section 4.5) by the vectors $\eta_p=(p\pi(\alpha^*)-\mathbf{1})^{-1}\eta(\alpha^*)$. In other words, we approximate the given cocycle by cocycles of the form

$$(\pi(a)-\delta(a)\mathbf{1})\eta_p$$

which are coboundaries. The case of H_δ is easily treated separately.

We remark that, assuming formally the existence of the VON NEUMANN series of $(\alpha^*-\mathbf{1})^{-1}$, it is easy to 'derive' the values of $\mathcal{O}$ on the generators. We just have to compute the commutation rules for this formal element of $\mathcal{A}$ by simple applications of Relations (a) – (e). Thus, omitting the approximation of $\mathcal{O}$ by $\mathcal{O}_p$, we can omit the use of a notion of strong convergence. Investigating the mappings $\pi \circ \mathcal{O}(\bullet)$ instead of $\mathcal{O}$ itself, we can omit the use of the C^*–completion of $\mathcal{A}_q$, because only geometric series of operators appear. Thus, the contents of Theorem 2.8 can indeed be expressed in terms of $\mathcal{A}_q$.

Furthermore, we remark that due to the construction of $\mathcal{T}_p$ any conditionally positive functional corresponding to the cocycle η can be approximated on K_2 by functionals of type

$$\psi_p(\bullet) = \langle \eta_p | \pi(\bullet) | \eta_p \rangle$$

with $\eta_p \in H$ (cf. Example 2.5). For the time being, this statement is clear only for cocycles with respect to representations of type π_2. However, by the remark following Proposition 5.15 we see that also cocycles with respect to representations of type π_1 are strong limits of coboundaries. In other words, the cone, spanned by the restrictions to K_2 of all states, is dense in the cone, consisting of the restrictions to K_2 of all conditionally positive functionals, with respect to pointwise convergence. It is this fact which is crucial to obtain the Lévy-Khintchine formula in Section 4.5.

Chapter 3

Representation theory

In this section we give a new treatment of the representation theory of $\mathcal{A}_q$. Our treatment is almost completely algebraic and we do not refer to the C^*–algebra structure at any time. As a corollary we obtain the irreducible representations and realize that a general representation decomposes into a direct integral over irreducible ones.

Vaksman and Soibelman proceed in the converse direction. They find the irreducible representations (cf. Theorem 1.3) without stating an explicit form for the general representation. From the irreducible representations and the norm introduced on $\mathcal{A}_q$ it is clear that the C^*–completion is a type I C^*–algebra, having the same representations. Thus, the general representation must be given by a direct integral over irreducible ones (cf. e.g. Dixmier [8]).

We obtain the new result that the irreducible representations ρ_φ can be expressed in terms of ρ_0 and a family I_φ of automorphisms of $\mathcal{A}_q$. We express these automorphisms in terms of the convolution product which arises from the coalgebra structure of $SU_q(2)$ (cf. also [16], where the convolution of irreducible representations was investigated independently). As a consequence of this result we are able to express any conditionally positive functional associated with a representation of type π_2 in terms of functionals associated with ρ_0.

3.1 Representations on H_2

For any $*$–representation π of $\mathcal{A}_q$ on a Hilbert space H there is a unique decomposition into the invariant subspaces H_1 and H_2 by Proposition 2.4. On H_1 the general representation is given by ρ_u as defined at the end of Section 2.2 where u is any unitary operator on H_1. Thus, in order to complete the representation theory we have to consider the remaining part H_2.

In this section we assume that $H = H_2$ and $\pi = \pi_2$, i.e. $\pi(\gamma)$ is injective and hence Lemma 1.6 is applicable. Furthermore, in order to simplify notation we write a for the image $\pi(a) \in \mathcal{B}(H)$ of $a \in \mathcal{A}_q$ under π.

We introduce the following two sequences of operators:

$$\begin{array}{rcll} P_k &=& {\alpha^*}^k(\mathbf{1}-q^2\gamma^*\gamma)^{-1}\cdots(\mathbf{1}-q^{2k}\gamma^*\gamma)^{-1}\alpha^k, & k\in\mathbf{N} \\ P_0 &=& \mathbf{1}, & k=0 \\ E_k &=& P_k - P_{k+1}, & k\in\mathbf{N}_0. \end{array}$$

We remark that

$$e_q(z) = \prod_{k=0}^{\infty}\frac{1}{1-q^k z}, \qquad z\in\mathbf{C}\backslash\{z|z=q^{-k}, k\in\mathbf{N}_0\}$$

is a well-known meromorphic function (see Appendix A). By Theorem A.4 this function is different from 0 everywhere, it is a strictly increasing function on the intervall $[0,1)$ and for fixed $z\in[0,1)$ the product for $k \to \infty$ is also increasing. Thus, we see that

$$1 \leq \|(\mathbf{1} - q^2\gamma^*\gamma)^{-1} \cdots (\mathbf{1} - q^{2k}\gamma^*\gamma)^{-1}\| \leq e_{q^2}(q^2).$$

Due to Lemma 1.6 we have for any $f \in H$

$$\langle f|P_k|f\rangle \leq e_{q^2}(q^2)\|\alpha^k f\|^2 \to 0 \quad \text{for} \quad k \to \infty,$$

hence,

$$\sum_{k=0}^{\infty} E_k = \lim_{k\to\infty} (\mathbf{1} - P_{k+1}) = \mathbf{1} \tag{3.1}$$

in the strong operator topology. For E_k we obtain by Relation (a)

$$\begin{aligned} E_k &= {\alpha^*}^k \Big\{(\mathbf{1} - q^2\gamma^*\gamma)^{-1} \cdots (\mathbf{1} - q^{2k}\gamma^*\gamma)^{-1} \\ &\quad - \alpha^*(\mathbf{1} - q^2\gamma^*\gamma)^{-1} \cdots (\mathbf{1} - q^{2(k+1)}\gamma^*\gamma)^{-1}\alpha\Big\}\alpha^k \\ &= {\alpha^*}^k \big\{\mathbf{1} - \alpha^*(\mathbf{1} - q^2\gamma^*\gamma)^{-1}\alpha\big\}(\mathbf{1} - q^2\gamma^*\gamma)^{-1} \cdots (\mathbf{1} - q^{2k}\gamma^*\gamma)^{-1}\alpha^k. \end{aligned}$$

For the term in curly brackets we obtain, using Relations (a) and (e),

$$\begin{aligned} E_0 = \mathbf{1} - \alpha^*(\mathbf{1} - q^2\gamma^*\gamma)^{-1}\alpha &= \lim_{\ell\to\infty} \left(\mathbf{1} - \alpha^* \sum_{k=0}^{\ell-1} q^{2k}(\gamma^*\gamma)^k\alpha\right) \\ &= \lim_{\ell\to\infty} \left(\mathbf{1} - (\mathbf{1} - \gamma^*\gamma)\sum_{k=0}^{\ell-1}(\gamma^*\gamma)^k\right) \\ &= \lim_{\ell\to\infty} (\gamma^*\gamma)^\ell \end{aligned}$$

where the limit is in the operator norm topology (cf. [18], where the convergence was only shown to be strong). Inserting this we arrive at

$$E_k = \lim_{\ell\to\infty} (q^{-2k}\gamma^*\gamma)^\ell P_k. \tag{3.2}$$

Obviously, P_k commutes with $\gamma^*\gamma$ and so does E_k. Of course, the operators P_k and E_k are self-adjoint. We show the following

Proposition 3.1 *P_k is an orthogonal projection.*

Proof It remains to show that ${P_k}^2 = P_k$ which we will prove by induction.

Clearly, the statement holds for $k = 0$. Now let us assume that it holds for k. We have

$$P_{k+1} = P_k - E_k = \lim_{\ell\to\infty} \left(\mathbf{1} - (q^{-2k}\gamma^*\gamma)^\ell\right) P_k.$$

If we square this expression and perform the ℓ-limits simultaneously, we obtain

$$\begin{aligned} {P_{k+1}}^2 &= \lim_{\ell\to\infty} \left(\mathbf{1} - (q^{-2k}\gamma^*\gamma)^\ell\right)^2 {P_k}^2 \\ &= \lim_{\ell\to\infty} \left(\mathbf{1} - 2(q^{-2k}\gamma^*\gamma)^\ell + (q^{-2k}\gamma^*\gamma)^{2\ell}\right) P_k. \end{aligned}$$

This yields the desired result. ■

Now we have the following crucial

Lemma 3.2 *The operators $(E_k)_{k\in\mathbf{N}_0}$ form a complete set of orthogonal projections.*

PROOF By Property (3.1) (i.e. actually Lemma 1.6) the E_k are complete. Therefore, it remains to show that

$$E_k E_\ell = E_k \delta_{k\ell}.$$

From Equation (3.2) and the foregoing Proposition it is immediate that ${E_k}^2 = E_k$. We assume without loss of generality that $k > \ell$. Performing the limits simultaneously, we obtain

$$\begin{aligned} E_k E_\ell &= \lim_{m\to\infty} (q^{-(k+\ell)}\gamma^*\gamma)^{2m} P_k P_\ell \\ &= \lim_{m\to\infty} ((q^{-2k}\gamma^*\gamma)^{2m} P_k) \lim_{m\to\infty} (q^{2m(k-\ell)} P_\ell) \\ &= E_k \cdot 0 = 0 \end{aligned}$$

which is the desired result. ■

For all $k \in \mathbf{N}_0$ let $\mathcal{H}_k = E_k H$ be the range of the projection E_k. From Equation (3.2) we see that the $\mathcal{H}_k$ are eigenspaces of $\gamma^*\gamma$ to eigenvalues q^{2k} respectively. Clearly, since γ and E_k commute, the restriction of $q^{-k}\gamma$ to $\mathcal{H}_k$ must be given by an operator U_k which is unitary on $\mathcal{H}_k$ (i.e. $U_k : H \to H$ with ${U_k}^* U_k = U_k {U_k}^* = E_k$). Thus, we can write

$$\gamma = \sum_{k=0}^{\infty} q^k U_k.$$

Next we obtain, using

$$\alpha {\alpha^*}^{k+1} = (\mathbf{1} - q^2\gamma^*\gamma){\alpha^*}^k = {\alpha^*}^k(\mathbf{1} - q^{2(k+1)}\gamma^*\gamma)$$

for $k \in \mathbf{N}_0$, that

$$\alpha P_{k+1} = P_k \alpha \quad \text{and clearly} \quad \alpha P_0 = P_0 \alpha.$$

This yields

$$\text{and} \quad \begin{aligned} \alpha E_{k+1} &= E_k \alpha, & \alpha E_0 &= 0 \\ E_{k+1}\alpha^* &= \alpha^* E_k, & E_0 \alpha^* &= 0. \end{aligned} \tag{3.3}$$

If we introduce the mappings $\varphi_{k+1}, \varphi_k^* : H \to H$, $k \in \mathbf{N}_0$, by setting

$$\varphi_{k+1} = \frac{\alpha E_{k+1}}{\sqrt{1 - q^{2(k+1)}}} \quad \text{and} \quad \varphi_k^* = \frac{\alpha^* E_k}{\sqrt{1 - q^{2(k+1)}}}$$

we can write

$$\alpha = \sum_{k=0}^{\infty} \sqrt{1 - q^{2(k+1)}}\, \varphi_{k+1} \quad \text{and} \quad \alpha^* = \sum_{k=0}^{\infty} \sqrt{1 - q^{2(k+1)}}\, \varphi_k^*.$$

Notice that the series for γ, α and α^* converge at least strongly. Now we show

Lemma 3.3 *For any $k \in \mathbf{N}_0$ the restriction of φ_k^* to $\mathcal{H}_k$ is an isomorphism onto $\mathcal{H}_{k+1}$ and φ_{k+1} is its inverse.*

PROOF From Properties (3.3) we see that the mappings φ_{k+1} and φ_k^* map to $\mathcal{H}_k$ and $\mathcal{H}_{k+1}$ respectively. By the same properties we find

$$\begin{aligned} \alpha^* E_k \alpha E_{k+1} &= \alpha^* \alpha E_{k+1} = (\mathbf{1} - \gamma^*\gamma) E_{k+1} \\ &= (1 - q^{2(k+1)}) E_{k+1} \end{aligned}$$

and similarly

$$\alpha E_{k+1} \alpha^* E_k = (1 - q^{2(k+1)}) E_k.$$

Thus, the restrictions of φ_{k+1} and φ_k^* to $\mathcal{H}_{k+1}$ and $\mathcal{H}_k$ are inverse mappings. The isometry conditions are also proved by an application of the above relations. ■

Now using $U_k = q^{-k}\gamma E_k$ we calculate

$$\begin{aligned}\varphi_k^* U_k \varphi_{k+1} &= \frac{q^{-k}}{1-q^{2(k+1)}}\alpha^*\gamma\alpha E_{k+1} = \frac{q^{-(k+1)}}{1-q^{2(k+1)}}\gamma(\mathbf{1}-\gamma^*\gamma)E_{k+1} \\ &= q^{-(k+1)}\gamma E_{k+1} = U_{k+1},\end{aligned}$$

i.e. all the $\mathcal{H}_k$ can be regarded as copies of the same $\mathcal{H}_0$ carrying the same unitary operator $U = U_0$. For making this explicit, we identify H with $h_0 \otimes \mathcal{H}_0$ by the following isomorphism (recall that h_0 is the HILBERT space with ONB $\{e_k\}_{k\in\mathbf{N}_0}$ which carries the irreducible representation ρ_0): First we identify $\mathcal{H}_0$ with $e_0 \otimes \mathcal{H}_0$ in the natural way

$$f \in \mathcal{H}_0 \longmapsto e_0 \otimes f.$$

Then for $k \in \mathbf{N}$ we identify $\mathcal{H}_k$ with $e_k \otimes \mathcal{H}_0$ such that

$$\varphi_k(e_k \otimes f) = (e_{k-1} \otimes f) \quad \text{for} \quad f \in \mathcal{H}_0.$$

Now it is clear that $\pi(\alpha)$ and $\pi(\gamma)$ are given by

$$\pi(\alpha) = \rho_0(\alpha) \otimes \mathbf{1} \quad \text{and} \quad \pi(\gamma) = \rho_0(\gamma) \otimes U. \tag{3.4}$$

On the other hand, if we are given any HILBERT space $\mathcal{H}_0$ with a unitary operator U acting on it, it is easy to check that a pair of operators on $h_0 \otimes \mathcal{H}_0$ defined by (3.4) can be extended to a representation of $\mathcal{A}_q$. Therefore, we have the following

Theorem 3.4 *Equations (3.4) establish a one-to-one correspondence between $*$–representations of type π_2 and unitary operators.*

We again denote π_2 also by π_U to indicate the classification by unitary operators.

Corollary 3.5 *The irreducible $*$–representations of $\mathcal{A}_q$ are given by the two families $\delta_\varphi, \rho_\varphi$.*

PROOF In both cases $\pi = \rho_u$ and $\pi = \pi_U$ the representations are classified by unitary operators, and in both cases the representations decompose into a direct sum on invariant subspaces if the corresponding unitary operators do. In other words, in both cases the unitary operators have to act on a one-dimensional HILBERT space. Thus, the irreducible representations must be of the stated form.

On the other hand, h_0 does not contain any invariant subspace, because a projection to any basis vector of h_0 can be approximated by representation operators, i.e. the given representations are indeed irreducible. ■

N.B.: Notice that Lemma 1.6 is the only result of the foregoing chapters used in this section. In order to prove Lemma 1.6 we did not need to know the irreducible representations.

3.2 Coalgebra structure of $\mathcal{A}_q$

We equip $\mathcal{A}_q$ with the coalgebra structure of $SU_q(2)$ by requiring the matrix $\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$ to be a corepresentation (cf. [35]). In other words, we have, written symbolically,

$$\Delta \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \otimes \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

which means

$$\begin{aligned}\Delta(\alpha) &= \alpha \otimes \alpha - q\gamma^* \otimes \gamma \\ \Delta(\gamma) &= \gamma \otimes \alpha + \alpha^* \otimes \gamma.\end{aligned} \tag{3.5}$$

We can summarize the structure imposed on $\mathcal{A}_q$ up to this point by considering $\mathcal{A}_q$ as the $*-$bialgebra having the matrix $\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$ as *unitary* corepresentation. Indeed, if we require this matrix to be unitary we get all the Relations (a)-(e).

We remark that we are mainly interested in representations up to unitary equivalence (which we shall denote by $\asymp$), because the set of all conditionally positive functionals associated with a given representation does not change under *any* equivalence transform. Keeping this in mind, we come back to representations of the form given by (3.4) and show

Proposition 3.6 *For any unitary operator V on $\mathcal{H}_0$ the mapping*

$$\alpha \longmapsto \rho_0(\alpha) \otimes V \quad \textit{and} \quad \gamma \longmapsto \rho_0(\gamma) \otimes U$$

can be extended to a $*$–representation of $\mathcal{A}_q$ which is unitarily equivalent to that defined by Equations (3.4).

Proof Consider the unitary transform $\mathcal{V}$ on $h_0 \otimes \mathcal{H}_0$ which maps $e_k \otimes f$ to $e_k \otimes V^k f$. If we apply $\mathcal{V}^{-1} \bullet \mathcal{V}$ to the operators in (3.4) we get the desired equivalence. ■

Now let π and $\widetilde{\pi}$ be two representations on H and $\widetilde{H}$ respectively. The convolution product defined by

$$\pi \star \widetilde{\pi} = (\pi \otimes \widetilde{\pi}) \circ \Delta$$

is a representation on $H \otimes \widetilde{H}$, since Δ is a homomorphism into $\mathcal{A}_q \otimes \mathcal{A}_q$ and $\pi \otimes \widetilde{\pi}$ is a representation of $\mathcal{A}_q \otimes \mathcal{A}_q$. For the two possible convolution products of ρ_0 and ρ_U we find using (3.5)

$$\begin{aligned} \rho_0 \star \rho_U(\alpha) &= \rho_0(\alpha) \otimes U, & \rho_0 \star \rho_U(\gamma) &= \rho_0(\gamma) \otimes U \\ \text{and} \quad \rho_U \star \rho_0(\alpha) &= U \otimes \rho_0(\alpha), & \rho_U \star \rho_0(\gamma) &= U^* \otimes \rho_0(\gamma). \end{aligned}$$

Thus, we obtain

Corollary 3.7 $$\pi_U \asymp \rho_0 \star \rho_U \asymp \rho_{U^*} \star \rho_0.$$

Now consider the unitary transform $U = e^{i\varphi}$ on $\mathbf{C}$. Since $\rho_{e^{i\varphi}} = \delta_\varphi$ and $\pi_{e^{i\varphi}} = \rho_\varphi$, we have

Corollary 3.8 $$\rho_\varphi \asymp \rho_0 \star \delta_\varphi \asymp \delta_{-\varphi} \star \rho_0.$$

We write $\hat{\rho}_\varphi$ for the family $\rho_0 \star \delta_\varphi$. Thus, we expressed the infinite-dimensional irreducible representations ρ_φ in terms of ρ_0 and δ_φ up to unitary equivalence. Now we like to carry this over to the mappings $\rho_\varphi \circ \mathcal{O}$ and $\rho_\varphi \circ \mathcal{T}$. However, it turns out that the above form is not yet suitable. (This is mainly due to the fact that an element of $K_1(K_2)$ is not necessarily mapped by Δ to the sum of elementary tensors $a \otimes b$ with $a, b \in K_1(K_2)$.)

We introduce the family

$$I_{\varphi_1\,\varphi_2} = \delta_{\varphi_1} \star Id \star \delta_{\varphi_2}$$

of automorphisms of $\mathcal{A}_q$. Then we obtain for the generators

$$I_{\varphi_1\,\varphi_2} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} e^{i(\varphi_2+\varphi_1)}\alpha \\ e^{i(\varphi_2-\varphi_1)}\gamma \end{pmatrix}.$$

The automorphisms $I_{\varphi_1\,\varphi_2}$ express the two fundamental invariances of $\mathcal{A}_q$, namely multiplying α or γ, respectively, by $e^{i\varphi}$. In particular, we are interested in the latter case which we obtain by setting $\varphi_2 = -\varphi_1 = \frac{\varphi}{2}$. We define

$$I_\varphi = I_{-\frac{\varphi}{2}\,\frac{\varphi}{2}}.$$

Thus, we can write

$$\rho_\varphi = \rho_0 \circ I_\varphi,\ \hat{\rho}_\varphi = \rho_0 \circ I_{0\,\varphi} \text{ (and } \delta_{-\varphi} \star \rho_0 = \rho_0 \circ I_{-\varphi\,0}).$$

The automorphisms I_φ have an additional property.

Proposition 3.9 $\mathcal{P} \circ I_\varphi = I_\varphi \circ \mathcal{P}$ *and* $(Id - \delta\mathbf{1}) \circ I_\varphi = I_\varphi \circ (Id - \delta\mathbf{1})$.

Proof Clearly, I_φ leaves K_1 and K_2 invariant (i.e. $I_\varphi K_i = K_i$). On the other hand, the elements $\mathbf{1}$ and $\frac{\alpha-\alpha^*}{2i}$ are not changed by I_φ at all. Now the statement is obvious. ■

Notice that

$$\frac{\mathbf{1}}{\mathbf{1} - p\alpha^{(*)}} = I_\varphi(\frac{\mathbf{1}}{\mathbf{1} - p\alpha^{(*)}})$$

for all φ.

Corollary 3.10 $I_\varphi \circ \mathcal{O} \;=\; \mathcal{O} \circ I_\varphi$ *and* $I_\varphi \circ \mathcal{T} \;=\; \mathcal{T} \circ I_\varphi$.

For any representation π we denote by $\mathcal{O}_\pi$ and $\mathcal{T}_\pi$ the mappings $\pi \circ \mathcal{O}$ and $\pi \circ \mathcal{T}$.

Theorem 3.11 $\mathcal{O}_{\rho_\varphi} = \mathcal{O}_{\rho_0} \circ I_\varphi$ *and* $\mathcal{T}_{\rho_\varphi} = \mathcal{T}_{\rho_0} \circ I_\varphi$.

Proof Using the foregoing proposition and corollary, we obtain

$$\mathcal{O}_{\rho_\varphi} = \rho_\varphi \circ \mathcal{O} = \rho_0 \circ I_\varphi \circ \mathcal{O} = \rho_0 \circ \mathcal{O} \circ I_\varphi = \mathcal{O}_{\rho_0} \circ I_\varphi$$

and, similarly, for $\mathcal{T}_{\rho_\varphi}$. ■

We conclude this section stating the obvious

Corollary 3.12 $\delta_{\varphi_1} \star \delta_{\varphi_2} = \delta_{\varphi_1+\varphi_2}$.

Thus, the convolution turns the set of all δ_φ into a convolution group of states with identity element δ and $\delta_{-\varphi}$ the inverse of δ_φ. We will treat the remaining convolution of irreducible representations, namely $\rho_0 \star \rho_0$, in Section 4.1.

Any unitary operator on a separable Hilbert space can be written as a *direct integral* over the *irreducible unitary operators* $e^{i\varphi}$. This fact extends to representations of $\mathcal{A}_q$ (see [8]). We will make this explicit in the next section.

3.3 Representations as direct integrals and consequences for conditionally positive functionals

Any unitary operator U on a Hilbert space H admits a *spectral representation* (cf. [9, X.5.3]), i.e. there is a family $\nu_\imath$ of finite (positive) regular measures on $S = [0, 2\pi]$ indexed by $\imath \in \mathcal{I}$ such that the identification

$$H = \bigoplus_{\imath \in \mathcal{I}} L^2(S, \nu_\imath)$$

can be made. Moreover, if $f_\imath : S \to \mathbf{C}$ is the component of $f \in H$ in $L^2(S, \nu_\imath)$, the restriction of U onto $L^2(S, \nu_\imath)$ is given by

$$U f_\imath(s) = e^{is} f_\imath(s).$$

In other words, H can be decomposed into a direct sum (of a possibly uncountable number) of L^2–function spaces where U is represented by multiplication with e^{is}.

We have at least two good reasons for assuming H to be separable, i.e. the set $\mathcal{I}$ to be countable. Firstly, our algebra $\mathcal{A}_q$ is finitely generated. Hence for any vector $\eta \in H$ the invariant subspace $\overline{\pi(\mathcal{A}_q)\eta}$ is separable. Secondly, for calculating our conditionally positive functionals we only need scalar products of the form $\langle \eta | \pi(a) | \eta \rangle$ with a fixed vector $\eta \in H$. This vector has components only in a countable number of subspaces $L^2(S, \nu_\imath)$ whose direct sum is separable.

Now we consider a mapping $\hat{\rho}_U : \mathcal{A}_q \to \mathcal{B}(H)$. It maps $a \in \mathcal{A}_q$ to an operator which is represented by multiplication with the function $\delta_s(a)$ of s, i.e.

$$\hat{\rho}_U(a) f_\imath(s) = \delta_s(a) f_\imath(s).$$

Clearly, this is a well-defined mapping, the representation property holds and it coincides with ρ_U on the generators. Therefore, we must have $\hat{\rho}_U = \rho_U$. This is the decomposition of ρ_U into irreducible representations.

Now we apply similar considerations to π_U on $H = h_0 \otimes \mathcal{H}_0$. We decompose $\mathcal{H}_0$ in the same manner and obtain

$$h_0 \otimes \mathcal{H}_0 \;=\; h_0 \otimes \left(\bigoplus_{\imath \in \mathcal{I}} L^2(S, \nu_\imath) \right) \;=\; \bigoplus_{\imath \in \mathcal{I}} (h_0 \otimes L^2(S, \nu_\imath)) \;=\; \bigoplus_{\imath \in \mathcal{I}} L^2_{h_0}(S, \nu_\imath)$$

where we use the notation $L^2_{h_0}(S, \nu_\imath) = h_0 \otimes L^2(S, \nu_\imath)$. These spaces can be interpreted as spaces of square integrable, h_0–valued, $\nu_\imath$–measurable functions on S. The natural isomorphism is given by

$$f_\imath = \sum_{k \in \mathbf{N}_0} e_k \otimes f_{\imath k}(s) \;\longmapsto\; f_\imath(s) = \sum_{k \in \mathbf{N}_0} e_k f_{\imath k}(s).$$

From this correspondence we see that

$$\pi_U(a) f_\imath(s) \;=\; \rho_s(a) f_\imath(s)$$

or shorter

$$\pi_U \;=\; \rho_s \;=\; \rho_0 \circ I_s.$$

This is the decomposition of π_U into irreducible representations. Obviously, the decomposition carries over to $\mathcal{O}_{\pi_U}$ and $\mathcal{T}_{\pi_U}$. We obtain by Theorem 3.11

$$\mathcal{O}_{\pi_U} = \mathcal{O}_{\rho_s} = \mathcal{O}_{\rho_0} \circ I_s \text{ and } \mathcal{T}_{\pi_U} = \mathcal{T}_{\rho_s} = \mathcal{T}_{\rho_0} \circ I_s.$$

The scalar product $\langle \bullet | \bullet \rangle_\imath$ on $L^2_{h_0}(S, \nu_\imath)$ is given by

$$\langle f_\imath | g_\imath \rangle_\imath = \int_S \langle f_\imath(s) | g_\imath(s) \rangle \, d\nu_\imath(s).$$

In order to obtain the scalar product on H we just have to sum this expression over $\imath \in \mathcal{I}$. Now we arrive at the main result of this section.

Theorem 3.13 *Let ψ be any conditionally positive functional associated (by Theorem 1.1) with a representation of type π_U. Then there is a family $\psi_{\imath s}$ of conditionally positive functionals, all associated with ρ_0 and defined for all $\imath \in \mathcal{I}$ and $\nu_\imath$–almost all $s \in S$, such that*

$$\psi \;=\; \sum_{\imath \in \mathcal{I}} \int_S \psi_{\imath s} \circ I_s \, d\nu_\imath(s) \;=\; \sum_{\imath \in \mathcal{I}} \int_S \delta_{-\frac{s}{2}} \star \psi_{\imath s} \star \delta_{\frac{s}{2}} \, d\nu_\imath(s)$$

on K_2.

Moreover, given any measurable family $\psi_{\imath s}$ of conditionally positive functionals associated with ρ_0, satisfying

$$\sum_{\imath \in \mathcal{I}} \int_S \psi_{\imath s}(\beta \beta^*) \, d\nu_\imath(s) < \infty,$$

there is a conditionally positive functional ψ whose values on K_2 are given by the above formula.

PROOF Let $\eta_\imath(s) \in H$ be the vector which generates the corresponding cocycle $\eta_{\imath s}$ via $\eta_{\imath s}(a) = \mathcal{O}_{\rho_s}(a)\eta_\imath(s)$. Defining

$$\psi_{\imath s} = \langle \eta_\imath(s) | \mathcal{T}_{\rho_0} | \eta_\imath(s) \rangle,$$

we obtain the desired family.

If, on the other hand, $\psi_{\imath s}$ is a family, satisfying the claimed conditions, we can define a state by $\varphi(\bullet) = \sum_{\imath\in\mathcal{I}} \int_S \langle \eta_{\imath s}(\beta^*) | \rho_0(\bullet) | \eta_{\imath s}(\beta^*) \rangle \, d\nu_\imath(s)$. The conditionally positive functional $\varphi \circ \mathcal{T}$ is on K_2 given by the stated formula. ∎

In order to complete the decomposition of a general conditionally positive functional, we return to representations of type ρ_U. They had been decomposed into the irreducible representations δ_s. One easily checks that

$$\begin{aligned} \mathcal{O}_{\delta_s}(a) &= \frac{\delta_s(a) - \delta(a)}{e^{-is} - 1} \\ \mathcal{T}_{\delta_s}(a) &= \frac{\delta_s(a) - \delta'(a)\sin s - \delta(a)}{|e^{-is} - 1|^2} \end{aligned}$$

for $s > 0$. On the other hand, for $s \to 0$ these mappings converge precisely to the elementary GAUSSian cocycle $\mathcal{O}_{\delta_0} = i\delta'$ and the elementary GAUSSian conditionally positive functional $\mathcal{T}_{\delta_0} = \frac{\delta''}{2}$ associated with $i\delta'$. The fact that all these mappings are scalar valued leads to the well-known LÉVY-KHINTCHINE formula for the one-dimensional torus.

Theorem 3.14 *Let ψ be any conditionally positive functional associated (by Theorem 1.1) with a representation of type ρ_U. Then there is a finite (positive) regular measure ν on S such that*

$$\psi = \int_S \mathcal{T}_{\delta_s} \, d\nu(s)$$

on K_2.

N.B.: Cf. also Example 4.3. It is not too surprising that we recover the results for the one-dimensional torus which is contained as a subgroup in any of the $SU_q(2)$. However, it is absolutely remarkable that the conditionally positive functionals on this *one*-parameter subgroup already contain the general GAUSSian part of the *quantization* of the *three*-parameter group $SU(2)$ (cf. Section 5.4).

PROOF We start with the spectral decomposition as given in the beginning of this section. For the moment, we identify $\mathcal{I}$ with the natural numbers $\mathbf{N}$. From the measures $\nu_\imath$ we construct the measure

$$\widetilde{\nu} = \sum_{\imath\in\mathcal{I}} \frac{1}{\imath^2} \frac{\nu_\imath}{\nu_\imath(S)}$$

with respect to which all $\nu_\imath$ are absolutely continuous. Therefore, we can find $\widetilde{\nu}$–integrable functions $\chi_\imath$ such that

$$d\nu_\imath = \chi_\imath \, d\widetilde{\nu}.$$

Now let $\eta = \eta_\imath(s)$ be the vector which generates the corresponding cocycle. For its norm we obtain

$$\begin{aligned} \|\eta\|^2 &= \sum_{\imath\in\mathcal{I}} \int_S \langle \eta_\imath(s) | \eta_\imath(s) \rangle \, d\nu_\imath(s) = \sum_{\imath\in\mathcal{I}} \int_S \langle \eta_\imath(s) | \eta_\imath(s) \rangle \chi_\imath(s) \, d\widetilde{\nu}(s) \\ &= \int_S \left(\sum_{\imath\in\mathcal{I}} \langle \eta_\imath(s) | \eta_\imath(s) \rangle \chi_\imath(s) \right) d\widetilde{\nu}(s). \end{aligned}$$

In the last step we used the theorem of *monotone convergence*. By setting

$$d\nu(s) = \sum_{\imath\in\mathcal{I}} \langle \eta_\imath(s) | \eta_\imath(s) \rangle \chi_\imath(s) \, d\widetilde{\nu}(s),$$

we define the measure ν which appears in the theorem. ■

Notice that we could have obtained this result directly, if we started from the spectral decompostion $U = \int_S e^{is}\, dE_s$. Clearly, the measure ν is given by $d\nu(s) = d\langle\eta|E_s|\eta\rangle$. But this derivation does not show explicitly why the family of possibly infinitely many measures $\nu_\imath$ can be reduced to a single one due to the fact that the δ_φ are one-dimensional representations.

The result can be reformulated as follows: The unitary operator U and the vector η, describing a conditionally positive functional associated with a representation of type π_1, can be chosen such that the spectrum of U is simple and that in the spectral representation η is given by $\eta(s) = 1$.

Chapter 4

Topology enters

4.1 The faithful representation $\rho_0 \star \rho_0$

Now we investigate the remaining not yet treated convolution of irreducible representations, namely $\rho_0 \star \rho_0$ acting on $h_0 \otimes h_0$ (cf. also [16]). Let $e_k \otimes e_\ell$ be a basis vector of $h_0 \otimes h_0$. According to (3.5) we have

$$\begin{aligned}
\rho_0 \star \rho_0(\alpha)(e_k \otimes e_\ell) &= \sqrt{1-q^{2k}}\sqrt{1-q^{2\ell}} e_{k-1} \otimes e_{\ell-1} \; - \; q^{k+\ell+1} e_k \otimes e_\ell \\
\rho_0 \star \rho_0(\gamma)(e_k \otimes e_\ell) &= q^k \sqrt{1-q^{2\ell}} e_k \otimes e_{\ell-1} \; + \; \sqrt{1-q^{2(k+1)}} q^\ell e_{k+1} \otimes e_\ell .
\end{aligned}$$

Any vector $c \in h_0 \otimes h_0$ can be written in the form

$$c = \sum_{k,\ell=0}^{\infty} c_{k\ell} e_k \otimes e_\ell .$$

If we define $c_{-1\,\ell} = c_{k\,-1} = 0$ we obtain

$$\begin{aligned}
\rho_0 \star \rho_0(\alpha)c &= \sum_{k,\ell=0}^{\infty} (c_{k+1\,\ell+1}\sqrt{1-q^{2(k+1)}}\sqrt{1-q^{2(\ell+1)}} \; - \; c_{k\ell} q^{k+\ell+1}) e_k \otimes e_\ell \\
\rho_0 \star \rho_0(\gamma)c &= \sum_{k,\ell=0}^{\infty} (c_{k\,\ell+1} q^k \sqrt{1-q^{2(\ell+1)}} \; + \; c_{k-1\,\ell}\sqrt{1-q^{2k}} q^\ell) e_k \otimes e_\ell .
\end{aligned}$$

Let us check if $\rho_0 \star \rho_0(\gamma)c$ can be 0. One easily finds by setting k equal to $0, 1, \ldots$ that $c_{0\ell+1}, c_{1\ell+2}, \ldots$ must be 0 for $\ell \in \mathbf{N}_0$, i.e. $c_{k\ell} = 0$ for $k < \ell$. On the other hand, for $k \geq \ell$ we have $|c_{k+1\,\ell+1}| > |c_{k\ell}|$. These cannot be components of a vector unless they vanish for all k, ℓ. Thus, $\rho_0 \star \rho_0$ must be of type π_2, hence must be unitarily equivalent to π_{U_0} for some U_0. We find

Theorem 4.1 *Let $\{\widetilde{e}_n\}_{n\in\mathbf{Z}}$ be an ONB of a* Hilbert *space $\mathcal{H}$ and U_0 be the unitary operator defined by*

$$U_0 \widetilde{e}_n = \widetilde{e}_{n+1} .$$

Then we have the following equivalence

$$\rho_0 \star \rho_0 \quad \asymp \quad \pi_{U_0} .$$

Proof In order to find U_0, we just have to identify the subspace $\mathcal{H}_0$ on which $\rho_0 \star \rho_0(\alpha)$ is 0. Then U_0 is unitarily equivalent to the restriction of $\rho_0 \star \rho_0(\gamma)$ to this subspace. For a vector $c \in \mathcal{H}_0$ we must have

$$c_{k+1\,\ell+1} = \frac{q^{k+\ell+1}}{\sqrt{1-q^{2(k+1)}}\sqrt{1-q^{2(\ell+1)}}} c_{k\ell} . \tag{4.1}$$

We immediately see that any $c_{k\ell}$ on the coordinate lines, i.e. c_{k0} or $c_{0\ell}$, determines the corresponding subdiagonal of $(c_{k\ell})$, i.e. the $c_{k+j\,j}$ or $c_{j\,\ell+j}$ for all $j \in \mathbf{N}_0$. The elements of such a diagonal decrease approximately like $q^{\frac{1}{2}(2k)^2}$, hence are components of a vector.

For $n \in \mathbf{Z}$ we require $e'_n \in \mathcal{H}_0$ to be the vector whose components fulfill Equation (4.1) and are such that $c_{k0} = c_{0\ell} = 0$ unless $k = n$ or $\ell = -n$. For all n this vector is unique if we choose c_{n0} and $c_{0\,-n}$ respectively in $\mathbf{R}_+$ such that e'_n has unit length. Obviously, $\{e'_n\}_{n\in\mathbf{Z}}$ is an ONB of $\mathcal{H}_0$. Moreover, one easily checks that $\rho_0 \star \rho_0(\gamma)$ maps any e'_n to e'_{n+1}. Therefore, π_{U_0} is unitarily equivalent to $\rho_0 \star \rho_0$. ■

In [35] Woronowicz shows (for $|q| \in (0, 1]$) that the set of all

$$\alpha^{\check{\ell}}\gamma^{*m}\gamma^n \quad \text{for} \quad \ell \in \mathbf{Z};\ m, n \in \mathbf{N}_0$$

is a basis for $\mathcal{A}_q$. (For the notation cf. the conventions). In order to prove the case $|q| \in (0, 1)$ he introduced precisely the representation π_{U_0}. In the course of this proof it becomes clear that π_{U_0} is a faithful representation.

We easily see that any element $a \in \mathcal{A}_q$ can be written uniquely in the form

$$a = \sum_{k,\ell\in\mathbf{Z}} \alpha^{\check{k}}\gamma^{\check{\ell}}P_{k\ell}(\gamma^*\gamma) \tag{4.2}$$

where $P_{k\ell}$ are polynomials and different from 0 only for a finite number of pairs $k, \ell \in \mathbf{Z}$. Thus, $\mathcal{A}_q$, as a vector space, has an obvious $\mathbf{Z} \times \mathbf{Z}$–graduation. The homogeneous elements are $\alpha^{\check{k}}\gamma^{\check{\ell}}P_{k\ell}(\gamma^*\gamma)$ with their degree d given by

$$d(\alpha^{\check{k}}\gamma^{\check{\ell}}P_{k\ell}(\gamma^*\gamma)) = (k, \ell).$$

Checking that

$$d(ab) = d(a)d(b) \quad \text{and} \quad d(a^*) = d(a)^{-1}$$

for all homogeneous $a, b \in \mathcal{A}_q$, we see that $\mathcal{A}_q$ is a $\mathbf{Z} \times \mathbf{Z}$–graded $*$–algebra.

4.2 $\rho_0 \star \rho_0$ as a C^*–algebra isomorphism

Now we consider the $*$–algebra $\pi_{U_0}(\mathcal{A}_q) \subset \mathcal{B}(h_0 \otimes \mathcal{H})$ equipped with the operator norm. In the sequel, we will see that the norms of $\mathcal{A}_q$ and this operator algebra coincide, hence the two algebras are isomorphic as pre–C^*–algebras. In this way, we obtain that $\rho_0 \star \rho_0(\mathcal{A})$ is an operator C^*–algebra isomorphic to $\mathcal{A}$.

The representations of $\mathcal{A}$ decompose into a direct integral over irreducible representations. Thus, $\|\pi(a)\|$ cannot be greater than the supremum over all irreducible representations. We obtain

$$\|a\| = \sup_{\pi \text{ irr.}} \|\pi(a)\| = \max\left(\sup_{s\in S} |\delta_s(a)|, \sup_{s\in S} \|\hat{\rho}_s(a)\|\right).$$

We show that we can forget about the first term in the maximum. Let $a \in \mathcal{A}_q$ be expanded according to the graduation (4.2). We have

$$|\delta_s(a)| = \Big|\sum_{k\in\mathbf{Z}} \delta_s(\alpha^{\check{k}})P_{k0}(0)\Big| = \Big|\sum_{k\in\mathbf{Z}} e^{isk}P_{k0}(0)\Big|.$$

Now let $c_n(\lambda) \in h_0$ for $\lambda \in (0, 1)$ be a sequence of unit vectors defined by

$$c_n(\lambda) = \sum_{k=n}^{\infty} c_{nk}(\lambda)e_k$$

with

$$c_{nk}(\lambda) = c_{n0}(\lambda) \frac{\lambda^k}{\sqrt{1-q^2} \cdots \sqrt{1-q^{2k}}}$$

and $c_{n0}(\lambda) \in \mathbf{R}_+$ such that the unit condition is fulfilled. (The denominator of c_{nk} can be estimated using the function $e_{q^2}(z)$ introduced in Appendix A.) By an easy calculation we obtain for these vectors

$$\langle c_n(\lambda) | \hat{\rho}_s(\alpha^{\check{k}}) | c_n(\lambda) \rangle = \lambda^{\check{k}} e^{isk}$$

independent of n. On the other hand,

$$\| \hat{\rho}_s(\gamma) c_n(\lambda) \| \leq q^n.$$

Thus, we have

$$\lim_{\lambda \to 1} \lim_{n \to \infty} \left| \langle c_n(\lambda) | \hat{\rho}_s(a) | c_n(\lambda) \rangle \right| = \left| \sum_{k \in \mathbf{Z}} e^{isk} P_{k0}(0) \right|.$$

In this manner, we obtain

$$|\delta_s(a)| \leq \| \hat{\rho}_s(a) \|$$

or in other words

$$\|a\| = \sup_{s \in S} \| \hat{\rho}_s(a) \|.$$

This result is closely related to that obtained by Coburn in [7]. He shows that the operator C^*–algebras generated by the one–sided shift operator and the direct sum of the one–sided shift and a unitary operator are isomorphic and, therefore, have the same norm. Recently, we received a preprint of Nica [19] in which a family T_q of C^*–algebras the so-called quantized Toeplitz algebra is investigated. We note that the operator C^*–algebras $\rho_\varphi(\mathcal{A})$ are isomorphic to T_q.

Returning to the foregoing formula, we can say that for all unitary operators U with spectrum e^{iS}, i.e. the whole unit circle, the operator norm on $\rho_0 \star \rho_U(\mathcal{A}_q)$ and the norm on $\mathcal{A}_q$ coincide. Clearly, U_0 fulfills this condition. (By considering the mapping $\widetilde{e}_n \mapsto e^{ins} \in L^2(S)$ we obtain the spectral representation of U_0 on $L^2(S)$. See the survey of Koornwinder [14] and [16].) Thus, we obtain

Theorem 4.2 *The algebras $\mathcal{A}_q$ and $\rho_0 \star \rho_0(\mathcal{A}_q)$ are isomorphic pre–C^*–algebras and the algebras $\mathcal{A}$ and $\rho_0 \star \rho_0(\mathcal{A})$ are isomorphic C^*–algebras. The isomorphism is given by $\rho_0 \star \rho_0$.*

Using the vectors $c_n(\lambda)$ we obtain the following

Corollary 4.3 *Any state associated with a representation of type π_1, and consequently any such conditionally positive functional, can be approximated pointwise by states associated with a representation of type π_2. More precisely, we have*

$$\langle \eta | \rho_U | \eta \rangle \quad = \quad \lim_{\lambda \to 1} \lim_{n \to \infty} \langle c_n(\lambda) \otimes \eta | \rho_0 \star \rho_U | c_n(\lambda) \otimes \eta \rangle.$$

Now we easily see that the structure maps of the coalgebra structure of $\mathcal{A}_q$ are continuous. δ is a one-dimensional representation. Δ is an algebra homomorphism into $\mathcal{A}_q \otimes \mathcal{A}_q$. Since $\mathcal{A}$ is of type I there is a unique C^*–algebra $\mathcal{A} \otimes \mathcal{A}$ which is the closure of $\mathcal{A}_q \otimes \mathcal{A}_q$ with respect to the unique norm (see [21, p.393]). We find the norm if we take the norm of the isomorphic operator algebra $\rho_0 \star \rho_0(\mathcal{A}) \otimes \rho_0 \star \rho_0(\mathcal{A})$. Now we can interpret Δ as a representation mapping to $\mathcal{B}(h_0 \otimes h_0 \otimes h_0 \otimes h_0)$. Hence, both δ and Δ are representations and therefore continuous.

By defining the antipode $\mathcal{S}$ by

$$\mathcal{S} \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \quad = \quad \begin{pmatrix} \alpha^* & \gamma^* \\ -q\gamma & \alpha \end{pmatrix}$$

and extending it as an anti-homomorphism, $\mathcal{A}_q$ is turned into a HOPF $*$–algebra (see [35]). From

$$\mathcal{S}(\gamma^{*k}) \quad = \quad (-q^{-1}\gamma^*)^k$$

we easily see that the antipode is not continuous. Now we close this section by giving the complete definition of $SU_q(2)$ equivalent to that introduced by WORONOWICZ [35, 36].

For $q \in [-1,1]$ the *matrix pseudo (quantum) group* $SU_q(2)$ is given by a C^*–bialgebra $\mathcal{A}$ and a $*$–bialgebra $\mathcal{A}_q$ for both of which the matrix

$$\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

is a unitary corepresentation. For $q \neq 0$ we can $\mathcal{A}_q$ turn into a HOPF $*$–algebra. The norm on $\mathcal{A}$ is given by the supremum on the operator norms of all representations.

4.3 Continuous cocycles and continuous conditionally positive functionals

In this section we solve the problem of finding all continuous cocycles and all continuous conditionally positive functionals on $\mathcal{A}_q$. Clearly, this means that we find all conditionally positive functionals on $\mathcal{A}$. This is because $Id - \delta\mathbf{1}$ is a continuous mapping onto $\overline{K_1}$, the completion of K_1, which, therefore, becomes a C^*–subalgebra of $\mathcal{A}$. Thus, since the restriction of any conditionally positive functional ψ on $\mathcal{A}$ to $\overline{K_1}$ is positive on $\overline{K_1}$, it must be a continuous mapping on $\overline{K_1}$. On the other hand, we have

$$\psi = \psi \circ (Id - \delta\mathbf{1}) + \psi(\mathbf{1})\delta$$

which is clearly a continuous mapping on $\mathcal{A}$.

For any representation π we have to find all vectors η in the representation space H for which the mappings $\mathcal{O}_\pi\eta$ *and* $\langle\eta|\mathcal{T}_\pi|\eta\rangle + r_1\delta + r_2\delta'$ are continuous or, equivalently, we have to find all η for which any bounded sequence $(a_n)_{n\in\mathbf{N}} \in \mathcal{A}_q$ by these two mappings is mapped to a bounded sequence in H and $\mathbf{C}$, respectively. First we show that we do not loose any continuous conditionally positive functional if we restrict ourselves to continuous cocycles.

Proposition 4.4 *Let η be any cocycle. Then we have*

$$\eta \text{ is not continuous} \Longrightarrow \psi \text{ is not continuous}$$

for any corresponding ψ.

PROOF Let (a_n) be a bounded sequence for which $\|\eta(a_n)\|^2$ is unbounded. Then for the sequence (b_n) with

$$b_n = (Id - \delta\mathbf{1})({a_n}^*)(Id - \delta\mathbf{1})(a_n),$$

$\psi(b_n) = \|\eta(a_n)\|^2$ is unbounded although (b_n) is bounded. ■

By the following lemma we reduce the problem of finding the continuous cocycles to that of investigating the bounded sequence $(\alpha^{*n})_{n\in\mathbf{N}}$.

Lemma 4.5 *Let π be any $*$–representation of $\mathcal{A}_q$ and η any vector in the representation space H. Then the following two conditions are equivalent:*

(i) $\eta \in \pi(\mathbf{1} - \alpha^*)H$, i.e. there is a vector $\zeta \in H$ such that $\eta = \pi(\mathbf{1} - \alpha^*)\zeta$.

(ii) There is a constant $C > 0$ such that $\|\mathcal{O}_\pi(\alpha^{*n})\eta\|^2 \leq C$ for all $n \in \mathbf{N}$.

Proof Let ζ be a vector such that the first condition is fulfilled. Then

$$(-\pi\zeta) \circ (Id - \delta\mathbf{1}) \;=\; \mathcal{O}_\pi \eta$$

is the cocycle generated by η which, therefore, is continuous. This yields the second condition.

Now let $C > 0$ be a constant such that the second condition is fulfilled. We split the proof according to the invariant subspaces H_1, and H_2.

We have $i\delta'(\alpha^{*n}) = n$, i.e. δ' is not continuous. Therefore, η cannot have a component in H_δ.

In the case of H_1, let u be given by its spectral decomposition

$$u = \int_0^{2\pi} e^{i\varphi}\, de_\varphi = \lim_{\lambda\to 0_+} \int_\lambda^{2\pi-\lambda} e^{i\varphi}\, de_\varphi.$$

The (strong) limit is due to the fact that 1 is not in the discrete spectrum of u (otherwise H_δ would not be the trivial subspace). For $\mathcal{O}_\pi(\alpha^{*n})$ we obtain

$$\mathcal{O}_\pi(\alpha^{*n}) = \int_0^{2\pi} \frac{e^{-in\varphi}-1}{e^{-i\varphi}-1}\, de_\varphi = \lim_{\lambda\to 0_+} \int_\lambda^{2\pi-\lambda} \frac{e^{-in\varphi}-1}{e^{-i\varphi}-1}\, de_\varphi.$$

Now the second condition reads

$$\int_\lambda^{2\pi-\lambda} \left|\frac{e^{-in\varphi}-1}{e^{-i\varphi}-1}\right|^2 d\mu(\varphi) \;\le\; C \text{ for all } \lambda \in [0,\pi] \text{ and } n \in \mathbf{N},$$

where we write

$$d\mu(\varphi) = d\langle \eta | e_\varphi | \eta \rangle.$$

We have

$$|e^{-i\varphi}-1|^2 \;=\; 2(1-\cos\varphi).$$

On the other hand, we find by summing the equation

$$\cos\nu\varphi\, \sin\frac{1}{2}\varphi \;=\; \frac{1}{2}\left(\sin(\nu+\frac{1}{2})\varphi - \sin(\nu-\frac{1}{2})\varphi\right)$$

for $\nu = 1,\ldots,n$ the well-known formula

$$\sum_{\nu=1}^n \cos\nu\varphi \;=\; \frac{1}{2}\frac{\sin(n+\frac{1}{2})\varphi}{\sin\frac{1}{2}\varphi} \;-\; \frac{1}{2}.$$

Using the estimate

$$\sin\frac{1}{2}\varphi \;\ge\; \frac{\varphi}{\pi} \text{ for } \varphi \in [0,\pi]$$

and a similar estimate for the interval $[\pi, 2\pi]$, we obtain

$$\frac{1}{n}\sum_{\nu=1}^n \cos\nu\varphi \;\le\; \frac{1}{2} \text{ for } \varphi \in [\tfrac{\pi}{n}, 2\pi - \tfrac{\pi}{n}].$$

Therefore, choosing n such that $\lambda \ge \frac{\pi}{n}$, we obtain

$$\begin{aligned} C \;&=\; \frac{1}{n}\sum_{\nu=1}^n C \;\ge\; \int_\lambda^{2\pi-\lambda} 2\frac{1-\frac{1}{n}\sum\limits_{\nu=1}^n \cos\nu\varphi}{|e^{-i\varphi}-1|^2}\, d\mu(\varphi) \;\ge \\ &\ge\; \int_\lambda^{2\pi-\lambda} \frac{1}{|e^{-i\varphi}-1|^2}\, d\mu(\varphi) \end{aligned}$$

for all $\lambda \in (0, \pi]$. This is nothing but the statement that

$$\zeta \;=\; -\lim_{\lambda \to 0_+} \int_\lambda^{2\pi-\lambda} \frac{1}{e^{-i\varphi}-1}\, de_\varphi \eta$$

is a vector in H_1. Clearly, ζ fulfills condition (i), for $H = H_1$.

Now we come to the case of H_2. First we give $\mathcal{O}(\alpha^{*n})$ in the explicit form

$$\mathcal{O}(\alpha^{*n}) \;=\; \sum_{\ell=1}^{n-1} \alpha^{*\ell}.$$

We introduce the renormed orthogonal basis $\{\bar{e}_k\}_{k\in\mathbf{N}_0}$ in the HILBERT space h_0 by

$$\bar{e}_k \;=\; \sqrt{1-q^2}\cdots\sqrt{1-q^{2k}}\, e_k$$

in which $\rho_0(\alpha^*)$ is given by

$$\rho_0(\alpha^*)\bar{e}_k \;=\; \bar{e}_{k+1}.$$

Let $\eta \in h_0 \otimes \mathcal{H}_0$ be given by the expansion

$$\eta \;=\; \sum_{k=0}^{\infty} \bar{e}_k \otimes c_k \text{ with } c_k \in \mathcal{H}_0.$$

We obtain (by putting c_ℓ equal to 0 if $\ell < 0$)

$$\mathcal{O}_\pi(\alpha^{*n})\eta \;=\; \sum_{\ell=1}^{n-1}\sum_{k=0}^{\infty} \bar{e}_{k+\ell} \otimes c_k \;=\; \sum_{k=0}^{\infty} \bar{e}_k \otimes \left(\sum_{\ell=k-n+1}^{k} c_\ell \right).$$

Therefore, condition (ii) reads

$$\sum_{k=0}^{K} (1-q^2)\cdots(1-q^{2k}) \left\| \sum_{\ell=k-n+1}^{k} c_\ell \right\|^2 \;\le C \text{ for all } n, K \in \mathbf{N}.$$

If we set $n = K + 1$, the inner sum starts from 0:

$$\sum_{k=0}^{K} (1-q^2)\cdots(1-q^{2k}) \left\| \sum_{\ell=0}^{k} c_\ell \right\|^2 \;\le C \text{ for all } K \in \mathbf{N}.$$

Thus, we see that

$$\zeta \;=\; \sum_{k=0}^{\infty} \bar{e}_k \otimes \left(\sum_{\ell=0}^{k} c_\ell \right)$$

is the claimed vector ζ in H_2. ■

We obtain as an immediate corollary

Theorem 4.6 *Let ψ be any continuous conditionally positive functional on $\mathcal{A}_q$. Then there are a $*$–representation π, a vector η_1 in the representation space H, and a number $r \in \mathbf{R}$ such that*

$$\psi \;=\; r\delta + \langle \eta_1 | \pi | \eta_1 \rangle \circ (Id - \delta \mathbf{1}).$$

The corresponding 1–cocycle η is given by

$$\eta \;=\; (\pi\eta_1) \circ (Id - \delta\mathbf{1}).$$

PROOF Let η be the cocycle associated with ψ by Theorem 1.1. Since ψ is continuous, η must be also continuous due to Proposition 4.4. Thus, choosing $\eta_1 = -\zeta$ by the foregoing Lemma we obtain that η is indeed of the claimed form.

On the other hand, one easily checks that $\langle\eta_1|\pi|\eta_1\rangle \circ (Id - \delta\mathbf{1})$ coincides with ψ on K_2 and is continuous on $\mathcal{A}$. According to Theorem 2.8, we can choose a real linear combination of δ and δ' in order to obtain ψ on the whole $\mathcal{A}$. However, δ is continuous and δ' is not. Therefore, the coefficient of δ' must be 0. ■

Notice that we obtain the number r by evaluating the functional ψ at the identity. The infinitesimal generators of white noises are the conditionally positive functionals vanishing at $\mathbf{1}$. We obtain them by setting r equal to 0.

By the above formulae we can associate with any given vector η_1 precisely one continuous cocycle and precisely one continuous infinitesimal generator.

Finally, we remark that the mappings $\mathcal{O}$ and $\mathcal{T}$ are not continuous (neither as mappings on $\mathcal{A}_q$ nor in the induced strong and weak topologies), i.e. there are cocycles which are not continuous.

EXAMPLE 4.1 *For representations π_U we find that any cocycle generated by a vector η of the form $\eta = e_k \otimes f$ with $f \in \mathcal{H}_0$ is not continuous.*

EXAMPLE 4.2 *For the representations ρ_u we find that the corresponding mappings $\mathcal{O}_{\rho_u}$ and $\mathcal{T}_{\rho_u}$ are continuous if and only if there is a 'gap' around 1 in the spectrum of u.*

4.4 Stronger norms on $\mathcal{A}_q$

In this section we equip $\mathcal{A}_q$ with two further norms in which all cocycles and all conditionally positive functionals, respectively, become continuous. To that goal we need some technical preparation.

Lemma 4.7 *For $p \in (0,1)$ we have*

$$\frac{\mathbf{1}}{\mathbf{1} - p\alpha^*}\frac{\mathbf{1}}{\mathbf{1} - p\alpha} \geq \frac{1}{4}\mathbf{1}.$$

PROOF Since $\|\mathbf{1} - p\alpha^*\| \leq 2$ for all $p \in (0,1)$ the element

$$A = \mathbf{1} - \lambda(\mathbf{1} - p\alpha^*)(\mathbf{1} - p\alpha)$$

is invertible in $\mathcal{A}$ for all $\lambda \in [0, \frac{1}{4})$. Therefore, we have

$$(\mathbf{1} - p\alpha)A^{-1}(\mathbf{1} - p\alpha^*) \quad = \quad \left(\frac{\mathbf{1}}{\mathbf{1} - p\alpha^*}\frac{\mathbf{1}}{\mathbf{1} - p\alpha} - \lambda\mathbf{1}\right)^{-1}.$$

This means that λ is not in the spectrum of our (positive) element of $\mathcal{A}$. ■

Now we can show the four fundamental inequalities.

Proposition 4.8

$$\begin{array}{lll} (i) & \|a\| \leq 2\|\mathcal{O}(a)\| & , a \in K_1 \\ (ii) & \|\mathcal{O}(a)\| \leq 2\|\mathcal{T}(a)\| & , a \in K_2 \\ (iii) & \|\mathcal{O}(ab)\| \leq 2\|\mathcal{O}(a)\|\,\|\mathcal{O}(b)\| & , a, b \in K_1 \\ (iv) & \|\mathcal{T}(ab)\| \leq 4\|\mathcal{T}(a)\|\,\|\mathcal{T}(b)\| & , a, b \in K_2. \end{array}$$

PROOF We identify elements of $\mathcal{A}$ with their images under the faithful representation ω. Because of the lemma we have for all $f \in H_\omega$, $p \in (0,1)$, and $a \in K_1$

$$\langle f|aa^*|f\rangle \quad \leq \quad 4\left\langle f\left|a\frac{\mathbf{1}}{\mathbf{1}-p\alpha^*}\frac{\mathbf{1}}{\mathbf{1}-p\alpha}a^*\right|f\right\rangle.$$

Since $a\frac{\mathbf{1}}{\mathbf{1}-p\alpha^*}$ converges to $\mathcal{O}(a)$ strongly for $a \in K_1$, we obtain (i) by performing the limit $p \to 1$, taking the supremum over $\|f\| = 1$, and taking the square root. (ii) follows similarly by using the inequality

$$\left\langle f\left|\frac{\mathbf{1}}{\mathbf{1}-p\alpha}a^*a\frac{\mathbf{1}}{\mathbf{1}-p\alpha^*}\right|f\right\rangle \quad \leq \quad 4\left\langle f\left|\frac{\mathbf{1}}{\mathbf{1}-p\alpha}a^*\frac{\mathbf{1}}{\mathbf{1}-p\alpha^*}\frac{\mathbf{1}}{\mathbf{1}-p\alpha}a\frac{\mathbf{1}}{\mathbf{1}-p\alpha^*}\right|f\right\rangle,$$

and taking into account that $\frac{\mathbf{1}}{\mathbf{1}-p\alpha}a\frac{\mathbf{1}}{\mathbf{1}-p\alpha^*}$ converges to $\mathcal{T}(a)$ strongly for $a \in K_2$. Since $\|\mathcal{T}(a)\| = \|\mathcal{T}(a^*)\|$, we obtain (iii) and (iv) directly from (2.1) and (2.2) together with (i) and (ii), respectively. ∎

Corollary 4.9 *(i) The mapping $a \mapsto \|a\|_{\mathcal{O}} = 2\|\mathcal{O}(a)\|$ is a norm on K_1. The completion $K_{\mathcal{O}}$ of K_1 with respect to this norm is a* BANACH *algebra.*

(ii) The mapping $a \mapsto \|a\|_{\mathcal{T}} = 4\|\mathcal{T}(a)\|$ is a norm on K_2. The completion $K_{\mathcal{T}}$ of K_2 with respect to this norm is an involutive BANACH *algebra.*

N.B.: The involution cannot be extended to a continuous mapping on $K_{\mathcal{O}}$. Consider the sequence $\{{\alpha^*}^k E_0\}_{k\in\mathbf{N}_0}$. (Here we identify E_0 with the unique element of $\overline{K_1}$ which is mapped to the projection E_0 by a faithful representation.) From

$$E_0\alpha^k\frac{\mathbf{1}}{\mathbf{1}-p\alpha^*} \quad = \quad E_0\alpha^k\sum_{i=0}^{k}(p\alpha^*)^i$$

we see that the elements ${\alpha^*}^k E_0$ and their adjoints are elements of $K_{\mathcal{O}}$. However, $\mathcal{O}({\alpha^*}^k E_0) = {\alpha^*}^k E_0$ is a bounded sequence and $\mathcal{O}(E_0\alpha^k) = E_0\alpha^k\sum\limits_{i=0}^{k}{\alpha^*}^i$ is not. One easily calculates in the irreducible representation ρ_0

$$\begin{aligned}\langle e_0|\mathcal{O}_{\rho_0}(E_0\alpha^k)\mathcal{O}_{\rho_0}(E_0\alpha^k)^*|e_0\rangle \quad &= \quad (1-q^2)\cdots(1-q^{2k})\sum_{i=0}^{k}(1-q^{2k})\cdots(1-q^{2(k-i+1)}) \geq \\ &\geq \quad k\frac{1}{e_{q^2}(q^2)^2}.\end{aligned}$$

Therefore, $K_{\mathcal{O}}$ cannot be an involutive BANACH algebra. Furthermore, we see that if $a^*, b \in K_{\mathcal{O}}$ then $ab \in K_{\mathcal{T}}$. Thus, E_0 is an element of $K_{\mathcal{T}}$.

So far we obtained the non-unital BANACH algebras $K_{\mathcal{O}}$ and $K_{\mathcal{T}}$. Now we show that we can extend the norms $\|\bullet\|_{\mathcal{O}}$, and $\|\bullet\|_{\mathcal{T}}$ to $\mathcal{A}_q$ such that $(\mathcal{A}_q, \|\bullet\|_{\mathcal{O}})$, and $(\mathcal{A}_q, \|\bullet\|_{\mathcal{T}})$ are a normed algebra and an involutive normed algebra, respectively. Together with the original C^*–norm the three norms show to be increasing.

Theorem 4.10 *The norms $\|\bullet\|_{\mathcal{O}}$ and $\|\bullet\|_{\mathcal{T}}$ can be extended to norms on $\mathcal{A}_q$. More precisely, we have:*

(i) The mapping $a \mapsto \|a\|_{\mathcal{O}} = |\delta(a)| + 2\|\mathcal{O}(a)\|$ is a norm on $\mathcal{A}_q$. The completion $\mathcal{A}_{\mathcal{O}}$ of $\mathcal{A}_q$ with respect to this norm is a BANACH *algebra with unit. Moreover, we have $\mathcal{A}_{\mathcal{O}} = \mathbf{C}\,\mathbf{1} \oplus K_{\mathcal{O}}$.*

(ii) The mapping $a \mapsto \|a\|_{\mathcal{T}} = |\delta(a)| + 2\|\mathcal{O}(\frac{\alpha-\alpha^*}{2i})\|\,|\delta'(a)| + 4\|\mathcal{T}(a)\|$ is a norm on $\mathcal{A}_q$. The completion $\mathcal{A}_{\mathcal{T}}$ of $\mathcal{A}_q$ with respect to this norm is an involutive BANACH *algebra with unit. Moreover, we have $\mathcal{A}_{\mathcal{T}} = \mathbf{C1} \oplus \mathbf{C}\,\frac{\alpha-\alpha^*}{2i} \oplus K_{\mathcal{T}}$.*

(iii) We have $\|a\|_{\mathcal{T}} \geq \|a\|_{\mathcal{O}} \geq \|a\|$, so that $\mathcal{A}_{\mathcal{T}} \subset \mathcal{A}_{\mathcal{O}} \subset \mathcal{A}$.

PROOF If we adjoin a unit to $K_{\mathcal{O}}$ in the usual way (see e.g. [8]), we obtain (i).

Next we show (iii). Let any $a \in \mathcal{A}_q$ be given in the canonical Expansion (1.7), i.e.

$$a = c_1\mathbf{1} + c_2\frac{\alpha - \alpha^*}{2i} + c$$

with $c \in K_2$ and $c_1, c_2 \in \mathbf{C}$. Using (ii) and (i) of Proposition 4.8, we obtain

$$\begin{aligned} \|a\|_{\mathcal{T}} &= |c_1| + 2|c_2|\,\|\mathcal{O}(\frac{\alpha-\alpha^*}{2i})\| + 4\|\mathcal{T}(c)\| \geq \\ &\overset{\text{(ii)}}{\geq} |c_1| + 2|c_2|\,\|\mathcal{O}(\frac{\alpha-\alpha^*}{2i})\| + 2\|\mathcal{O}(c)\| \geq \\ &\geq |c_1| + 2\|\mathcal{O}(c_2\frac{\alpha-\alpha^*}{2i} + c)\| = \|a\|_{\mathcal{O}} \geq \\ &\overset{\text{(i)}}{\geq} |c_1| + \|c_2\frac{\alpha-\alpha^*}{2i} + c\| \geq \|a\|, \end{aligned}$$

In order to prove (ii) we first show that $\mathbf{C}\,\frac{\alpha-\alpha^*}{2i} \oplus K_{\mathcal{T}}$ is an involutive BANACH algebra with the norm described in (ii). The only property of an involutive BANACH algebra which still has to be shown is the product inequality $\|ab\| \leq \|a\|\,\|b\|$. It suffices to prove it for $a, b \in K_1$. We obtain

$$\begin{aligned} \|ab\|_{\mathcal{T}} &= 4\|\mathcal{T}(ab)\| \leq \\ &\leq 4\|\mathcal{O}(a^*)\|\,\|\mathcal{O}(b)\| = \|a^*\|_{\mathcal{O}}\|b\|_{\mathcal{O}} \leq \\ &\leq \|a\|_{\mathcal{T}}\|b\|_{\mathcal{T}} \end{aligned}$$

where we used (iii). To this algebra we can adjoin a unit as in (i). ■

N.B.: Notice that $\{{\alpha^*}^k\}_{k\in\mathbf{N}_0}$ is a sequence which is bounded in $\mathcal{A}$ but unbounded in $\mathcal{A}_{\mathcal{O}}$ (cf. Section 4.3), and that $\{(\alpha^k - \mathbf{1})(\alpha^* - \mathbf{1})\}_{k\in\mathbf{N}_0}$ is a sequence which is bounded in $\mathcal{A}_{\mathcal{O}}$ but unbounded in $\mathcal{A}_{\mathcal{T}}$ (use (2.1) and (2.2) to reduce the latter case to the foregoing). Therefore, not any two of the three unital BANACH algebras coincide. Furthermore, notice that the adjoint of the latter sequence is bounded in $\mathcal{A}_{\mathcal{O}}$, too. Therefore, even $\mathcal{A}_{\mathcal{O}} \cap {\mathcal{A}_{\mathcal{O}}}^* \neq \mathcal{A}_{\mathcal{T}}$. If we equip this intersection with the norm $\max(\|\bullet\|_{\mathcal{O}}, \|(\bullet)^*\|_{\mathcal{O}})$, we obtain another involutive BANACH algebra which itermediates $\mathcal{A}_{\mathcal{T}}$ and $\mathcal{A}_{\mathcal{O}}$. Notice further that (again using (2.1) and (2.2)) $\mathcal{A}_q K_{\mathcal{O}} \mathcal{A}_q = K_{\mathcal{O}}$ and $\mathcal{A}_q K_{\mathcal{T}} \mathcal{A}_q = K_{\mathcal{T}}$, and $K_{\mathcal{O}}$ is a left ideal in $\mathcal{A}$. But neither $K_{\mathcal{O}}$ nor $K_{\mathcal{T}}$ are ideals in $\mathcal{A}$. We mention also that the projection $\mathcal{P}$ is continuous on $\mathcal{A}_{\mathcal{O}}$, because $|\delta'(\bullet)| = |\delta \circ \mathcal{O}(\bullet)| \leq \|\mathcal{O}(\bullet)\|$. Of course, all cocycles and conditionally positive functionals are continuous mappings on $\mathcal{A}_{\mathcal{O}}$ and $\mathcal{A}_{\mathcal{T}}$, respectively.

We further mention that the operations of $\mathcal{O}$ and $\mathcal{T}$ are in analogy to the operations of the first and second derivative of the functions $a(\varphi) = \delta_\varphi(a)$, respectively. In both cases we have that the uniform convergence of the first (second) derivative of a sequence a_n, implies the convergence of the sequence (the first derivative of the sequence), if only the convergence at one single basis vector, here $\mathbf{1}$ ($\frac{\alpha-\alpha^*}{2i}$), is guaranteed.

4.5 Lévy-Khintchine formula for $SU_q(2)$

At the end of Chapter 2 we mentioned that the problem of finding cocycles consists mainly in 'inverting' the element β^*. Of course, we cannot invert β^* as an element of an algebra, because it *is* not invertible. But by looking carefully at the formulae, one realizes that we actually inverted the operations of multiplication with β^* from the right in the case of $\mathcal{O}$, and conjugation with β in the case of $\mathcal{T}$.

Theorem 4.11 *(i) The algebra $\mathcal{A}_q\beta^* \subset K_1$ is dense in $K_{\mathcal{O}}$. Moreover, $\mathcal{A}\beta^* = K_{\mathcal{O}}$ and $\mathcal{O} : \mathcal{A}\beta^* \to \mathcal{A}$ is a BANACH space isomorphism with the inverse given by $a \mapsto a\beta^*$.*

(ii) *The $*$–algebra $\beta\mathcal{A}_q\beta^* \subset K_2$ is dense in $K_{\mathcal{T}}$. Moreover, $\beta\mathcal{A}\beta^* = K_{\mathcal{T}}$ and $\mathcal{T} : \beta\mathcal{A}\beta^* \to \mathcal{A}$ is an involutive* Banach *space isomorphism with the inverse given by $a \mapsto \beta a \beta^*$.*

Proof Obviously, $\mathcal{A}\beta^*$ is the closure of $\mathcal{A}_q\beta^*$ and, therefore, a subset of $K_{\mathcal{O}}$. And, obiously, $\mathcal{O}$ is an isometry from $\mathcal{A}\beta^*$ onto $\mathcal{A}$ and multiplication with β^* from the right its inverse. On the other hand, we also have $\mathcal{O}(K_{\mathcal{O}}) = \mathcal{A}$ and, therefore, $K_{\mathcal{O}} = \mathcal{A}\beta^*$. This proves (i). The proof of (ii) is completely anologous. ■

Corollary 4.12 $$K_{\mathcal{O}}{}^* K_{\mathcal{O}} = K_{\mathcal{T}}.$$

We explained how to reverse the action of β^* on the algebraic level. Another possibility of giving sense to ${\beta^*}^{-1}$ is on the level of representations. It consists in either restricting the domain of $\pi(\beta^*)^{-1}$ such that the range coincides with H (this will be done rather for β than for β^*), or enlarging the range such that $\pi(\beta^*)^{-1}f$ has a precise meaning for any $f \in H$. Clearly, for this it is necessary that β^* and β are mapped to injective operators. Each of these two conditions is equivalent to the condition that the invariant subpace H_δ is given by the nullspace. Therefore, in the remainder of this section we assume these conditions to be fulfilled, and we identify the elements of $\mathcal{A}$ with their images under the representation π on H.

We introduce the triplet of Hilbert spaces

$$H_{\beta^*} \supset H \supset H_\beta$$

as follows. $H_\beta = \beta H$ and the scalar product $\langle \bullet | \bullet \rangle_\beta$ on H_β is such that $\beta : H \to H_\beta$ becomes a Hilbert space isomorphism, i.e.

$$\langle \bullet | \bullet \rangle_\beta \quad = \quad \langle \beta^{-1} \bullet | \beta^{-1} \bullet \rangle.$$

H_{β^*} is the completion of H with respect to the scalar product

$$\langle \bullet | \bullet \rangle_{\beta^*} \quad = \quad \langle \beta^* \bullet | \beta^* \bullet \rangle.$$

Clearly, β^* can be extended continuously to a Hilbert space isomorphism $\beta^* : H_{\beta^*} \to H$. Moreover, H is dense in H_{β^*} by definition. By Lemma 2.1 and the remark following Proposition 5.15 we realize that H_β is dense in H, too.

Now we can ask two questions. Firstly, what is the completion of the space consisting of all continuous linear functionals on H_β of the form $\langle f | \bullet \rangle, f \in H$? Secondly, which of the functionals $\langle g | \bullet \rangle, g \in H$ can be extended continuously to H_{β^*}? The answer is given by

Proposition 4.13 *The spaces H_{β^*} and H_β are dual in a canonical way. More precisely, let $\{f_n\}_{n \in \mathbf{N}}$ with $f_n \in H$ be a representative of $f \in H_{\beta^*}$.*

(i) *By setting*

$$F(g) \quad = \quad \lim_{n \to \infty} \langle f_n | g \rangle, g \in H_\beta,$$

we establish a one-to-one correspondence between continuous linear functionals F on H_β and elements $f \in H_{\beta^*}$. We have $\|F\| = \|f\|_{\beta^*}$.

(ii) *By setting*

$$G(f) \quad = \quad \lim_{n \to \infty} \langle g | f_n \rangle, f \in H_{\beta^*},$$

we establish a one-to-one correspondence between continuous linear functionals G on H_{β^*} and elements $g \in H_\beta$. We have $\|G\| = \|g\|_\beta$.

PROOF For any pair (f,g) of vectors $f \in H_{\beta^*}$ and $g \in H_\beta$ there is a (unique) pair (f',g') of vectors $f',g' \in H$ such that $f = {\beta^*}^{-1}f'$ and $g = \beta g'$, and $\beta^* f_n$ converges in H to f'. Therefore, we have

$$F(g) \;=\; \langle f'|g'\rangle \;=\; \overline{G(f)},$$

i.e. the given linear functionals $F(g)$ and $G(f)$ have the whole spaces H_β and H_{β^*}, respectivly, as their domain. For their norms we obtain

$$\|F\| = \sup_{\|g\|_\beta=1} \langle f'|g'\rangle = \sup_{\|g'\|=1} \langle f'|g'\rangle = \|f'\| = \|f\|_{\beta^*}$$

and

$$\|G\| = \sup_{\|f\|_{\beta^*}=1} \langle g'|f'\rangle = \sup_{\|f'\|=1} \langle g'|f'\rangle = \|g'\| = \|g\|_\beta,$$

respectively. Therefore, F and G are indeed continuous.

On the other hand, given any two continuous linear functionals F and G on H_β and H_{β^*}, there are vectors $f'' \in H_\beta$ and $g'' \in H_{\beta^*}$, such that $F(\bullet) = \langle f''|\bullet\rangle_\beta$ and $G(\bullet) = \langle g''|\bullet\rangle_{\beta^*}$, respectively. Obviously, the vectors $f = {\beta^*}^{-1}\beta^{-1}f'' \in H_{\beta^*}$ and $g = \beta\beta^* g'' \in H_\beta$ generate F and G in the stated way. ■

Corollary 4.14 *For any representation π of $\mathcal{A}$ with $\pi(\beta^*)$ injective we have*

(i) For any $a \in \mathcal{A}\beta^*$ the mapping $\pi(a) \in \mathcal{B}(H,H)$ can be extended (uniquely) to a mapping $\pi_{\beta^*}(a) \in \mathcal{B}(H_{\beta^*},H)$ with $\|\pi_{\beta^*}(a)\| = \|\mathcal{O}_\pi(a)\|$. For any $f \in H_{\beta^*}$ we have

$$\pi_{\beta^*}(a)f \;=\; \mathcal{O}_\pi(a)\beta^* f.$$

(ii) If we interprete for $a \in \beta\mathcal{A}$ the element $\pi(a) \in \mathcal{B}(H,H)$ as an element of $\mathcal{B}(H,H_\beta)$, we indicate this by writing $\pi_\beta(a)$. For any $a \in \beta\mathcal{A}\beta^*$ the mapping $\pi_\beta(a) \in \mathcal{B}(H,H_\beta)$ can be extended (uniquely) to a mapping $\pi_{\beta\beta^*}(a) \in \mathcal{B}(H_{\beta^*},H_\beta)$ with $\|\pi_{\beta\beta^*}(a)\| = \|\mathcal{T}_\pi(a)\|$. For any $f,g \in H_{\beta^*}$ we have

$$F(\pi_{\beta\beta^*}(a)g) \;=\; \langle \beta^* f|\mathcal{T}_\pi(a)|\beta^* g\rangle.$$

PROOF The statements are straightforward applications of the fact that an operator $B : \mathcal{D}_1 \to H_2$ from a pre-HILBERT space $\mathcal{D}_1$ which is dense in H_1 to a HILBERT space H_2 allows a continuous extension to H_1 if and only if there is a constant C, such that

$$|\langle g|Bf\rangle_2| \le C\|g\|_2\|f\|_1$$

for all $f \in \mathcal{D}_1$, $g \in H_2$. The smallest of these constants is the norm $\|B\|$ of $B \in \mathcal{B}(H_1,H_2)$. ■

We also write $F(\pi_{\beta\beta^*}(a)g) = \langle f|\pi_{\beta\beta^*}(a)|g\rangle$.

As an immediate consequence we obtain a new formulation of Theorem 2.8, at least for the conditional positive functionals without GAUSSian parts.

Proposition 4.15 *Let π be a representation of $\mathcal{A}$ with $\pi(\beta^*)$ injective. Then*

$$\eta(a) \;=\; \pi_{\beta^*} \circ (Id - \delta\mathbf{1})(a)\eta_1$$

establishes a one-to-one correspondence between cocycles η with respect to π and elements η_1 in H_{β^*}. By choosing arbitrary real numbers r_1, r_2, we obtain all conditionally positive functionals ψ fulfilling property (1.2) by

$$\psi(a) \;=\; r_1\delta(a) + r_2\delta'(a) + \langle \eta_1|\pi_{\beta\beta^*} \circ \mathcal{P}(a)|\eta_1\rangle.$$

The connection between η_1 and η_{α^*} is given by $\eta_1 = {\beta^*}^{-1}\eta_{\alpha^*}$.

We ommit the subscript $\beta\beta^*$, because no confusion about the domain and range of π can arise, and give a formula for the conditionally positive functionals which may be considered as one of the main results of these notes: the analogue of the classical LÉVY-KHINTCHINE formula.

Theorem 4.16 *Any conditionally positive functional ψ can be written in the form*

$$\psi(a) \;=\; \psi_\delta(a) + \langle \eta_1 | \pi \circ \mathcal{P} | \eta_1 \rangle,$$

where π is a representation with $\pi(\beta^*)$ injective, $\eta_1 \in H_{\beta^*}$, and ψ_δ is a GAUSSian part according to Corollary 2.11. The correspondence between conditionally positive functionals and triples $(\pi, \eta_1, \psi_\delta)$ is one-to-one up to unitary equivalence.

We explained already in the introduction, why this formula is in formal analogy to the classical case (see also Section 5.4). However, we mention again that the GAUSSian part ψ_δ and the projection $\mathcal{P}$, written explicitly, are in formal anology rather to the one-parameter case than to the classical $SU(2)$.

Furthermore, we want to emphasize that the calculation of this simpler looking expressions is by no means less difficult than the calculation of our original formulation in Theorem 2.8. The elements of H_{β^*} are given as sequences $f_n \in H$ such that $\beta^* f_n$ converges to an element of H. That means that in the latter formulation, in contrast to the first formulation, the limit has to be performed for any classifying pair (π, η_1) separately. Cf. Example 2.5.

EXAMPLE 4.3 *Consider a representation $\rho_U = \int \delta_s \, dE_s$ without GAUSSian part. For a vector $\eta_{\alpha^*} \in H$ we can define a conditionally positive functional*

$$\psi \;=\; \int \mathcal{T}_{\delta_s} \, d\nu(s),$$

where $d\nu(s) = d\langle \eta_{\alpha^*} | E_s | \eta_{\alpha^*} \rangle$. On the other hand, our new formula reads

$$\psi \;=\; \int \delta_s \circ \mathcal{P} \, d\mu(s),$$

where $d\mu(s) = d\langle \eta_1 | E_s | \eta_1 \rangle$ (which is to be understood as the extension of measures of the form $d\langle \eta | E_s | \eta \rangle$ to vectors in H_{β^*}). The two formulae coincide, because $|e^{-is} - 1|^2 \, d\mu(s) = d\nu(s)$. However, notice that $\delta \circ \mathcal{P} = 0$. This shows that, in contrast to our first formulation, it is not possible to include a GAUSSian part in the new formulation.

Chapter 5

The exceptional cases $q = -1, 0, 1$

In this chapter we investigate the remaining cases $q = 0$, $q = 1$, and $q = -1$. The results for $q = 0$ coincide completely with those for $|q| \in (0, 1)$ obtained in the preceeding chapters. Just the proofs of some lemmas and propositions must be *adapted* carefully to this case. It is known that the C^*-completions $\mathcal{A}$ are isomorphic for all $q \in (-1, 1)$ (see [35]). We give a new algebraic proof of this fact. It turns out that the algebras $\mathcal{A}_\mathcal{O}, \mathcal{A}_\mathcal{T}, \mathcal{K}_\mathcal{O}, \mathcal{K}_\mathcal{T}$, considered as subalgebras of $\mathcal{A}$ are also independent of $q \in (-1, 1)$. However, notice that the norms $\| \bullet \|_\mathcal{O}, \| \bullet \|_\mathcal{T}$ depend on q.

Although at least the classical case $q = 1$ is known, we also investigate the cases $q = -1, 1$ with our methods, to have a unified reference for the results of Chapter 6. The anti-classical case $q = -1$ seems to be still unknown.

5.1 The case $q = 0$

The very first step in the main arguments of the preceeding chapters was to split up the representation spaces into a part H_1, where γ is mapped to 0, and a part H_2, where γ is mapped to an injective operator. These subspaces showed to be invariant subspaces and on H_2 we could apply Lemma 1.6. However, we see from Relation (d) that in the case when $q = 0$ either α or α^* must be different from 0 if γ is different from 0. On the other hand, by the adjoint of Relation (a) or (b) it follows that γ cannot be injective at all. Nevertheless, it is still possible to decompose a given representation into invariant subspaces H_1 and H_2 such that $\pi_1(\gamma) = 0$ and Lemma 1.6 holds for $\pi_2(\alpha)$. Since it is not possible to give another simple characterization of these subspaces (such as $\pi_2(\gamma)$ is injective), we postpone the proof of this statement. In order to proceed as in Chapter 2 we only need to know the following

Proposition 5.1 *By replacing in the expressions of π_{U_0} the parameter q formally by 0 (with the convention $0^0 = 1$), a representation of $\mathcal{A}_0$, again denoted by π_{U_0}, is defined. This representation is faithful and Lemma 1.6 holds.*

Proof It is easy to see that the expressions of any representation of type π_2 extended formally to $q = 0$ define a representation of $\mathcal{A}_0$. In the case of π_{U_0} the element α is mapped to a multiple of the one-sided shift operator and, therefore, Lemma 1.6 holds. The faithfulness of π_{U_0} follows from the proof of the next proposition. ■

Proposition 5.2 *The elements ${\alpha^*}^k \gamma^{\check{m}} \alpha^\ell$ for $k, \ell \in \mathbf{N}_0$ and $m \in \mathbf{Z}$ form a basis of $\mathcal{A}_0$.*

Proof First we have to make clear that any monomial in $\mathcal{A}_0$ can be expressed as a (finite) linear combination of the given vectors. Notice that due to the relations α^* is an isometry, i.e. $\alpha\alpha^* = \mathbf{1}$, and $\gamma^*\gamma$ is a projection with the additional property that $\gamma^{(*)}\gamma^*\gamma = \gamma^{(*)}$. Thus, any factor in the monomial consisting of an *accumulation* of γ's and γ^*'s, reduces to either $\gamma^{\check{m}}, m \neq 0$ or $\gamma^*\gamma$. The latter can be reduced by Relation (e) to the difference of two monomials with no γ's at this position.

The remaining *accumulations* of α's and α^*'s between the $\gamma^{\check{m}}$'s can be brought into the form ${\alpha^*}^k \alpha^\ell, k, \ell \in \mathbf{N}_0$ by the isometry property. But, any such factor having $k \neq 0$ and $\ell \neq 0$ whose position is not on the extreme left and the extreme right, respectively, cancels with the neighbouring γ's. Thus, the given set of vectors indeed spans $\mathcal{A}_0$.

The linear independence is still to be shown. For this aim we introduce the following linear functionals $\chi_{km\ell}$ on $\mathcal{A}_0$. Let

$$\chi_{km\ell} \;=\; \langle e_k \otimes \widetilde{e}_m | \pi_{U_0} | e_\ell \otimes \widetilde{e}_0 \rangle$$

for $m \neq 0$ and

$$\chi_{k0\ell} \;=\; \langle e_k \otimes \widetilde{e}_0 | \pi_{U_0} | e_\ell \otimes \widetilde{e}_0 \rangle - \langle e_{k-1} \otimes \widetilde{e}_0 | \pi_{U_0} | e_{\ell-1} \otimes \widetilde{e}_0 \rangle,$$

where we defined $e_{-1} = 0$. The functionals fulfill

$$\chi_{k'm'\ell'}({\alpha^*}^k \gamma^{\check{m}} \alpha^\ell) \;=\; \delta_{k'k}\delta_{m'm}\delta_{\ell'\ell}$$

for all $k, k', \ell, \ell' \in \mathbf{N}_0$ and $m, m' \in \mathbf{Z}$. This proves the linear independence. Moreover, it is now clear that π_{U_0} must be a faithful representation. ■

If we combine this with the theorem of Woronowicz given in [35], we obtain

Corollary 5.3 *The elements ${\alpha^*}^k \gamma^{\check{m}} \alpha^\ell$ for $k, \ell \in \mathbf{N}_0$ and $m \in \mathbf{Z}$ form a basis of $\mathcal{A}_q$ for any $q \in [-1, 1]$.*

Proof Let q be different from 0. We have

$$\begin{aligned} \alpha^*\alpha({\alpha^*}^{k-1}\alpha^{k-1}) \;&=\; (\mathbf{1} - \gamma^*\gamma){\alpha^*}^{k-1}\alpha^{k-1} \;=\; {\alpha^*}^{k-1}(\mathbf{1} - q^{2(k-1)}\gamma^*\gamma)\alpha^{k-1} \;= \\ &=\; (1 - q^{2(k-1)}){\alpha^*}^{k-1}\alpha^{k-1} + {\alpha^*}^k\alpha^k. \end{aligned}$$

Thus, we can show that $(\alpha^*\alpha)^n$ and, henceforth, $(\gamma^*\gamma)^n$ is a linear combination of $\{{\alpha^*}^k\alpha^k\}_{k=0,\ldots,n}$ by induction. By Woronowicz's theorem we know that for all $\ell \in \mathbf{N}, n \in \mathbf{N}_0$ and $m \in \mathbf{Z}$ any of the sets $V_{\ell n} = \{{\alpha^*}^\ell \gamma^{\check{m}} (\gamma^*\gamma)^k\}_{k=0,\ldots,n}$, $V_{-\ell n} = \{\gamma^{\check{m}} (\gamma^*\gamma)^k \alpha^\ell\}_{k=0,\ldots,n}$, and $V_{0n} = \{\gamma^{\check{m}} (\gamma^*\gamma)^k\}_{k=0,\ldots,n}$ is linearly independent. Clearly, this must hold for any of the sets $V'_{\ell n} = \{{\alpha^*}^{\ell+k} \gamma^{\check{m}} \alpha^k\}_{k=0,\ldots,n}$, $V'_{-\ell n} = \{{\alpha^*}^k \gamma^{\check{m}} \alpha^{\ell+k}\}_{k=0,\ldots,n}$, and $V'_{0n} = \{{\alpha^*}^k \gamma^{\check{m}} \alpha^k\}_{k=0,\ldots,n}$, because $V_{\ell n}$ and $V'_{\ell n}$ span the same $n+1$–dimensional vector space for any $\ell \in \mathbf{Z}$. Now the statement is obvious, because $\mathcal{A}_q$ is spanned by the union of all these sets which mutually linearly independent. ■

Let us return to the case $q = 0$. It is easy to check that decomposition (1.7), the formula for the projection $\mathcal{P}$ onto K_2, and the proofs of Lemma 2.1, and Theorems 2.2 and 2.3 remain unchanged, if q is replaced by 0. For the values of $\mathcal{O}$ on the generators we obtain $\mathcal{O}(\gamma^{(*)}) = -\gamma^{(*)}$, $\mathcal{O}(\alpha) = -\alpha$, and, of course, $\mathcal{O}(\alpha^*) = \mathbf{1}$. This means that in this case $\mathcal{O}$ is a mapping onto $\mathcal{A}_0$. This corresponds to the fact that the values of any cocycle η on the generators can be calculated from its value on β^* by direct use of Relations (1.6), i.e. without arguments like injectivity or invertibility of operators. Therefore, Lemma 2.6 remains true, too. Finally, we obtain the analgue of Theorem 2.8.

Now we come to the representations of $\mathcal{A}_0$. As in Chapter 3 we introduce (for any representation) the operators E_k with q replaced by 0. This yields

$$E_k \;=\; {\alpha^*}^k\alpha^k - {\alpha^*}^{k+1}\alpha^{k+1} \;=\; {\alpha^*}^k\gamma^*\gamma\alpha^k.$$

From the relations it is immediate that the E_k form again a set of orthogonal projections, and the commutation relations with the generators do not change. Therefore, the subspace onto which $\sum_{k=0}^{\infty} E_k$ projects is an invariant subspace which we denote by H_2. Since $\gamma E_k = \gamma\delta_{0k}$, we obtain that γ is mapped to 0 on the orthogonal complement of H_2 which we denote by H_1. By the same arguments as used in Chapter 3 (actually the arguments are simpler because the formulae simplify) we obtain again Theorem 3.4, where now ρ_0 denotes the irreducible representation ρ_0 extended to $q = 0$. Also the rest of the discussion in the last two sections of Chapter 3 does not change if q is replaced by 0.

Investigating $\rho_0 \star \rho_0$ the main problem lies in proving that $\rho_0 \star \rho_0$ is a representation of type π_2. However, we have

$$\rho_0 \star \rho_0(\alpha)(e_k \otimes e_\ell) \;=\; e_{k-1} \otimes e_{\ell-1}.$$

We immediately see that there cannot be a vector $f \in H$ such that

$$\|\rho_0 \star \rho_0(\alpha^k) f\| \;=\; \|f\|$$

for all $k \in \mathbf{N}_0$. In other words, the invariant subspace H_1 is the nullspace. The further steps become simpler and we obtain again that

$$\rho_0 \star \rho_0 \;\asymp\; \pi_{U_0}$$

is a C^*–isomorphism.

In the remaining part of Chapter 4, q appears only in some normalization factors which become simply 1 if $q = 0$. The proofs of all results contain only statements which have been proved to remain valid for $q = 0$ in this section. As we have seen the case $q = 0$ is in many respects simpler than the general case. In the next section we will see that all of the completions, considered in Section 4.4, coincide for different q and, henceforth, our results can be extended to $q \in (-1, 1)$.

5.2 The q–dependence of $SU_q(2)$

In [35] Woronowicz showed that for all $q \in (-1, 1)$ the C^*–completions of $\mathcal{A}_q$ are isomorphic. We had this in mind when we denoted this C^*–algebra by $\mathcal{A}$. In this section we will give another proof of Woronowicz's result by establishing isometric embeddings of $\mathcal{A}_q$ into $\mathcal{A}$ explicitly. If we want to distinguish between different values q, q', we indicate this by adding $'$ to those symbols connected with q'. E.g. $\mathcal{A}$ and $\mathcal{A}'$ denote the completions $\mathcal{A}_q$ and $\mathcal{A}_{q'}$, respectively. We will see that most of our results are independent of $q \in (-1, 1)$.

The operators P_k, E_k are identified via π_{U_0} with elements of $\mathcal{A}$. Our aim is to find for any $q' \in (-1, 1)$ pairs of elements $\alpha_{q'}, \gamma_{q'}$ of $\mathcal{A}$ which satisfy the q'–relations and span a dense subalgebra of $\mathcal{A}$. (To clarify notation: a pair of elements of $\mathcal{A}'$ fulfilling the q–relations will be denoted by α'_q, γ'_q.) We obtain a first hint how to proceed by observing that according to our representation theory the series

$$\sum_{k=1}^{\infty} \sqrt{\frac{1-q'^{2k}}{1-q^{2k}}}\, \pi_{U_0}(\alpha E_k)$$

$$= \lim_{\ell \to \infty} \left(\sum_{k=1}^{\ell} \left(\sqrt{\frac{1-q'^{2k}}{1-q^{2k}}} - \sqrt{\frac{1-q'^{2(k-1)}}{1-q^{2(k-1)}}} \right) \pi_{U_0}(\alpha P_k) - \sqrt{\frac{1-q'^{2\ell}}{1-q^{2\ell}}}\, \pi_{U_0}(\alpha P_{\ell+1}) \right) \tag{5.1}$$

where we set $\frac{1-q'^0}{1-q^0} = 0$ converges strongly to the operator $\pi'_{U_0}(\alpha')$. Since the convergence is only strong, we cannot be sure that this operator is an element of $\pi_{U_0}(\mathcal{A})$. However, we see that the last term under the limit converges strongly to 0, hence, can be neglected. In the sequel the remaining series will indeed show to converge in norm. Similarly, we see that for q different from 0 the series

$$\sum_{k=0}^{\infty} \left(\frac{q'}{q} \right)^k \pi_{U_0}(\gamma E_k) \tag{5.2}$$

converges to $\pi'_{U_0}(\gamma')$ in norm. In other words, in terms of the representation π_{U_0} we obtain an isometric embedding of $\mathcal{A}_{q'}$ into $\mathcal{A}$ whose closure is onto $\mathcal{A}$. We want to prove this more directly on a purely algebraic level without reference to the general representation theory. We only mention that any homomorphism $\mathcal{A}_{q'} \to \mathcal{A}$ is continuous due to the fact that any C^*–algebra has an isomorphic representation as an operator algebra and the definition of the norm on $\mathcal{A}_{q'}$.

It is not difficult to see that the coefficients appearing in (5.1) can be estimated by

$$\left|\sqrt{\frac{1-q'^{2(k+1)}}{1-q^{2(k+1)}}} - \sqrt{\frac{1-q'^{2k}}{1-q^{2k}}}\right| \quad \leq \quad \max(|q|,|q'|)^{2k} C_{qq'}$$

for k sufficiently large, where $C_{qq'}$ is a positive constant depending on q and q'. Therefore,

$$P'_\alpha \quad = \quad \sum_{k=0}^{\infty}\left(\sqrt{\frac{1-q'^{2(k+1)}}{1-q^{2(k+1)}}} - \sqrt{\frac{1-q'^{2k}}{1-q^{2k}}}\right) P_k$$

converges in norm to an element of $\mathcal{A}$. Furthermore, we have for $q \neq 0$

$$\begin{aligned}\left(\frac{q'}{q}\right)^k \gamma E_k \quad &= \quad \left(\frac{q'}{q}\right)^k \gamma \alpha^{*k} E_0 (\mathbf{1} - q^2\gamma^*\gamma)^{-1} \cdots (\mathbf{1} - q^{2k}\gamma^*\gamma)^{-1}\alpha^k \\ &= \quad q'^k \alpha^{*k}\gamma E_0 (\mathbf{1} - q^2\gamma^*\gamma)^{-1} \cdots (\mathbf{1} - q^{2k}\gamma^*\gamma)^{-1}\alpha^k \\ &= \quad q'^k \frac{\alpha^{*k}\gamma\alpha^k}{(1-q^2)\cdots(1-q^{2k})} E_k.\end{aligned}$$

In this form the series (5.2) is norm convergent even for $q = 0$. Thus, we can define

$$\begin{aligned}\alpha_{q'} \quad &= \quad P'_\alpha \alpha \quad = \quad \sum_{k=1}^{\infty}\left(\sqrt{\frac{1-q'^{2k}}{1-q^{2k}}} - \sqrt{\frac{1-q'^{2(k-1)}}{1-q^{2(k-1)}}}\right)\alpha P_k \\ \gamma_{q'} \quad &= \quad \sum_{k=0}^{\infty} q'^k \frac{\alpha^{*k}\gamma\alpha^k}{(1-q^2)\cdots(1-q^{2k})} E_k.\end{aligned}$$

Proposition 5.4 *The elements $\alpha_{q'}, \gamma_{q'}$ of $\mathcal{A}$ fulfill the q'–relations. We have $\alpha_q = \alpha$, and $\gamma_q = \gamma$.*

PROOF First we show the latter statement. In the expression for α_q only the summand for $k = 1$ does not vanish and, indeed, gives α. If $q \neq 0$ then $\gamma_{q'}$ is given by the simpler expression

$$\gamma_{q'} \quad = \quad \sum_{k=0}^{\infty}\left(\frac{q'}{q}\right)^k \gamma E_k.$$

Setting $q' = q$ we obtain

$$\begin{aligned}\gamma_q \quad &= \quad \sum_{k=0}^{\infty}\gamma E_k \\ &= \quad \lim_{k\to\infty}\gamma(\mathbf{1} - P_{k+1}) \quad = \quad \gamma,\end{aligned}$$

because $\|\gamma P_{k+1}\| \leq q^{k+1}$. If $q = q' = 0$ only the summand for $k = 0$ is different from 0 and gives γ.

Notice that we have, in addition to the orthogonality relations $E_k E_\ell = E_k \delta_{k\ell}$, the following properties

$$\begin{aligned}E_k P_\ell \quad &= \quad \begin{cases} E_k & \text{for } k \geq \ell \\ 0 & \text{otherwise}\end{cases} \\ P_k P_\ell \quad &= \quad P_k \qquad \text{for } k \geq \ell.\end{aligned}$$

They follow immediately from $P_k = \mathbf{1} - E_0 - \ldots - E_{k-1}$. The first relation which is obviously fulfilled is Relation (c). We easily obtain

$$\gamma^*_{q'}\gamma_{q'} \quad = \quad \gamma_{q'}\gamma^*_{q'} \quad = \quad \sum_{k=0}^{\infty} q'^{2k} E_k.$$

Let us write $\alpha_{q'}$ in a different way.

$$\begin{aligned}\alpha_{q'} &= \sum_{k=1}^{\infty}\left(\sqrt{\frac{1-q'^{2k}}{1-q^{2k}}}-\sqrt{\frac{1-q'^{2(k-1)}}{1-q^{2(k-1)}}}\right)\alpha P_k \\ &= \lim_{\ell\to\infty}\left(\sum_{k=1}^{\ell}\sqrt{\frac{1-q'^{2k}}{1-q^{2k}}}\alpha E_k \;+\; \sqrt{\frac{1-q'^{2\ell}}{1-q^{2\ell}}}\alpha P_{\ell+1}\right) \\ &= \lim_{\ell\to\infty}\left(\sum_{k=1}^{\ell}\sqrt{\frac{1-q'^{2k}}{1-q^{2k}}}\alpha E_k \;+\; \alpha P_{\ell+1}\right).\end{aligned} \tag{5.3}$$

(Since P_ℓ is a bounded sequence, the ℓ–limit of $\frac{1-q'^{2\ell}}{1-q^{2\ell}} \to 1$ can be performed first.) Now we want to calculate $\alpha^*_{q'}\alpha_{q'}$ and $\alpha_{q'}\alpha^*_{q'}$. Both factors are given as norm limits in ℓ. We perform these ℓ–limits simultaneously in both factors and obtain by repeated use of the relations (fulfilled by the projections E_k and P_k) which are written above and in Chapter 3

$$\begin{aligned}\alpha^*_{q'}\alpha_{q'} &= \lim_{\ell\to\infty}\left(\sum_{k=1}^{\ell}\frac{1-q'^{2k}}{1-q^{2k}}\alpha^*\alpha E_k \;+\; \alpha^*\alpha P_{\ell+1}\right) \\ \alpha_{q'}\alpha^*_{q'} &= \lim_{\ell\to\infty}\left(\sum_{k=0}^{\ell}\frac{1-q'^{2(k+1)}}{1-q^{2(k+1)}}\alpha\alpha^* E_k \;+\; \alpha\alpha^* P_{\ell+1}\right).\end{aligned}$$

We insert $P_{\ell+1} = \mathbf{1} - \sum_{k=0}^{\ell} E_k$ and obtain, taking into account that $\alpha^*\alpha E_k = (1-q^{2k})E_k$,

$$\begin{aligned}\alpha^*_{q'}\alpha_{q'} &= \alpha^*\alpha + \sum_{k=1}^{\infty}(q^{2k}-q'^{2k})E_k \;=\; \alpha^*\alpha + \gamma^*\gamma - \gamma^*_{q'}\gamma_{q'} \\ &= \mathbf{1} - \gamma^*_{q'}\gamma_{q'},\end{aligned}$$

and similarly with $\alpha\alpha^* E_k = (1-q^{2(k+1)})E_k$

$$\alpha_{q'}\alpha^*_{q'} \;=\; \mathbf{1} - q'^2\gamma^*_{q'}\gamma_{q'}.$$

This yields Relations (d) and (e).

$\gamma_{q'}$ was given as the sum over $g_k = q'^k \frac{\alpha^{*k}\gamma\alpha^k}{(1-q^2)\cdots(1-q^{2k})}E_k$. For $q \neq 0$ this simplifies to $g_k = \left(\frac{q'}{q}\right)^k \gamma E_k$. If $q=0$ we obtain $g_k = q'^k\alpha^{*k}\gamma\alpha^k$. In both cases we have

$$\alpha_{q'}g^{(*)}_{k+1} \;=\; q' g^{(*)}_k \alpha_{q'} \quad\text{and}\quad \alpha_{q'}g^{(*)}_0 \;=\; 0.$$

Therefore, we obtain Relations (a) and (b). ■

Corollary 5.5 *The mapping*

$$\begin{aligned}\alpha' &\longmapsto \alpha_{q'} \\ \gamma' &\longmapsto \gamma_{q'}\end{aligned}$$

can be extended (uniquely) to a homomorphism $I_{q'} : \mathcal{A}' \to \mathcal{A}$.

In the opposite direction we write I'_q. Denote by $E_{kq'}$ the projections as defined in Chapter 3, calculated in terms of $\alpha_{q'}$ and $\gamma_{q'}$. In other words, we have

$$E_{kq'} \;=\; I_{q'}(E'_k),$$

where E'_k denotes the projections in $\mathcal{A}'$.

Proposition 5.6 *We have*

$$E_{kq'} \;=\; E_k.$$

Proof From

$$E_k \;=\; \frac{\alpha^{*k} E_0 \alpha^k}{(1-q^2)\cdots(1-q^{2k})}$$

we see that the E_k fulfill the following recursion formula

$$E_{k+1} \;=\; \frac{\alpha^* E_k \alpha}{1-q^{2(k+1)}}.$$

We have

$$E_{0q'} \;=\; \lim_{\ell\to\infty}(\gamma_{q'}^*\gamma_{q'})^\ell \;=\; \lim_{\ell\to\infty}\sum_{k=0}^{\infty} q'^{2k\ell} E_k \;=\; E_0.$$

Now assume that the statement is true for k, i.e. $E_{kq'} = E_k$. We obtain

$$E_{(k+1)q'} \;=\; \frac{\alpha_{q'}^* E_{kq'} \alpha_{q'}}{1-q'^{2(k+1)}} \;=\; \sqrt{\frac{1-q'^{2(k+1)}}{1-q^{2(k+1)}}}\,\frac{\alpha^* E_k \alpha}{1-q'^{2(k+1)}}\sqrt{\frac{1-q'^{2(k+1)}}{1-q^{2(k+1)}}} \;=\; E_{k+1},$$

where we made use of $E_k\alpha_{q'} = \sqrt{\frac{1-q'^{2(k+1)}}{1-q^{2(k+1)}}}E_k\alpha$. ■

Corollary 5.7 *We have*

$$I_{q'}(\alpha'_q) \;=\; \alpha$$
$$I_{q'}(\gamma'_q) \;=\; \gamma.$$

Proof According to (5.3), α'_q is given by

$$\alpha'_q \;=\; \lim_{\ell\to\infty}\left(\sum_{k=1}^{\ell}\sqrt{\frac{1-q^{2k}}{1-q'^{2k}}}\alpha' E'_k \;+\; \alpha' P'_{\ell+1}\right).$$

$I_{q'}$ maps this to

$$\begin{aligned} I_{q'}(\alpha'_q) \;&=\; \lim_{\ell\to\infty}\left(\sum_{k=1}^{\ell}\sqrt{\frac{1-q^{2k}}{1-q'^{2k}}}\alpha_{q'} E_k \;+\; \alpha_{q'} P_{\ell+1}\right) \\ &=\; \lim_{\ell\to\infty}\left(\sum_{k=1}^{\ell}\alpha E_k \;+\; \alpha P_{\ell+1} \;+\; (\alpha_{q'}-\alpha)P_{\ell+1}\right) \;=\; \alpha. \end{aligned}$$

This is true because we can write, using (5.3), $(\alpha_{q'} - \alpha) = \sum_{k=1}^{\infty}(\sqrt{\frac{1-q'^{2k}}{1-q^{2k}}} - 1)\alpha E_k$ and, henceforth,

$$(\alpha_{q'} - \alpha)P_{\ell+1} = \sum_{k=\ell+1}^{\infty}(\sqrt{\frac{1-q'^{2k}}{1-q^{2k}}} - 1)\alpha E_k \to 0.$$

For γ'_q we obtain in a similar manner

$$I_{q'}(\gamma'_q) \;=\; \sum_{k=0}^{\infty}\left(\frac{q}{q'}\right)^k \gamma_{q'} E_k \;=\; \gamma$$

for $q' \neq 0$, and

$$
\begin{aligned}
I_{q'}(\gamma'_q) &= \sum_{k=0}^{\infty} q^k {\alpha^*_{q'}}^k \gamma_{q'} {\alpha_{q'}}^k = \sum_{k=0}^{\infty} q^k {\alpha^*_{q'}}^k \gamma E_0 {\alpha_{q'}}^k \\
&= \sum_{k=0}^{\infty} q^k \sqrt{\frac{1}{(1-q^2)\cdots(1-q^{2k})}} \alpha^{*k} \gamma \alpha^k \sqrt{\frac{1}{(1-q^2)\cdots(1-q^{2k})}} E_k = \gamma
\end{aligned}
$$

for $q' = 0$. ∎

Now the following theorem is a simple corollary.

Theorem 5.8 *The mapping $I_{q'}$ is an isomorphism from $\mathcal{A}'$ onto $\mathcal{A}$ and I'_q is its inverse.*

PROOF Consider the endomorphism $I'_q \circ I_{q'}$ of $\mathcal{A}'$. $I_{q'}$ mapps α', γ' to $\alpha_{q'}, \gamma_{q'}$. On the other hand, changing the role of q and q' in the foregoing corollary, we see that $\alpha_{q'}, \gamma_{q'}$ are mapped back to α', γ' by I'_q. Therefore, the restriction of $I'_q \circ I_{q'}$ to $\mathcal{A}_{q'}$ is the identity. Clearly, this extends to $\mathcal{A}'$. By changing q and q', we see that $I_{q'} \circ I'_q$ is the identity on $\mathcal{A}$. Since $\|I(\bullet)\| \leq \|\bullet\|$ for any homomorphism I between C^*–algebras, $I_{q'}$ and I'_q must be isomorphisms. ∎

Notice that obviously $\rho'_U = \rho_U \circ I_{q'}$ and $\pi'_U = \pi_U \circ I_{q'}$ for any unitary operator U (actually $I_{q'}$ was constructed such that the second condition is fulfilled). Therefore, even the strong and weak topologies on $\mathcal{A}$ and $\mathcal{A}'$ coincide. In the sequel we identify $\mathcal{A}'$ with $\mathcal{A}$, i.e. $E'_0 = E_0$, $\alpha_{q'} = \alpha'$, $\alpha'_q = \alpha$, and similarly for γ.

Now we investigate wether the completions $\mathcal{A}_{\mathcal{O}}$, $\mathcal{A}_{\mathcal{T}}$, $K_{\mathcal{O}}$, and $K_{\mathcal{T}}$ depend on q, and to which extend the cocycles and the conditionally positive functionals change with q. (Since the representations do not change, the latter question is equivalent to the question what happens to the mappings $\mathcal{O}$ and $\mathcal{T}$.) The key for an answer to these questions lies in the following

Lemma 5.9 *There is a (unique) invertible element $B' \in \mathcal{A}$ (depending on q and q'), satisfying*

$$B'\beta^* = \beta'^*.$$

PROOF The statement of the lemma is equivalent to the statement that β'^* is an element of $K_{\mathcal{O}}$. In this case B' is given by

$$B' = \mathcal{O}(\beta'^*),$$

and, by symmetry in q and q', its inverse must be given by

$$B'^{-1} = \mathcal{O}'(\beta^*).$$

Notice that $\beta'^* - \beta^* = \alpha'^* - \alpha^* = \alpha^*(P'_\alpha - \mathbf{1})$. Since β^* is an element of $K_{\mathcal{O}}$ and $K_{\mathcal{O}}$ is a left ideal in $\mathcal{A}$, it is sufficient to show that $P'_\alpha - \mathbf{1} \in K_{\mathcal{O}}$. We obtain by calculations similar to (5.3)

$$
\begin{aligned}
P'_\alpha - \mathbf{1} &= \lim_{\ell \to \infty} \left(\sum_{k=0}^{\ell} \sqrt{\frac{1-q'^{2(k+1)}}{1-q^{2(k+1)}}} E_k + P_{\ell+1} \right) - \mathbf{1} \\
&= \sum_{k=0}^{\infty} \left(\sqrt{\frac{1-q'^{2(k+1)}}{1-q^{2(k+1)}}} - 1 \right) E_k.
\end{aligned}
$$

We have $\|\mathcal{O}(E_k)\| \leq (k+1)$, the factors $\sqrt{\frac{1-q'^{2(k+1)}}{1-q^{2(k+1)}}} - 1$ can again be estimated from above by a multiple of $\max(|q|, |q'|)^{2k}$, and $E_k \in K_{\mathcal{O}}$. Therefore, the series converges in $\mathcal{O}$–norm to an element of $K_{\mathcal{O}}$. ∎

We mention that $P'_\alpha - \mathbf{1}$ is an element even of $K_\mathcal{T}$. This can be seen similarly by the estimate $\|\mathcal{T}(E_k)\| \leq (k+1)^2$. Therefore, $\frac{\alpha-\alpha^*}{2i}$ and $\frac{\alpha'-\alpha'^*}{2i}$ differ only by an element of $K_\mathcal{T}$, i.e.

$$\mathbf{C}\frac{\alpha'-\alpha'^*}{2i} \oplus K_\mathcal{T} \quad = \quad \mathbf{C}\frac{\alpha-\alpha^*}{2i} \oplus K_\mathcal{T}$$

(as subsets of $\mathcal{A}$). Furthermore, we have $\mathcal{A}\beta'^* = \mathcal{A}B'\beta^* = \mathcal{A}\beta^*$ and similarly for $\beta'\mathcal{A}\beta'^*$. Notice that $\mathcal{O}'$ can be uniqely characterized by $\mathcal{O}'(\bullet)\beta'^* = Id_{K_\mathcal{O}}(\bullet) = \mathcal{O}(\bullet)\beta^*$, and similarly for $\mathcal{T}'$. Thus, we obtain

Theorem 5.10 *We have*

$$\begin{aligned} K_{\mathcal{O}'} &= K_\mathcal{O} \\ K_{\mathcal{T}'} &= K_\mathcal{T} \\ \mathcal{A}_{\mathcal{O}'} &= \mathcal{A}_\mathcal{O} \\ \mathcal{A}_{\mathcal{T}'} &= \mathcal{A}_\mathcal{T}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{O}'B' &= \mathcal{O} \\ B'^*\mathcal{T}'B' &= \mathcal{T} \text{ on } K_\mathcal{T} \\ \|\bullet\|_{\mathcal{O}'} &\sim \|\bullet\|_\mathcal{O} \\ \|\bullet\|_{\mathcal{T}'} &\sim \|\bullet\|_\mathcal{T}. \end{aligned}$$

In other words, the extension of any conditionally positive functional ψ on $\mathcal{A}_q$ to $\mathcal{A}_\mathcal{T}$ coincides with the extension of a conditionally positive functional ψ' on $\mathcal{A}_{q'}$. Clearly, the correspondence between ψ and ψ' is one-to-one, and the sets containing the extensions of all conditionally positive fuctionals ψ and ψ', respectively, both coincide with the set of all conditionally positive functionals on $\mathcal{A}_\mathcal{T}$. The same holds for the extensions of cocycles from $\mathcal{A}_q$ and $\mathcal{A}_{q'}$, respectively, to $\mathcal{A}_\mathcal{O}$.

Now we investigate how the classifications for q and q' are related to each other. First notice that δ does not depend on q, and, consequently, so does the projection $Id - \delta\mathbf{1}$. Unfortunately, it is not possible to find an element $a_\alpha \in K_\mathcal{O}$ such that $\delta'(a_\alpha) \neq 0$ and $a_\alpha \in \bigcap_{q\in(-1,1)} \mathcal{A}_q$. This means that we are not able to define a projection from $\mathcal{A}_q$ onto K_2 in a q–independent way. However, in all our formulae for conditionally positive functionals the projection $\mathcal{P}$ appears. Therefore, whatever we choose for $\pi, \eta_{\alpha^*}, \pi', \eta_{\alpha'^*}$, the functionals $\langle\eta_{\alpha^*}|\pi|\eta_{\alpha^*}\rangle \circ \mathcal{T}$ and $\langle\eta_{\alpha'^*}|\pi'|\eta_{\alpha'^*}\rangle \circ \mathcal{T}'$ *cannot* coincide on $\mathcal{A}_\mathcal{T}$, even, if they coincide on $K_\mathcal{T}$, unless both vanish on $(\alpha-\alpha') - (\alpha^* - \alpha'^*)$ which is in general not the case. One way to avoid this q–dependence lies in the restriction to $\mathcal{K}_\mathcal{T}$. The projection simply dissapears in the formulae. Another possibility is to fix the projection e.g. by choosing $\mathcal{P}_0$ where q is replaced by 0. The classification in Theorem 2.8 remains unchanged with the exception that the parameter r_2 is shifted by a (q–dependend) constant c. We only have to keep in mind that in general $\mathcal{P}_0$ does not map $\mathcal{A}_q$ to K_2 but to $K_2 + \mathbf{C}\frac{(\alpha-\alpha_0)-(\alpha^*-\alpha_0^*)}{2i} \subset K_\mathcal{T}$.

Now the following corollary is obvious.

Corollary 5.11 *Let π be a representation of $\mathcal{A}$ and $\eta_{\alpha^*}, \eta_{\alpha'^*}$ vectors in the representation space H. We have*

$$\begin{aligned} (\pi\eta_{\alpha^*})\circ\mathcal{O} &= (\pi\eta_{\alpha'^*})\circ\mathcal{O}' && \text{on } \mathcal{A}_\mathcal{O} \\ \langle\eta_{\alpha^*}|\pi|\eta_{\alpha^*}\rangle\circ\mathcal{T} &= \langle\eta_{\alpha'^*}|\pi|\eta_{\alpha'^*}\rangle\circ\mathcal{T}' && \text{on } K_\mathcal{T} \end{aligned}$$

if and only if

$$\eta_{\alpha'^*} \quad = \quad \pi(B')\eta_{\alpha^*}.$$

So far we see that the classification by Theorem 2.8 depends on the choice of q. However, it follows from the invertibility of $\pi(B')$ that all the spaces H_{β^*} coincide as sets whatever q is, because their scalar products are equivalent. We denote this space by H^* and its dual by H_*. These two spaces are a q–independend pair of dual topological vector spaces whose topologies can be described by scalar products. Consequently $\pi_{\beta\beta^*}$ becomes a q–independent mapping $\pi_* \in \mathcal{L}_0(H^*, H_*)$, the set of all continuous linear operators from H^* to H_*. Clearly, the topology on $\mathcal{L}_0(H^*, H_*)$ coincides for any $q \in (-1, 1)$ with the topology induced by the norm on $\mathcal{B}(H_{\beta^*}, H_\beta)$. If no confusion about the domain and range of π can arise, we ommit the various subscripts. We immediately obtain a final, q–independenent formulation of the LÉVY-KHINTCHINE formula in Theorem 4.16.

Theorem 5.12 *Any conditionally positive functional ψ on $\mathcal{A}_\mathcal{T}$ can be written in the form*

$$\psi \;=\; \psi_\delta + \langle \eta_1 | \pi \circ \mathcal{P}_0 | \eta_1 \rangle,$$

where π is a representation without GAUSS*ian part, $\eta_1 \in H^*$, and ψ_δ is a* GAUSS*ian part according to Corollary 2.11. The correspondence between conditionally positive functionals and triples $(\pi, \eta_1, \psi_\delta)$ is one-to-one up to unitary equivalence. The restrictions of such functionals to $\mathcal{A}_q$ form the set of all conditionally positive functionals on $\mathcal{A}_q$.*

5.3 Remarks

Let us summarize the results. $(\mathcal{A}, \delta)$ is a pair consisting of a $*$–algebra and a homomorphism, and we can investigate its conditionally positive functionals. However, by Section 4.3 we know that these are given (more or less) by multiples of states. This is because the topology on $\mathcal{A}$ is too weak and, henceforth, $\mathcal{A}$ is too big. The $*$–algebra $\mathcal{A}^0$ generated by $\bigcup_{q\in(-1,1)} \mathcal{A}_q$ is a dense subalgebra. On the one hand, $\mathcal{A}^0$ does not depend on q. On the other hand, $\mathcal{A}^0$ is the smallest subalgebra of $\mathcal{A}$, containing all the $\mathcal{A}_q$. (In other words, if it is possible to describe the conditionally positive functionals on $\mathcal{A}_q$ in a q–independent manner, it must be possible to extend such functionals at least to $\mathcal{A}^0$.) The set of all conditionally positive functionals on $\mathcal{A}^0$ induces a topology on $\mathcal{A}^0$ which is equivalent to the topology given by any of the norms $\| \bullet \|_\mathcal{T}$ and, henceforth, stronger than the topology on $\mathcal{A}$. The closure with respect to the new toplogy yields $\mathcal{A}_\mathcal{T}$.

Now the question arises *where* the q–dependence of $SU_q(2)$ actually lies. The answer is: in the comultiplication. We know e.g. that a the representation ρ_0 does not depend on q. However, $\rho_0 \star_q \rho_0$ *does* depend on q although ρ_0 does not. On the other hand, we know that $\rho_0 \star_q \rho_0$ is unitarily equivalent to π_{U_0} for any q. This implies that all results which are only up to unitary equivalence are essentially independend of q.

EXAMPLE 5.1 *In [35] the* HAAR *measure on $\mathcal{A}_q$ was given as*

$$h \;=\; \frac{1}{1-q^2} \sum_{k=0}^{\infty} q^{2k} \langle e_k \otimes \widetilde{e}_0 | \pi_{U_0} | e_k \otimes \widetilde{e}_0 \rangle.$$

It is not difficult to see (cf. also [18]) that this can be written as

$$h \;=\; \langle \eta_q | \pi_{U_0 \otimes U_0} | \eta_q \rangle,$$

where η_q is cyclic for $q \neq 0$ and given by

$$\eta_q \;=\; \frac{1}{\sqrt{1-q^2}} \sum_{k=0}^{\infty} q^k e_k \otimes \widetilde{e}_k \otimes \widetilde{e}_0.$$

(Notice that the GNS*–representation $\pi_{U_0 \otimes U_0}$ is unitarily equivalent to $\rho_0 \star_q \rho_0 \star_q \rho_0$ as can be seen by simple calculations, using our rules for convoluting representations.) Since h depends on q essentially, we can be sure that there is no unitary equivalence transform on $h_0 \otimes \mathcal{H}_0 \otimes \mathcal{H}_0$, mapping η_q to $\eta_{q'}$ and leaving $\pi_{U_0 \otimes U_0}$ invariant.*

We should remind the reader of the fact that the comultiplication is a mapping into the *algebraic* tensor product of a linear space with itself. However, it is not difficult to see that if $q \neq q'$.

$$\Delta_q(\mathcal{A}_{q'}) \;\not\subset\; \mathcal{A} \underline{\otimes} \mathcal{A}.$$

Actually, there seems to be no subalgebra of $\mathcal{A}$ bigger than $\mathcal{A}_q$, which still is a bialgebra with comultiplication Δ_q.

5.4 The case $q = 1$

In this section we give, to some extend, an introduction to the classical theory of conditionally positive functionals (i.e. infinitesimal generators of stochastic processes) on a *compact* Lie *group* (cf. [11, 13]) in the case of $SU(2)$ (i.e. $\mathcal{A}_1$). We recover these results, using methods motivated by the techniques of Chapter 2.

An element of $U \in SU(2)$ is given by a unitary matrix $U = (u_{ij})_{i,j=1,2}$ with unit determinant. Consider the unital $*$–algebra $\mathcal{A}_\mathrm{f}$, which is generated by the coefficient functions $f_{ij} : U \mapsto u_{ij}$ on $SU(2)$. In the usual parametrization of $SU(2)$ which is non-singular at the identity, we have

$$(f_{ij}(\varphi, x, y))_{ij} = \begin{pmatrix} \sqrt{1-x^2-y^2}e^{i\varphi} & -(x-iy) \\ x+iy & \sqrt{1-x^2-y^2}e^{-i\varphi} \end{pmatrix},$$

with $\varphi \in [-\pi, \pi)$ and $x^2 + y^2 \le 1$. On $\mathcal{A}_\mathrm{f}$ we have a natural Hopf $*$–algebra structure given by

$$\begin{aligned} \Delta_\mathrm{f}(f)(U,V) &= f(UV) \\ \delta_\mathrm{f}(f) &= f(I) \\ \mathcal{S}_\mathrm{f}(f)(U) &= f(U^{-1}), \end{aligned}$$

for $U, V \in SU(2)$ and I being the identity of $SU(2)$. Obviously $(f_{ij})_{i,j=1,2}$ is a unitary corepresentation of $\mathcal{A}_\mathrm{f}$. It is easy to see that the irreducible representations of $\mathcal{A}_\mathrm{f}$ are given by $\rho_U(f) = f(U)$, where U can be any point in $SU(2)$, i.e. they are in one-to-one correspondence with the group elements. On the other hand, the convolution of two irreducible representations ϱ_U, ϱ_V, associated with elements U, V of $SU(2)$, yields another irreducible representation ϱ_{UV},

$$\begin{aligned} \varrho_{UV}(f_{ij}) &= (\varrho_U \otimes \varrho_V) \circ \Delta_\mathrm{f}(f_{ij}) = (\varrho_U \otimes \varrho_V)(\sum_{k=1}^{2} f_{ik} \otimes f_{kj}) \\ &= f_{ij}(UV), \end{aligned}$$

associated with UV. In other words, the convolution of irreducible representations gives us nothing but the group structure of $SU(2)$. (Obviously, we have $\varrho_{U^{-1}} = \varrho_U \circ \mathcal{S}_\mathrm{f}$.) We introduce the usual supremum norm on $\mathcal{A}_\mathrm{f}$ by

$$\|f\| = \sup_{U \in SU(2)} |f(U)|.$$

By an application of Stone-Weierstrass theorem $\mathcal{A}_\mathrm{f}$ is dense in $C(SU(2))$, the $*$–algebra of continuous functions on $SU(2)$, which, therefore, is the completion of $\mathcal{A}_\mathrm{f}$.

The generators f_{ij} of $\mathcal{A}_\mathrm{f}$ satisfy $f_{11} = f_{22}^*$, $f_{12} = -f_{21}^*$, and $f_{11}^* f_{11} + f_{21}^* f_{21} = \mathbf{1}$. Thus, by

$$\begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \longmapsto \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

we define a $*$–algebra homomorphism from $\mathcal{A}_1$ onto $\mathcal{A}_\mathrm{f}$. It is easily checked that the irreducible representations of $\mathcal{A}_1$ are given by the three-parameter family $\delta_{\varphi xy}$, with

$$\delta_{\varphi xy}\begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \varrho_U \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{1-x^2-y^2}e^{i\varphi} & -(x-iy) \\ x+iy & \sqrt{1-x^2-y^2}e^{-i\varphi} \end{pmatrix},$$

where φ, x, y describes $U \in SU(2)$. The general representation π_E of $\mathcal{A}_1$ is given by

$$\pi_E = \int_{SU(2)} \delta_{\varphi xy} \, dE_{\varphi xy},$$

where dE is an arbitrary spectral measure on $SU(2)$. Notice that $\pi_E(a) = \int_{SU(2)} f(U) \, dE_{\varphi xy}$, if f is the funtion in $\mathcal{A}_\mathrm{f}$ corresponding to $a \in \mathcal{A}_1$. Once again it follows from the proof of Woronowiczs theorem that $\mathcal{A}_1$ has a faithful representation. Therefore, $\|\bullet\| = \sup_{\pi} \|\pi(\bullet)\|$ defines a C^*–norm on $\mathcal{A}_1$. Clearly, this norm coincides with the norm on $\mathcal{A}_{\mathrm{f}}$. Thus, $\mathcal{A}_{\mathrm{f}}$ and $\mathcal{A}_1$ are isometrically isomorphic pre–C^*–algebras, and $C(SU(2))$ can be identified with the C^*–completion of $\mathcal{A}_1$. This result follows also from an immediate application of the results obtained by GLOCKNER and VON WALDENFELS in [10]. Notice that a representation is a C^*–algebra isomorphism, if the spectrum of the spectral measure dE is the whole $SU(2)$. If we e.g. choose the HAAR measure H on $SU(2)$, we obtain a representation π_0 on $L^2(SU(2), H)$. We observe that for any $U \in SU(2)$ the representation $\pi_0 \star \varrho_U$ is unitarily equivalent to π_0 and the unitary equivalence transform is the shift by U.

Now we come to the cocycles and conditionally positive functionals on $\mathcal{A}_1$. One main difference to the case $q \in (-1, 1)$ lies in the GAUSSian parts. Consider the three mappings $\delta'^{\varphi} = \partial_{\varphi}\delta_{000}$, $\delta'^{x} = \partial_x \delta_{000}$, and $\delta'^{y} = \partial_y \delta_{000}$. These mappings are cocycles with respect to $\delta = \delta_{000}$. Since

$$\begin{aligned} \delta'^{\varphi} \begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} &= i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \delta'^{x} \begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ \delta'^{y} \begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} &= i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

we see that the three cocycles are linearly independent. In particular, we see that any linear combination of δ'^{x} and δ'^{y} is a cocycle different from 0 but vanishing on α^*, i.e. Lemma 2.6 is no longer valid. Let $r \in \mathbf{R}^3$ be a vector with components (φ, x, y). Setting

$$\delta'^{r} = \left. \frac{d\delta_{(t\varphi)(tx)(ty)}}{dt} \right|_{t=0}, \tag{5.4}$$

we define a three parameter family of cocycles with respect to δ which consists of all real linear combinations of δ'^{φ}, δ'^{x}, and δ'^{y}. Setting

$$\frac{\delta''^{r}}{2} = \frac{1}{2} \left. \frac{d^2\delta_{(t\varphi)(tx)(ty)}}{dt^2} \right|_{t=0},$$

we obtain a conditionally positive functional fulfilling (1.2).

Now we see by Relations (1.6) that any GAUSSian cocycle, i.e. a cocycle η_δ with respect to a representation of the form $\delta \mathbf{1}_{H_\delta}$ on a HILBERT space H_δ, is defined by its values on $\alpha - \alpha^*$, γ, and γ^* which, on the other hand, can be chosen arbitrarily. In other words, we obtain

Theorem 5.13 *By*

$$\eta_\delta = \delta'^{\varphi}\eta_\varphi + \delta'^{x}\eta_x + \delta'^{y}\eta_y, \tag{5.5}$$

we establish a one-to-one correspondence between GAUSS*ian cocycles η_δ and triplets $(\eta_\varphi, \eta_x, \eta_y)$ of vectors $\eta_\varphi, \eta_x, \eta_y \in H_\delta$.*

Since η_δ must vanish on K_2, we have $\alpha - \alpha^*, \gamma, \gamma^* \notin K_2$ and any basis of K_2 can be extended by $\mathbf{1}, \frac{\alpha-\alpha^*}{2i}, \frac{\gamma+\gamma^*}{2}, \frac{\gamma-\gamma^*}{2i}$ to a basis of $\mathcal{A}_1$. Setting

$$\mathcal{P}_1 = Id - \delta\mathbf{1} - \delta'^{\varphi}\frac{\alpha - \alpha^*}{2i} - \delta'^{x}\frac{\gamma + \gamma^*}{2} - \delta'^{y}\frac{\gamma - \gamma^*}{2i},$$

we obtain a projection onto K_2.

The form (5.5) of the GAUSSian cocycles is not yet suitable to see the form of the GAUSSian conditionally positive functionals. In the following theorem we find a more practicable one. Moreover, it turns out that, similar to our counter Example 2.3, not *all* cocycles determine the values of a conditionally positive functional on K_2.

Theorem 5.14 *For any* GAUSS*ian cocycle* η_δ *which determines the values of a conditionally positive functional on* K_2 *there are three orthogonal vectors* $\eta_1, \eta_2, \eta_3 \in H_\delta$*, having length 1 or 0, and three vectors* $r_1, r_2, r_3 \in \mathbf{R}^3$ *such that*

$$\eta_\delta \quad = \quad \delta'^{r_1}\eta_1 + \delta'^{r_2}\eta_2 + \delta'^{r_3}\eta_3.$$

Choosing an arbitrary real number r_0 *and an arbitrary vector* $r \in \mathbf{R}^3$ *we obtain all conditionally positive functionals* ψ_δ*, fulfilling (1.2), in the form*

$$\psi_\delta \quad = \quad r_0\delta + \delta'^r + \frac{\delta''^{r_1}}{2} + \frac{\delta''^{r_2}}{2} + \frac{\delta''^{r_3}}{2}.$$

PROOF If a cocycle η_δ has the stated form, the form of the conditionally positive functionals follows straightforwardly. Thus, it remains to show that a cocycle which determines the values of a conditionally positive functional on K_2 has to be of this form.

An arbitrary cocycle η_δ given in the form (5.5) maps to the subspace of H_δ spanned by $\eta_\varphi, \eta_x, \eta_y$. This subspace is at most three-dimensional. Therefore, we can find three ortogonal vectors η_1, η_2, η_3 whose span contains the range of η_δ. The components $\langle \eta_i | \eta_\delta \rangle$ of η_δ in the directions of the $\eta_i, i = 1, 2, 3$ themselves are one-dimensional cocycles.

Now assume that η_δ determines the values of a conditionally positive functional on K_2. There is nothing to prove, if all vectors $\eta_\varphi, \eta_x, \eta_y$ are 0. Therefore, we assume, without loss of generality, that $\eta_x \neq 0$ and choose $\eta_1 = \frac{\eta_x}{\|\eta_x\|}$. It is easy to conclude from the commutativity of $\frac{\alpha-\alpha^*}{2i}, \frac{\gamma+\gamma^*}{2}, \frac{\gamma-\gamma^*}{2i}$ that the numbers $\langle \eta_1 | \eta_\varphi \rangle, \langle \eta_1 | \eta_y \rangle$ are real numbers. In other words, we can find $r_1 \in \mathbf{R}^3$, such that the one-dimensional cocycle $\langle \eta_1 | \eta_y \rangle = \delta'^{r_1}$. We know that this cocycle defines a conditionally positive functional. Therefore, also its 'orthogonal complement' defined by $\tilde{\eta}_\delta = \eta_\delta - \eta_1 \delta'^{r_1}$ must define a conditionally positive functional. Furthermore, there are vectors $\tilde{\eta}_\varphi, \tilde{\eta}_y$ orthogonal to η_1, such that $\tilde{\eta}_\delta = \delta'^\varphi \tilde{\eta}_\varphi + \delta'^y \tilde{\eta}_y$. If $\tilde{\eta}_y \neq 0$, we continue our argument in the same manner, by setting $\eta_2 = \frac{\tilde{\eta}_y}{\|\tilde{\eta}_y\|}$. If this is not so, the proof is complete. ■

We see that in the case $q = 1$ there is a much bigger variety in the GAUSSian parts. This is due to the fact that the counit δ can be approached in essentially three ways, corresponding to the three group parameters. The derivatives in the three *directions* of the *parameter space* $SU(2)$ are linearly independent linear functionals, corresponding to the existence of three linearly independent vectors in K_1/K_2 on which in general a GAUSSian cocycle can assume arbitrary values. In the sequel, we will restrict ourselves to representations without GAUSSian part.

Clearly, if the spectral measure of a general representation π has an atom at identity (i.e. the point $(0, 0, 0)$) it decomposes into a subspace H_δ, on which π is given by a multiple of δ, and its orthogonal complement, on which $\pi(\beta^*)$ (and, of course, $\pi(\beta)$) is given by an injective operator. Now we consider the latter case, where $H_\delta = \{0\}$. Such a representation is given by

$$\pi \quad = \quad \int_{SU(2)} \delta_{\varphi xy}\, dE_{\varphi xy} \quad = \quad \lim_{\epsilon \to 0} \int_{SU(2)\setminus U_\epsilon(I)} \delta_{\varphi xy}\, dE_{\varphi xy},$$

where $U_\epsilon(I)$ denotes an ϵ–neighbourhood of the identity and the limit is strong.

Notice that π_0 is of this type. If we choose π_0 to be the representation ω which induces the strong and weak topology, we see by the following proposition that Proposition 2.1 remains valid for $q = 1$.

Proposition 5.15 *For any representation* π *of* $\mathcal{A}_1$*, with* $\pi(\beta^*)$ *injective, we have*

$$\lim_{\substack{p \to 1 \\ p \in [0,1)}} \frac{\mathbf{1} - \pi(\alpha^*)}{\mathbf{1} - p\pi(\alpha^*)} \quad = \quad \mathbf{1}$$

in the strong operator topology.

PROOF Let z be a complex number on the closed unit disk and $p \in (0, 1)$. We have

$$\begin{aligned} \left| 1 - \frac{1-z}{1-pz} \right| \quad &= \quad (1-p)\frac{|z|}{|1-pz|} \\ &\leq \quad (1-p)\frac{|z|}{1-p|z|} \;\leq\; (1-p)\frac{1}{1-p1} \quad = \quad 1. \end{aligned}$$

Therefore, the strong ϵ–limit

$$\mathbf{1} - \frac{\mathbf{1} - \pi(\alpha^*)}{\mathbf{1} - p\pi(\alpha^*)} \quad = \quad \lim_{\epsilon \to 0} \int_{SU(2)\setminus U_\epsilon(I)} \left(1 - \frac{1 - \sqrt{1 - x^2 - y^2}e^{-i\varphi}}{1 - p\sqrt{1 - x^2 - y^2}e^{-i\varphi}} \right) dE_{\varphi xy}$$

is uniform in p. On the other hand, for fixed ϵ the integral becomes small in norm, if p is sufficiently close to 1. Thus, we can conclude that the integral converges to 0 strongly, if p goes to 1. ■

N.B.: If we replace x and y by 0 we obtain the same statement for a representation of type π_1 of $\mathcal{A}_q$ for $q \in (-1, 1)$. Therefore, it is justified to say that any cocycle on $\mathcal{A}_q$, even with respect to a representation of type π_1 without GAUSSian part, can be approximated strongly by coboundaries.

By commutativity we can easily conclude that

Corollary 5.16 *Lemma 2.6 remains valid for $q = 1$ if and only if $\pi(\beta^*)$ is injective.*

If we now try to find a mapping $\mathcal{O}$ we see the other main difference to the case $q \in (-1, 1)$. In the picture of the function algebra $\mathcal{A}_\mathrm{f}$ we should obtain

$$[\mathcal{O}(a)](U) \quad = \quad \lim_{p \to 1} \frac{\delta_{\varphi xy}(a) - \delta(a)}{p\sqrt{1 - x^2 - y^2}e^{-i\varphi} - 1} \quad = \quad \frac{\delta_{\varphi xy}(a) - \delta(a)}{\sqrt{1 - x^2 - y^2}e^{-i\varphi} - 1}.$$

However, by setting $\varphi = 0$ and performing $|x + iy| \to 0$, we see that this function is unbounded around the identity for any non-vanishing linear combination of γ and γ^*. On the other hand, $[\mathcal{O}(\alpha)](U)$ is a continuous function on $SU(2)\setminus\{I\}$. Therefore, by

$$\mathcal{O}_\pi(a) \quad = \quad \int_{SU(2)} [\mathcal{O}(a)](U)\, dE_{\varphi xy}$$

we define a possibly unbounded operator with dense domain. By the cocycle property of $[\mathcal{O}(a)](U)$ we conclude that a maximal common dense domain $\mathcal{D}$ of all the operators $\mathcal{O}_\pi(\mathcal{A}_1)$ is given by

$$\mathcal{D} \quad = \quad \mathcal{D}_\gamma \quad = \quad \mathcal{D}_{\gamma^*},$$

where $\mathcal{D}_{\gamma^{(*)}}$ denote the domains of $\pi(\gamma^{(*)})$. Obviously, $\mathcal{D}$ consists of all vectors η, for which

$$\int_{SU(2)} \left| [\mathcal{O}(\gamma^{(*)})](U) \right|^2 d\langle \eta | E_{\varphi xy} | \eta \rangle \quad = \quad \int_{SU(2)} \frac{x^2 + y^2}{|\sqrt{1 - x^2 - y^2}e^{-i\varphi} - 1|^2} d\langle \eta | E_{\varphi xy} | \eta \rangle \ < \ \infty.$$

Clearly, for any vector $\eta_{\alpha^*} \in \mathcal{D}$ we can define a cocycle η by

$$\eta \quad = \quad \mathcal{O}_\pi \eta_{\alpha^*} \tag{5.6}$$

for which $\eta(\alpha^*) = \eta_{\alpha^*}$ holds. Now we generalize the notion of strong convergence. We say a sequence $\{B_n\}_{n \in \mathbf{N}}$ of bounded operators on H converges strongly to a possibly unbounded operator B on H with domain $\mathcal{D}_B$, if for any $f \in \mathcal{D}_B$ the sequence $B_n f$ converges to Bf. It is not difficult to see, by arguments as in the proof of Proposition 5.15, that

$$\mathcal{O}_\pi \quad = \quad \lim_{p \to 1} \pi \circ (Id - \delta\mathbf{1})(p\pi(\alpha^*) - \mathbf{1})^{-1}$$

pointwise in this strong topology.

On the other hand, if η is a given cocycle assuming the value η_{α^*} on α^*, it follows immediately from Relations (1.6) that

$$\eta(\gamma^{(*)}) \quad = \quad \lim_{\epsilon \to 0} \int_{SU(2)\setminus U_\epsilon(I)} [\mathcal{O}(\gamma^{(*)})](U)\, dE_{\varphi xy} \eta_{\alpha^*},$$

i.e. $\eta_{\alpha^*} \in \mathcal{D}$. Thus, we obtain

Theorem 5.17 *Let π be a representation of $\mathcal{A}_1$ on a* HILBERT *space H with $\pi(\beta^*)$ injective, and $\mathcal{D}$ the (dense) subspace of H as defined above. By (5.6) we establish a one-to-one corresondence between cocycles with respect to π and vectors $\eta_{\alpha^*} \in \mathcal{D}$.*

Proceeding as in Chapter 2, we define

$$\mathcal{T}_\pi(a) \quad = \quad \int_{SU(2)} [\mathcal{T}(a)](U)\, dE_{\varphi xy},$$

where the function $\mathcal{T}(a)$ on $SU(2)\backslash\{I\}$ is given by

$$[\mathcal{T}(a)](U) \quad = \quad \frac{\delta_{\varphi xy} \circ \mathcal{P}_1(a)}{|\sqrt{1-x^2-y^2}e^{-i\varphi} - 1|^2}.$$

Clearly, $\mathcal{T}_\pi$ and $\mathcal{O}_\pi$ fulfill Equation (2.2). Therefore, we obtain again that any cocycle (5.6) defines via (1.2) the values of a conditionally positive functional ψ on K_2. (Notice that the domain of $\mathcal{T}_\pi$ is smaller than $\mathcal{D}$. However, it is obvious that this domain can be extended to $\mathcal{D}$ if we interprete $\mathcal{T}_\pi$ as mapping into $\mathcal{D}^*$ being the dual of $\mathcal{D}$.) By

$$d\mu_{\varphi xy} \quad = \quad \frac{d\langle \eta_{\alpha^*} | E_{\varphi xy} | \eta_{\alpha^*} \rangle}{|\sqrt{1-x^2-y^2}e^{-i\varphi} - 1|^2}$$

we define a positive regular not necessarily finite measure on $SU(2)$. We obtain HUNTs result for $SU(2)$.

Theorem 5.18 *The* LÉVY-KHINTCHINE *formula*

$$\psi \quad = \quad \psi_\delta + \int_{SU(2)} \delta_{\varphi xy} \circ \mathcal{P}_1 \, d\mu_{\varphi xy}$$

establishes a one-to-one correspondence between conditionally positive functionals on $\mathcal{A}_1$, and pairs (ψ_δ, μ) consisting of a GAUSS*ian part ψ_δ and a positive regular measure μ on $SU(2)\backslash I$, fulfilling*

$$\int_{SU(2)} (x^2+y^2)\, d\mu_{\varphi xy} \;<\; \infty \quad \textit{and} \quad \int_{SU(2)} |\sqrt{1-x^2-y^2}e^{-i\varphi} - 1|^2 \, d\mu_{\varphi xy} \;<\; \infty.$$

5.5 The case $q = -1$

Now we investigate the anti-classical case, where $q = -1$. We will obtain a result looking very similar to Theorem 5.18. In addition to the GAUSSian part we find another part, the so-called anti-GAUSSian part, which has to be written down separately. In the integral part of the classical case the term $\delta_{\varphi xy}$ runs over those states which have an irreducible GNS-representation. With some smaller exceptions this result remains true also for the anti-classical case: We obtain that the irreducible representations are more or less given by the family $\hat{\delta}_{\varphi xy}$ of two-dimensional representations. The family of states $\delta_{\varphi xy}$ has to be replaced by the family of states $\operatorname{Tr} \widehat{m}(\varphi, x, y)\hat{\delta}_{\varphi xy}$ where $\widehat{m}(\varphi, x, y)$ is a measurable function on $SU(2)$ with values in the positive 2×2–matrices of unit trace.

First let us agree on some notation. Let π be a representation of $\mathcal{A}_q$. By $\underline{\pi}$ we denote the representation defined by

$$\underline{\pi}(\alpha) \;=\; -\,\pi(\alpha) \quad \text{and} \quad \underline{\pi}(\gamma) \;=\; -\,\pi(\gamma).$$

Notice that in the case of $\mathcal{A}_{\pm 1}$ the roles of α and γ are interchangeable. Therefore, we can define another representation $\bar{\pi}$ by

$$\bar{\pi}(\alpha) \;=\; \pi(\gamma) \quad \text{and} \quad \bar{\pi}(\gamma) \;=\; \pi(\alpha).$$

Since the actions of $\_$ and $^-$ commute, $\underline{\bar{\pi}}$ has also unique meaning.

As we have seen in the foregoing section the classical case was from a conceptual point of view more complicated than the cases when $q \in (-1, 1)$. This was mainly due to the fact that in relations $(\widetilde{\text{a}})$ and $(\widetilde{\text{b}})$ the term $(1-q)\gamma$ disappears for $q = 1$. As a consequence of this, γ and γ^* are no longer elements of K_2, the GAUSSian parts of the cocycles are not classified by its values on α^*, and, in the remaining parts, the operator mapping $\eta(\alpha^*)$ to $\eta(\gamma^{(*)})$ is not a bounded operator. Almost all of these difficulties arise also in the anti-classical case.

Clearly, we have again a one-parameter family δ_φ of homomorphisms, mapping α to $e^{i\varphi}$ and γ to 0, and the derivatives δ' and δ''. By the same arguments as for $q \in (-1, 1)$ it follows that K_2 is of codimension 1 in K_1 and the projection $\mathcal{P}$, extended to $q = -1$, is again a projection onto K_2. We denote this projection by $\mathcal{P}_{-1}$ in order to emphasize that the domains of $\mathcal{P}$ and $\mathcal{P}_{-1}$ are completely different, whereas the domains $\mathcal{A}_\mathcal{O}$ of $\mathcal{P}$ for different $q, q' \in (-1, 1)$ coincide.

The main difference compared to all the other cases becomes apparent if we have a look at $\underline{\delta} = \delta_\pi$. A representation $\underline{\delta}\mathbf{1}_{H_{\underline{\delta}}}$ on a HILBERT space $H_{\underline{\delta}}$, a cocycle with respect to such a representation, and a conditionally positive functional associated with such a cocycle will be called anti-GAUSSian. Obviously, for any choice of $\eta_{\alpha^*}, \eta_\gamma, \eta_{\gamma^*} \in H_{\underline{\delta}}$ we define an anti-GAUSSian cocycle η, by setting

$$\eta(\alpha^*) \;=\; \eta_{\alpha^*} \quad , \quad \eta(\alpha) \;=\; -\,\eta_{\alpha^*} \quad , \quad \eta(\gamma^{(*)}) \;=\; \eta_{\gamma^{(*)}},$$

and $\eta(a) = 0$ for $a \in K_2$. Therefore, Lemma 2.6 cannot be true for a representation containing an anti-GAUSSian part.

Without any change in the proof of Proposition 2.4 we see that also in the case $q = -1$ a decomposition of any representation of $\mathcal{A}_{-1}$ into invariant subspaces H_1 and H_2 is possible, such that $\pi(\gamma)$ is 0 on H_1 and injective on H_2. Clearly, the eigenspaces H_δ, and $H_{\underline{\delta}}$ to eigenvalues 1, and -1 of $\pi(\alpha)$ are invariant subspaces and the restriction of π to these subspaces is purely GAUSSian, and anti-GAUSSian respectively.

Proposition 5.19 *Lemma 2.6 remains valid for those representations of $\mathcal{A}_{-1}$, which contain no anti-GAUSSian part.*

PROOF We proceed smilarly as in the proof of Lemma 2.6. In the case of H_1 α is mapped to a unitary operator u. From Equation (2.7) we can conclude that u has eigenvalue -1 if $\eta(\gamma^{(*)})$ is different from 0.

In the case of H_2 nothing changes till we arrive at Equation (2.10). The right-hand side becomes $\pi(\beta^*)\epsilon$ and $\pi(\beta^*)$ is not neccesarily an invertible operator. However, $\pi(\beta^*)$ is injective and, nevertheless, ϵ is determined by the left-hand side. ■

As an immediate consequence we obtain again the form of the GAUSSian parts as in Corollary 2.11.

We define a family of representations $\hat{\delta}_{\varphi xy}$ on $\mathbf{C}^2$ which is labeled by elements of $SU(2)$, by setting

$$\begin{aligned} \hat{\delta}_{\varphi xy}(\alpha) &= \sqrt{1 - x^2 - y^2}\, e^{i\varphi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \hat{\delta}_{\varphi xy}(\gamma) &= (x + iy) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

These representations are irreducible if and only if $x^2 + y^2 \neq 0$ and $x^2 + y^2 \neq 1$ and it is easy to check that in this case two different members $\hat{\delta}_{\varphi xy}, \hat{\delta}_{\varphi' x' y'}$ of the family are unitarily equivalent if and only if either $\varphi' - \varphi = (2n+1)\pi$ or $x' = -x, y' = -y$ or both. (A unitary equivalence transform leaves the determinant invariant. Therefore, in two dimensions the factors in front of the matrices can only differ by sign. On the other hand, choosing the unitary transforms $u = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $u = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ we indeed obtain the sign changes of $\hat{\delta}_{\varphi xy}(\gamma)$ and $\hat{\delta}_{\varphi xy}(\alpha)$, respectively.)

The representation belonging to the identity (of $SU(2)$) will be denoted by

$$\hat{\delta} \;=\; \hat{\delta}_{000}.$$

The partial derivatives of $\hat{\delta}_{\varphi xy}$ at 0 are defined in the same manner as those of $\delta_{\varphi xy}$ in the preceeding section. Notice that $\hat{\delta} = \delta \oplus \underline{\delta}$. We denote a basis of $\mathbf{C}^2$ by $\hat{e}_1 = (1, 0)$ and $\hat{e}_2 = (0, 1)$. Clearly, we have $\hat\delta\hat e_1 = \hat e_1\delta$ and $\hat\delta\hat e_2 = \hat e_2\underline{\delta}$. Therefore,

$$a \quad\longmapsto\quad \langle\hat e_2|\hat\delta'^r(a)|\hat e_1\rangle,$$

defines, for any $r\in\mathbf{R}^3$ with components (φ,x,y), a cocycle with respect to $\underline{\delta}$. Clearly, $\langle\hat e_1|\hat\delta'^r|\hat e_1\rangle$ is a cocycle with respect to δ, hence, must be given by a multiple of δ'. By evaluating at α, we obtain $\langle\hat e_1|\hat\delta'^r|\hat e_1\rangle = \varphi\delta'$, with φ being the first component of r. Furthermore, we have for $a,b\in K_1$ that

$$\begin{aligned}\langle\hat e_1|\hat\delta''^r(ab)|\hat e_1\rangle &= 2\sum_{i=1}^{2}\langle\hat e_1|\hat\delta'^r(a)|\hat e_i\rangle\langle\hat e_i|\hat\delta'^r(b)|\hat e_1\rangle \\ &= 2\langle\hat e_2|\hat\delta'^r(a^*)|\hat e_1\rangle^*\langle\hat e_2|\hat\delta'^r(b)|\hat e_1\rangle \;+\; \varphi^2\delta''(ab).\end{aligned} \tag{5.7}$$

In other words,

$$\frac{\langle\hat e_1|\hat\delta''^r|\hat e_1\rangle}{2} - \varphi^2\frac{\delta''}{2}$$

is a conditionally positive functional fulfilling (1.2).

In the classical case the cocycles corresponding to $r=(1,0,0),(0,1,0),(0,0,1)$ are linearly independent. However, we immediately see that in the anti-classical case the cocycle $\langle\hat e_2|\hat\delta'^\varphi|\hat e_1\rangle$ is identically 0. We can obtain a third linearly independent one-dimensional cocycle with respect to $\underline{\delta}$ by

$$\underline{\delta}\circ(Id-\delta\mathbf{1}) \;=\; \langle\hat e_2|\hat\delta|\hat e_2\rangle\circ(Id-\delta\mathbf{1}) \;=\; \underline{\delta}-\delta.$$

(Cf. the preliminaries. Both $\underline{\delta}\circ(Id-\delta\mathbf{1})$ or $\underline{\delta}\circ\mathcal{P}_{-1}$ may serve as associated conditionally positive functionals.) We obtain the analogue of (5.5).

Theorem 5.20 *By*

$$\eta_{\underline{\delta}} \;=\; \underline{\delta}\circ(Id-\delta\mathbf{1})\eta_\varphi + \langle\hat e_2|\hat\delta'^x|\hat e_1\rangle\eta_x + \langle\hat e_2|\hat\delta'^y|\hat e_1\rangle\eta_y,$$

*we establish a one-to-one correspondence between anti-*GAUSS*ian cocycles $\eta_{\underline{\delta}}$ and triplets $(\eta_\varphi,\eta_x,\eta_y)$ of vectors $\eta_\varphi,\eta_x,\eta_y\in H_{\underline{\delta}}$.*

In the case of a one-dimensional cocycle we again find by commutativity of γ,γ^* that the complex numbers η_x and η_y must have the same phase factor if $\eta_{\underline{\delta}}$ determines the values of a conditionally positive functional on K_2. Now let $r=(\varphi,x,y)$ be an element of $\mathbf{C}\times\mathbf{R}^2$. By straightforward verification on elements of K_1 we see that

$$\left\langle\hat e_2\,\middle|\hat\delta_{0(tx)(ty)}\middle|\,\frac{\hat e_1+t\varphi\hat e_2}{t}\right\rangle\circ(Id-\delta\mathbf{1}) \quad\longrightarrow\quad \underline{\eta}_r$$

as $t>0$ tends to 0 where $\underline{\eta}_r$ is the cocycle $\eta_{\underline{\delta}}$ having $\eta_\varphi=\varphi\in\mathbf{C}$ and $\eta_{x/y}=x/y\in\mathbf{R}$. Notice that

$$\left\langle\hat e_1\,\middle|\hat\delta_{0(tx)(ty)}\middle|\,\frac{\hat e_1+t\varphi\hat e_2}{t}\right\rangle\circ(Id-\delta\mathbf{1}) \;\longrightarrow\; \langle\hat e_1|\hat\delta'^{(0,x,y)}|\hat e_1\rangle+\varphi\langle\hat e_1|\hat\delta|\hat e_2\rangle\circ(Id-\delta\mathbf{1}) \;=\; 0$$

for $t\to0$. Therefore, we find by computations similar to (5.7) that

$$\underline{\psi}_r \;=\; \lim_{t\to0}\left\langle\frac{\hat e_1+t\varphi\hat e_2}{t}\,\middle|\hat\delta_{0(tx)(ty)}\middle|\,\frac{\hat e_1+t\varphi\hat e_2}{t}\right\rangle\circ\mathcal{P}_{-1}$$

defines a conditionally positive functional fulfilling (1.2). We obtain by a proof completely similar to that of Theorem 5.14

Theorem 5.21 *For any anti-*GAUSS*ian cocycle $\eta_{\underline{\delta}}$ which determines the values of a conditionally positive functional on K_2 there are three orthogonal vectors $\eta_1,\eta_2,\eta_3\in H_{\underline{\delta}}$, having lenth 1 or 0, and three vectors $r_1,r_2,r_3\in\mathbf{C}\times\mathbf{R}^2$ such that*

$$\eta_{\underline{\delta}} \;=\; \underline{\eta}_{r_1}\eta_1+\underline{\eta}_{r_2}\eta_2+\underline{\eta}_{r_3}\eta_3.$$

Choosing arbitrary real numbers r_0, r_α we obtain all conditionally positive functionals fulfilling (1.2) in the form

$$\psi_{\underline{\delta}} \quad = \quad r_0\delta + r_\alpha\delta' + \underline{\psi}_{r_1} + \underline{\psi}_{r_2} + \underline{\psi}_{r_3}.$$

Now we come to the representation theory. Notice that $\pi(\gamma^2)$ is a normal operator, commuting with $\pi(\alpha)$ and $\pi(\gamma)$. Using its spectral measure, it is possible to define $\sqrt[n]{\pi(\gamma^2)}$, also commuting with everything, and such that $\sqrt[nm]{\pi(\gamma^2)}^m = \sqrt[n]{\pi(\gamma^2)}$. If we define $\sqrt[n]{\pi(\gamma^{*2})} = \sqrt[n]{\pi(\gamma^2)}^*$, then also $\sqrt[n]{\pi(\gamma^2)}^* \sqrt[n]{\pi(\gamma^2)} = \sqrt[\frac{n}{2}]{\pi(\gamma^*\gamma)}$ where the right-hand side is the usual root of a positive operator. Actually the following lemma does not depend on the special choice of this n–th root. To be explicit we fix, on the scalar level, an n–th root by setting $\sqrt[n]{z} = \sqrt[n]{r}e^{i\frac{\varphi}{n}}$ for $z = re^{i\varphi}, \varphi \in [0, 2\pi)$.

Lemma 5.22 *Let π be a representation of the unital $*$–algebra generated by two normal anti-commuting indeterminants α, γ (i.e. Relations (a)-(d)) as bounded operators on a* Hilbert *space H. Then the representation $\pi^\oplus$ on $\mathbf{C}^2 \otimes H$, defined by*

$$\begin{aligned}
\pi^\oplus(\alpha) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \pi(\alpha) \\
\pi^\oplus(\gamma) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \sqrt{\pi(\gamma^2)},
\end{aligned}$$

is unitarily equivalent to $\pi \oplus \underline{\pi}$.

Morover, if $\pi(\alpha)$ and $\pi(\gamma)$ are injective and (S_1, S_2) is any partition of $\mathbf{C}_0 = \mathbf{C}\backslash\{0\}$ into Borel *sets, such that $z \in S_1$ implies $-z \in S_2$, then the restrictions of $\pi^\oplus$ to the invariant orthogonal subspaces H_i onto which the projections $\mathbf{1}_{M_{2\times 2}} \otimes E_i$ with*

$$E_i \quad = \quad \int_{S_i} dE_z^\alpha$$

project both are unitarily equivalent to π. In particular,

$$\pi \quad \asymp \quad \underline{\pi}.$$

The proof of this lemma can be found in Appendix C. We obtain the irreducible representations of $\mathcal{A}_{-1}$ as a simple corollary.

Corollary 5.23 *The irreducible representations of $\mathcal{A}_{-1}$ are given by the three families $\hat{\delta}_{\varphi xy}$ with $0 < x^2 + y^2 < 1$, δ_φ, and $\bar{\delta}_\varphi$.*

Proof If π is irreducible, then we have necessarily that either $\pi(\gamma) = 0$, or $\pi(\alpha) = 0$, or both are injective. (Otherwise π could be decomposed into non-trivial invariant subspaces.) Obviously, the first case leads to a representation δ_φ and the second, by exchanging the roles of α and γ, to $\bar{\delta}_\varphi$.

By Lemma 5.22 we know that in the remaining case π is unitarily equivalent to the restriction of $\pi^\oplus$ to H_1. Therefore, π is irreducible if and only if E_1 projects to a one-dimensional subspace. From Relation (e) we see that $\pi = \hat{\delta}_{\varphi xy}$ for some $(\varphi, x, y) \in SU(2)$ and $0 < x^2 + y^2 < 1$. ■

Theorem 5.24 *Let π be any $*$–representation of $\mathcal{A}_{-1}$ on a* Hilbert *space H. There is a spectral measure $dE_{\varphi xy}$ on $SU(2)$ with values in $\mathcal{B}(H)$, such that the representation $\widehat{\pi}$ on $\mathbf{C}^2 \otimes H$ defined by*

$$\widehat{\pi} \quad = \quad \int_{SU(2)} \hat{\delta}_{\varphi xy} \otimes dE_{\varphi xy}$$

is unitarily equivalent to $\pi \oplus \underline{\pi}$.

On the subspace H_0, where $\pi(\alpha)$ and $\pi(\gamma)$ have no eigenvalue 0, we even have

$$\pi \asymp \underline{\pi} \quad \textit{and} \quad \pi \asymp (\mathbf{1} \otimes E_1)\widehat{\pi}.$$

On the subspace H_0^α, where $\pi(\gamma)=0$, we have

$$\pi \asymp \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbf{1} \right\} \widehat{\pi}.$$

On the subspace H_0^γ, where $\pi(\alpha)=0$, we have

$$\pi \asymp \left\{ \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \mathbf{1} \right\} \widehat{\pi}.$$

Proof Let dE_z^α, and $dE_w^{\sqrt{\gamma 2}}$ be the spectral measures of $\pi(\alpha)$, and $\sqrt{\pi(\gamma^2)}$ respectively. By $dE_{zw} = dE_z^\alpha \, dE_w^{\sqrt{\gamma 2}}$ we denote the joined spectral measure of these two commuting operators. By Relation (e) we see that dE_{zw} has to be concentrated on the subset

$$\left\{ (\sqrt{1-x^2-y^2}e^{i\varphi}, x+iy) \,|(\varphi,x,y) \in SU(2) \right\} \subset \mathbf{C}^2.$$

On the subspace $H_0 \oplus H_0^\alpha$ we define $dE_{\varphi xy}$ to be the spectral measure on $SU(2)$, induced by dE_{zw}. By Lemma 5.22 we obtain the statements concerning this subspace.

On H_0^γ, the remaining subspace, $\pi(\gamma)$ is a unitary operator with spectral measure dE_φ. In this case we define $dE_{\varphi xy}$ to be the measure concentrated on the sphere consisting of all points $(0,\cos\varphi,\sin\varphi) \in SU(2)$ and such that $dE_{0\cos\varphi\sin\varphi} = dE_\varphi$. This representation is a direct sum of $\int \overline{\delta}_\varphi \, dE_\varphi$ and $\int \underline{\overline{\delta}}_\varphi \, dE_\varphi$. Since $\frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ is *the* projection onto the eigenspace to the eigenvalue 1 of the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, we obtain the last statement. ■

In the sequel instead of the set $S_1 \subset \mathbf{C}_0$ we rather use the corresponding subset $S_1 \subset SU(2)$ on which the spectral measure dE_{zw} on $S_1 \times \mathbf{C}$ has to be concentrated. By restricting to elements of $SU(2)^0 = \{(\varphi,x,y) \in SU(2) | 0 < x^2+y^2 < 1\}$, we include a projection onto H_0. Notice that by our choice of the square root the spectral measure $dE_{\varphi xy}$ vanishes for $\arg(x+iy) \geq \pi$. It is convenient to fix S_1 by

$$S_1 = \left\{ (\varphi,x,y) \in SU(2)^0 \middle| \, \varphi \in [-\tfrac{\pi}{2},\tfrac{\pi}{2}), 0 \leq \arg(x+iy) < \pi \right\}.$$

Correspondingly $E_1 = \int_{S_1} dE_{\varphi xy}$. We mention that S_1 provides a partition of $SU(2)^0$ into four subsets, each containing any two-dimensional irreducible representation precisely once, by

$$\begin{aligned} SU(2)^0 = S_1 & \bigcup \left\{ (\varphi,x,y) \in SU(2)^0 \middle| (\varphi+\pi,x,y) \in S_1 \right\} \\ & \bigcup \left\{ (\varphi,x,y) \in SU(2)^0 \middle| (\varphi,-x,-y) \in S_1 \right\} \\ & \bigcup \left\{ (\varphi,x,y) \in SU(2)^0 \middle| (\varphi+\pi,-x,-y) \in S_1 \right\}. \end{aligned}$$

N.B.: Consider the C^*–algebra $C(SU(2),M_{2\times 2})$ of continous functions on $SU(2)$ with values in $M_{2\times 2}$, equipped with the supremum norm $\|f\| = \sup\limits_{SU(2)} \|f(U)\|$. Clearly, the subalgebra $\mathcal{A}_{-\mathrm{f}}$ of $C(SU(2)$,$M_{2\times 2})$ generated by $f_\alpha(U) = \hat{\delta}_{\varphi xy}(\alpha)$ and $f_\gamma(U) = \hat{\delta}_{\varphi xy}(\gamma)$ can be identified with $\mathcal{A}_{-1}$ equipped with the norm $\|\bullet\| = \sup\limits_\pi \|\pi(\bullet)\|$. By

$$\pi_0 = \int_{SU(2)} \hat{\delta}_{\varphi xy} \, dH$$

we define a faithful representation on $\mathbf{C}^2 \otimes L^2(SU(2),H)$. Again the representation remains faithful if the Haar measure is replaced by any other measure, whose spectrum is $SU(2)$. Notice that $\mathcal{A}_{-1}$ has an obvious Z_2–graduation. An element of $a \in \mathcal{A}_{-1}$ is called even, i.e. a has degree 1, if $\pi_0(a)$ is diagonal, and a is called odd, i.e. a has degree -1, if the diagonal entries of $\pi_0(a)$ are 0.

If $\pi(\alpha), \pi(\gamma)$ are injective, the restriction $\int_{S_1} \hat{\delta}_{\varphi xy} \otimes dE_{\varphi xy}$ of $\widehat{\pi}$ to the subspace $H_1 = \mathbf{C}^2 \otimes E_1 H$ of $\mathbf{C}^2 \otimes H$ can be identified with the original representation π. For $\eta \in H$ denote the corresponding vector in H_1 by (η_1, η_2) with $\eta_i \in E_1 H$. We obtain for the diagonal element

$$\begin{aligned} \langle \eta | \pi | \eta \rangle &= \sum_{i,j=1}^{2} \left\langle \hat{e}_i \otimes \eta_i \middle| \int_{S_1} \hat{\delta}_{\varphi xy} \otimes dE_{\varphi xy} \middle| \hat{e}_j \otimes \eta_j \right\rangle \\ &= \mathrm{Tr} \int_{S_1} \hat{\delta}_{\varphi xy} \, d\hat{\nu}^0_{\varphi xy}, \end{aligned}$$

where we introduced the measure valued, self-adjoint matrix $(d\hat{\nu}^0_{\varphi xy})_{ij} = d\langle \eta_j | dE_{\varphi xy} | \eta_i \rangle$. Notice that the non-diagonal entries are complex. By CAUCHY-SCHWARTZ inequality we have

$$|(d\hat{\nu}^0_{\varphi xy})_{12}| \;\leq\; \sqrt{(d\hat{\nu}^0_{\varphi xy})_{11} (d\hat{\nu}^0_{\varphi xy})_{22}} \;\leq\; \frac{1}{2}((d\hat{\nu}^0_{\varphi xy})_{11} + (d\hat{\nu}^0_{\varphi xy})_{22}).$$

Therefore, the matrix entries $(d\hat{\nu}^0_{\varphi xy})_{ij}$ are all absolutely continuous with respect to the (positive, regular, finite) measure $d\nu^0_{\varphi xy} = (d\hat{\nu}^0_{\varphi xy})_{11} + (d\hat{\nu}^0_{\varphi xy})_{22}$. By an application of the theorem of RADON-NIKODYM we can write

$$d\hat{\nu}^0 \;=\; \hat{n}^0 \, d\nu^0,$$

where $\hat{n}^0$ is a ν^0–measurable function on S_1 with values in the positive 2×2–matrices of unit trace.

If $\pi(\alpha)$ or $\pi(\gamma)$ are not injective, we have to take into account the contributions of the subspaces H^γ_0 and H^α_0. By Theorem 5.24 we easily see that this is done by adding a measure of the form

$$d\hat{\nu}^\alpha \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} d\nu^\alpha,$$

where $d\nu^\alpha$ is a (positive, regular, finite) measure concentrated on $\{(\varphi, x, y) \in SU(2) | x^2 + y^2 = 0\}$, for H^α_0, and by adding a measure of the form

$$d\hat{\nu}^\gamma \;=\; \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} d\nu^\gamma,$$

where $d\nu^\gamma$ is a (positive, regular, finite) measure concentrated on $\{(\varphi, x, y) \in SU(2) | x^2 + y^2 = 1\}$, for H^γ_0.

Clearly, also the sum

$$d\hat{\nu} \;=\; d\hat{\nu}^0 + d\hat{\nu}^\alpha + d\hat{\nu}^\gamma$$

fulfills CAUCHY-SCHWARTZ inequality. Therefore, we can write

$$d\hat{\nu} \;=\; \hat{n} \, d\nu,$$

where

$$d\nu \;=\; (d\hat{\nu})_{11} + (d\hat{\nu})_{22} \;=\; d\nu^0 + d\nu^\alpha + d\nu^\gamma$$

and $\hat{n}$ is a ν–measurable function on $SU(2)$ with values in the positive 2×2–matrices of unit trace. We summarize.

Theorem 5.25 *Any state φ on $\mathcal{A}_{-1}$ can be expressed as*

$$\varphi \;=\; \int_{SU(2)} \mathrm{Tr}\; \hat{n}(\varphi, x, y) \hat{\delta}_{\varphi xy} \, d\nu_{\varphi xy}, \tag{5.8}$$

where ν is a probability measure on $SU(2)$ and $\hat{n}$ is a ν–measurable function on $SU(2)$ with values in the positive 2×2–matrices of unit trace, fulfilling the following conditions.

(i) The restriction of ν to $SU(2)^0$ is concentrated on S_1.

(ii) For $x^2+y^2=0$ we have

$$\hat{n}(\varphi,x,y) \quad = \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(iii) For $x^2+y^2=1$ we have

$$\hat{n}(\varphi,x,y) \quad = \quad \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

This is the decomposition of an arbitrary state into its irreducible components. However, notice that a state of the form Tr $\hat{n}(\varphi,x,y)\hat{\delta}_{\varphi xy}$ in general can be decomposed into two irreducible components, because any positive matrix can be written as the sum of in general two dyadic products of the form $|c\rangle\langle c|$.

We proceed now precisely as in the foregoing section. Assume that a given representation π has no GAUSSian and anti-GAUSSian part. We introduce the mapping $\hat{\mathcal{O}} : SU(2)\backslash\{I,(\pi,0,0)\} \to M_{2\times 2}$ by

$$\begin{aligned}
[\hat{\mathcal{O}}(a)](U) \quad &= \quad (\hat{\delta}_{\varphi xy}(a)-\delta(a)\mathbf{1}_{M_{2\times 2}})\left[\sqrt{1-x^2-y^2}e^{-i\varphi}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]^{-1} \\
&= \quad \frac{\hat{\delta}_{\varphi xy}(a)-\delta(a)\mathbf{1}_{M_{2\times 2}}}{(1-x^2-y^2)e^{-2i\varphi}-1}\begin{pmatrix} 1+\sqrt{1-x^2-y^2}e^{-i\varphi} & 0 \\ 0 & 1-\sqrt{1-x^2-y^2}e^{-i\varphi} \end{pmatrix}
\end{aligned}$$

and define the possibly unbounded operator

$$\hat{\mathcal{O}}_\pi(a) \quad = \quad \int_{SU(2)} [\hat{\mathcal{O}}(a)](U) \otimes dE_{\varphi xy}.$$

Similar to the classical case, a vector $\eta_{\alpha^*} = (\eta_{\alpha^*1}, \eta_{\alpha^*2}) \in H$ defines a cocycle with respect to π, assuming this vector on α^*, by setting

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \quad = \quad \hat{\mathcal{O}}_\pi \begin{pmatrix} \eta_{\alpha^*1} \\ \eta_{\alpha^*2} \end{pmatrix},$$

if and only if the corresponding vector η_{α^*} is an element of $\widehat{\mathcal{D}}$, the domain of $\hat{\mathcal{O}}_\pi(\gamma^{(*)})$. This condition reads

$$\int_{SU(2)} \frac{x^2+y^2}{|\sqrt{1-x^2-y^2}e^{-i\varphi}-1|^2}\,(d\hat{\nu}_{\varphi xy})_{11} + \int_{SU(2)} \frac{x^2+y^2}{|\sqrt{1-x^2-y^2}e^{-i\varphi}+1|^2}\,(d\hat{\nu}_{\varphi xy})_{22} \quad < \quad \infty.$$

The integrands of both integrals are bounded on the substets of $SU(2)$, for which $x^2+y^2=0$ and $x^2+y^2=1$, respectively, and that the integrand of the second integral is bounded on S_1. Therefore, the condition is actually a condition only on $(d\hat{\nu})_{11}$.

Notice, that also the mapping $\hat{\mathcal{O}}_\pi$ can be obtained as a strong limit of the mappings

$$\pi \circ (Id-\delta\mathbf{1})\frac{\mathbf{1}}{p\pi(\alpha^*)-\mathbf{1}_H}$$

for $p \to 1$. The proof is comletely analogous to that of the classical case.

By

$$\psi \quad = \quad \left\langle \begin{pmatrix} \eta_{\alpha^*1} \\ \eta_{\alpha^*2} \end{pmatrix} \middle| \int_{SU(2)} [\hat{\mathcal{T}}(a)](U) \otimes dE_{\varphi xy} \middle| \begin{pmatrix} \eta_{\alpha^*1} \\ \eta_{\alpha^*2} \end{pmatrix} \right\rangle,$$

where we defined

$$[\hat{\mathcal{T}}(a)](U) = \begin{pmatrix} 1+\sqrt{1-x^2-y^2}e^{i\varphi} & 0 \\ 0 & 1-\sqrt{1-x^2-y^2}e^{i\varphi} \end{pmatrix} \frac{\hat{\hat{\delta}}_{\varphi xy}\circ\mathcal{P}_{-1}(a)}{|(1-x^2-y^2)e^{-2i\varphi}-1|^2} \begin{pmatrix} 1+\sqrt{1-x^2-y^2}e^{-i\varphi} & 0 \\ 0 & 1-\sqrt{1-x^2-y^2}e^{-i\varphi} \end{pmatrix},$$

we find a conditionally positive functional satisfying (1.2).

We introduce the measure valued matrix $d\hat{\mu}_{\varphi xy}$ by

$$\begin{aligned}
(d\hat{\mu}_{\varphi xy})_{11} &= \frac{(d\hat{\nu}_{\varphi xy})_{11}}{|1-\sqrt{1-x^2-y^2}e^{-i\varphi}|^2} \\
(d\hat{\mu}_{\varphi xy})_{22} &= \frac{(d\hat{\nu}_{\varphi xy})_{22}}{|1+\sqrt{1-x^2-y^2}e^{-i\varphi}|^2} \\
(d\hat{\mu}_{\varphi xy})_{12} &= \frac{(d\hat{\nu}_{\varphi xy})_{12}}{(1-\sqrt{1-x^2-y^2}e^{-i\varphi})(1+\sqrt{1-x^2-y^2}e^{i\varphi})} \\
(d\hat{\mu}_{\varphi xy})_{21} &= \overline{(d\hat{\mu}_{\varphi xy})_{12}}.
\end{aligned}$$

Notice that the entries of $d\hat{\mu}_{\varphi xy}$ also fulfill the CAUCHY-SCHWARTZ inequality. Therefore, we obtain again a (positive, regular) measure μ and a μ–measurable function $\widehat{m}$ with values in the positive 2×2–matrices of unit trace, such that $d\hat{\mu} = \widehat{m}d\mu$, and by Equation (5.8)

$$\psi = \int_{SU(2)} \mathrm{Tr}\ \widehat{m}(\varphi,x,y)\hat{\hat{\delta}}_{\varphi xy}\circ\mathcal{P}_{-1}\, d\mu_{\varphi xy}.$$

This yields the LÉVY-KHINTCHINE formula.

Theorem 5.26 *The formula*

$$\psi = \psi_\delta + \psi_{\underline{\delta}} + \int_{SU(2)} \mathrm{Tr}\ \widehat{m}(\varphi,x,y)\hat{\hat{\delta}}_{\varphi xy}\circ\mathcal{P}_{-1}\, d\mu_{\varphi xy}$$

establishes a one-to-one correspondence between conditionally positive functionals on $\mathcal{A}_{-1}$ and triples $(\psi_\delta+\psi_{\underline{\delta}},\mu,\widehat{m})$ consisting of the sum of a GAUSSian part ψ_δ and an anti-GAUSSian part $\psi_{\underline{\delta}}$, a (not necessarily finite) measure μ on $SU(2)\backslash\{I,(\pi,0,0)\}$, and a μ–measurable, positive 2×2–matrix valued function $\widehat{m}$ of unit trace, fulfilling the following conditions:

(i) The restriction of μ to $SU(2)^0$ is concentrated on S_1.

(ii) For $x^2+y^2=0$ we have

$$\widehat{m}(\varphi,x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(iii) For $x^2+y^2=1$ we have

$$\widehat{m}(\varphi,x,y) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

(iv) The measure μ fulfills

$$\int_{SU(2)} (x^2+y^2)(\widehat{m}(\varphi,x,y))_{11}\, d\mu_{\varphi xy} < \infty$$

$$\int_{SU(2)} |\sqrt{1-x^2-y^2}e^{-i\varphi}-1|^2(\widehat{m}(\varphi,x,y))_{11}\, d\mu_{\varphi xy} < \infty$$

$$\int_{SU(2)} (\widehat{m}(\varphi,x,y))_{22}\, d\mu_{\varphi xy} < \infty.$$

N.B.: Since Tr $\widehat{m} = 1$, it is a necessary (but not sufficient) condition that μ fulfills

$$\int_{SU(2)} (x^2 + y^2)\, d\mu_{\varphi xy} \;<\; \infty \quad \text{and} \quad \int_{SU(2)} |\sqrt{1 - x^2 - y^2} e^{-i\varphi} - 1|^2\, d\mu_{\varphi xy} \;<\; \infty.$$

We will need this in the next chapter.

5.6 Unifying description

In this section we want to point out that the three cases are more similar than they may look at first sight. We will find as a further main result of these notes that for any $q \in [-1, 1]$ the cone consisting of all conditionally positive functionals on $\mathcal{A}_q$ can be obtained by a simple completion of a cone consisting of states on $\mathcal{A}_q$ combined with an arbitrary projection $\mathcal{Q}$ onto K_2. But first we give a list of what coincides and what is different for the three cases $|q| < 1$, $q = 1$, and $q = -1$.

In order to establish mappings $\mathcal{O}$ and $\mathcal{T}$, fulfilling (2.1), (2.2), and $\mathcal{O}(\alpha^*) = \mathbf{1}$, we had in each of the cases to enlarge the original algebra (with $q = 0$ being the only exception). For $|q| < 1$ it was sufficient to consider the C^*–completion which yielded automatically bounded representing operators even for the enlarged algebra. In the cases $q = \pm 1$ it was necessary to allow elements of infinite norm. In both cases the enlarging was performed most easily in the picture of the function algebras $\mathcal{A}_{\mathrm{f}}$ and $\mathcal{A}_{-\mathrm{f}}$, respectively. The functions are $\mathbf{C}$–valued for $q = 1$ and $M_{2\times 2}$–valued for $q = -1$. In particular, the functions $\mathcal{O}(\gamma^{(*)})$ become essentially unbounded in both cases.

After having established the mappings $\mathcal{O}$ and $\mathcal{T}$, we obtain for any vector in the domain of all operators $\mathcal{O}_\pi(\mathcal{A}_q)$ a cocycle, mapping α^* to this vector, and a corresponding conditionally positive functional. For $|q| < 1$ the domain is the complete HILBERT space. In the other two cases we only obtained a dense pre-HILBERT space.

In all three cases for 'most' of the representations a cocycle is already determined by its value on α^*. For $|q| < 1$ this is true for all representations. In the cases $q = 1$ and $q = -1$ the GAUSSian and the anti-GAUSSian part, respectively, have to be treated separately. Therefore, excluding these parts, we obtain a classification of the cocycles by vectors in the respective domains of $\mathcal{O}_\pi(\gamma^{(*)})$.

In the case $|q| < 1$ we found that for a given cocycle η there *is* a conditionally positive functional ψ fulfilling (1.2). In the cases $q = \pm 1$ we found that this may be not so. The values of ψ on K_2 are determined by η. The classification of the restriction of the functionals to K_2 is equivalent to the classification of the states which have to be applied to $\mathcal{T}$ in order to obtain the functional. The restriction to the dense subspace in the case of the cocycles for $q = \pm 1$ corresponds to the restriction to those states which can be extended to all possibly unbounded functions in $\mathcal{T}(\mathcal{A}_{\pm 1})$ in the case of the conditionally positive functionals.

The codimension of K_2 in $\mathcal{A}_q$ is 2 for $|q| < 1$ *and* $q = -1$, and is 4 for $q = 1$. This corresponds to the fact that there are 2 and 4 linearly independent functionals, vanishing on K_2. In the next chapter this fact will cause a subtle limit procedure when we try to approximate conditionally positive functionals on $\mathcal{A}_1$ by conditionally positive functionals on $\mathcal{A}_q$.

In all three cases the results obtained can be reinterpreted in terms of a LÉVY-KHINTCHINE formula. On the level of the GNS–representation this description corresponds to introducing a new toplogy on the space of cyclic vevtors. For $|q| < 1$ the resulting space is bigger than the original HILBERT space. However, for $q = \pm 1$ the HILBERT space is neither a super- nor a subset of the new topological space.

Now we come to the claimed unification. It is the description in terms of LÉVY-KHINTCHINE formulae which allows us to define a procedure to find any conditionally positive functional on $\mathcal{A}_q$ in a unified way for all $q \in [-1, 1]$. Let $\mathcal{Q}$ be an arbitrary projection onto K_2. Let

$$\Phi_{\mathcal{Q}} \quad = \quad \{\varphi \circ \mathcal{Q} \,|\, \varphi \text{ positive}\}$$

be the cone consisting of all positive functionals φ on $\mathcal{A}_q$ combined with $\mathcal{Q}$. Denote by $\overline{\Phi}$ the completion of the cone $\bigcup_{\mathcal{Q}} \Phi_{\mathcal{Q}}$ with respect to pointwise convergence. Clearly, any element ψ of $\overline{\Phi}$ is a conditionally positive functional. We show that also the converse is true.

Theorem 5.27 *For any $q \in [-1,1]$ the cone $\overline{\Phi}$ consists of all conditionally positive functionals on $\mathcal{A}_q$.*

PROOF Let ψ be a conditionally positive functional on $\mathcal{A}_q$. First, we assume that $\psi \circ \mathcal{P} = \psi$. If ψ has no GAUSSian part (and also no anti-GAUSSian part if $q = -1$), then the LÉVY-KHINTCHINE formula tells us that $\psi \in \overline{\Phi}$.

We obtained the restrictions of the GAUSSian parts ψ_δ of the functionals to K_2 as second derivatives of families of states φ_t with respect to t at $t = 0$ for which $\lim_{t\to 0} \varphi_t = \delta$ holds. Therefore, if $\psi_\delta \circ \mathcal{P} = \psi_\delta$, we have

$$\psi_\delta \quad = \quad \lim_{t\to 0} \frac{\varphi_t \circ \mathcal{P}}{t^2}$$

which means that also $\psi_\delta \in \overline{\Phi}$. Clearly, the same holds for the anti-GAUSSian part of a functional on $\mathcal{A}_{-1}$, because it was given as limit of the same form.

A general conditionally positive functional has the form

$$\psi \quad = \quad \psi \circ \mathcal{P} + r_1\delta + r_\alpha \delta'$$

for $q \neq 1$ and

$$\psi \quad = \quad \psi \circ \mathcal{P}_1 + r_1\delta + \delta'^r$$

for $q = 1$ where $r_1, r_\alpha \in \mathbf{R}$ and $r \in \mathbf{R}^3$. We first consider the case $q \neq 1$. Notice that the general projection onto K_2 has the form

$$\begin{aligned} \mathcal{Q} \quad &= \quad Id - (\mathbf{1} + c_1)\delta - \left(\frac{\alpha - \alpha^*}{2i} + c_\alpha\right)\delta' \\ &= \quad \mathcal{P} - c_1\delta - c_\alpha\delta' \end{aligned}$$

with c_1, c_α being arbitrary elements of K_2. There exist a state φ on $\mathcal{A}_q$ and $c \in K_2$ such that $\varphi(c) = 1$. Therefore, by setting $c_1 = -\frac{r_1}{t}c$ and $c_\alpha = -\frac{r_\alpha}{t}c$, we obtain

$$\lim_{t\to 0} t\varphi \circ \mathcal{Q} \quad = \quad r_1\delta + r_\alpha\delta',$$

i.e. $r_1\delta + r_\alpha\delta' \in \overline{\Phi}$. The proof for $q = 1$ is analogous. ■

N.B.: It might be an interesting suggestion to ask for which pairs $(\mathcal{A}, \delta)$ consisting of a $*$–algebra $\mathcal{A}$ and a homomorphism δ the conditionally positive functionals can be described in that way.

Chapter 6

The cases $q = \pm 1$ as limits $q \to \pm 1$

If we put $q = \pm 1$ formally in expressions of the general representation of $\mathcal{A}_q$, we obtain representations of $\mathcal{A}_{\pm 1}$ which map either α or γ to 0, or to a direct sum of both. We use the notions of Appendices A and B in order to define q–dependent families of states and conditionally positive functionals which converge in some sense to states and conditionally positive functionals on $\mathcal{A}_{\pm 1}$, respectively, as q tends to ± 1.

First we give a precise meaning to 'convergence of a family of mappings' for different q. Let $\mathcal{F}$ denote the free unital $*$–algebra generated by the non-commutative indeterminants α and γ. For any $q \in [-1, 1]$ the ideal generated by Relations (1.4) is denoted by $\mathcal{I}_q$. By $\mathcal{E}_q$ we denote the canonical homomorphism from $\mathcal{F}$ onto $\mathcal{A}_q$. If now φ_q is a family of mappings from $\mathcal{A}_q$ into the same topological space, we can raise it to a family Φ_q of mappings on $\mathcal{F}$ by setting

$$\Phi_q = \varphi_q \circ \mathcal{E}_q.$$

We say that $\lim_{q \to q_0} \varphi_q = \varphi_{q_0}$ (or, in other words, φ_q is continuous at q_0) if and only if $\lim_{q \to q_0} \Phi_q = \Phi_{q_0}$ pointwise on $\mathcal{F}$. We also say that a sequence φ_n of mappings on $\mathcal{A}_{q_n}$ converges to a mapping φ on $\mathcal{A}_q$, if $\lim_{n \to \infty} q_n = q$ and $\lim_{n \to \infty} \Phi_n = \Phi$ pointwise.

6.1 States

Proposition 6.1 *Let φ_q be a family of states on $\mathcal{A}_q$ for $q \in (-1, 1)$. If the limits $\Phi_{\pm 1} = \lim_{q \to \pm 1} \Phi_q$ exist, there are states $\varphi_{\pm 1}$ on $\mathcal{A}_{\pm 1}$, such that $\varphi_{\pm 1} = \lim_{q \to \pm 1} \varphi_q$.*

Proof A mapping φ_q is a state on $\mathcal{A}_q$ if and only if Φ_q is a state on $\mathcal{F}$. Therefore, the limits $\Phi_{\pm 1}$ are states on $\mathcal{F}$.

It remains to show that $\Phi_{\pm 1}$ vanishes on $\mathcal{I}_{\pm 1}$. Suppose that $a_{\pm 1}$ is an element of $\mathcal{I}_{\pm 1}$. It can be written as

$$a_{\pm 1} = \sum_{i \in \{a, \dots, e\}} a_{i\pm} R_{i \pm 1} b_{i\pm},$$

where $a_{i\pm}, b_{i\pm} \in \mathcal{F}$ and $R_{i\,q}$ denotes the i–relation of Relations (1.4) (e.g. $R_{a\,q} = \alpha\gamma - q\gamma\alpha$). By replacing $R_{i\,\pm 1}$ with $R_{i\,q}$ we easily see that $a_{\pm 1}$ differs from an element of $\mathcal{I}_q$ by $\Delta_{q\pm}$ which is of the form

$$\Delta_{q\pm} = (\pm 1 - q) \sum_{j=1}^{n} f_{j\pm}(q) c_{j\pm},$$

where $f_{j\pm}(q)$ are bounded (continuous) functions of q and $c_{j\pm}$ are q–independent elements of $\mathcal{F}$. Thus, we obtain

$$\lim_{q \to \pm 1} \Phi_q(\Delta_{q\pm}) = 0$$

and consequently our statement. ■

Let λ, μ be complex numbers. From $|\lambda - \overline{\mu}|^2 > 0$ if and only if $\lambda \neq \overline{\mu}$ we can conclude that

$$\lim_{t\to\infty} \frac{e^{t\lambda\overline{\mu}} e^{t\overline{\lambda}\mu}}{e^{t|\lambda|^2} e^{t|\mu|^2}} = \begin{cases} 1 & \text{for } \lambda = \overline{\mu} \\ 0 & \text{otherwise.} \end{cases}$$

We prove a q–analogue (cf. Appendix A).

Proposition 6.2 *For $\lambda, \mu \in U_1(0)$ the q–exponential fulfills*

$$\lim_{q\to 1} \frac{e_q^{\lambda\overline{\mu}} e_q^{\overline{\lambda}\mu}}{e_q^{|\lambda|^2} e_q^{|\mu|^2}} = \begin{cases} 1 & \textit{for } \lambda = \overline{\mu} \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF For $\lambda = \overline{\mu}$ the statement is clear. Thus, let $\lambda \neq \overline{\mu}$. By the product representation of e_q^z we obtain

$$\frac{e_q^{\lambda\overline{\mu}} e_q^{\overline{\lambda}\mu}}{e_q^{|\lambda|^2} e_q^{|\mu|^2}} = \prod_{k=0}^{\infty} \frac{(1 - q^k|\lambda|^2)(1 - q^k|\mu|^2)}{(1 - q^k\lambda\overline{\mu})(1 - q^k\overline{\lambda}\mu)} = \prod_{k=0}^{\infty} \left(1 - \frac{q^k(\lambda - \mu)(\overline{\lambda - \mu})}{(1 - q^k\lambda\overline{\mu})(1 - q^k\overline{\lambda}\mu)}\right). \tag{6.1}$$

All factors lie in $[0, 1)$. Therefore, so does the product. On the other hand, for $q \to 1$ the factors converge to

$$1 - \frac{(\lambda - \mu)(\overline{\lambda - \mu})}{(1 - \lambda\overline{\mu})(1 - \overline{\lambda}\mu)}$$

which is less than 1. Since the product contains an arbitrary number of factors close to this limit, if only q is sufficiently close to 1, the product will be smaller than any given positive number. ■

Roughly speaking, the unit vectors

$$\hat{e}_{q^2}(\lambda) = \frac{e_{q^2}(\lambda)}{\sqrt{e_{q^2}^{|\lambda|^2}}}$$

'become orthogonal' if q tends to ± 1. We use this property in order to approximate irreducible states on $\mathcal{A}_{\pm 1}$.

Proposition 6.3 *Let (φ, x, y) be a point in $SU(2)$ such that $0 < x^2 + y^2 < 1$. Furthermore, let $c = (c_1, c_2)$ be a unit vector in $\mathbf{C}^2$. Denote by $c \odot \hat{e}_{q^2}(\lambda)$ the unitvector*

$$c \odot \hat{e}_{q^2}(\lambda) = \frac{c_1\hat{e}_{q^2}(\lambda) + c_2\hat{e}_{q^2}(-\lambda)}{\sqrt{1 + (\overline{c_1}c_2 + \overline{c_2}c_1)\frac{e_{q^2}^{-|\lambda|^2}}{e_{q^2}^{|\lambda|^2}}}}$$

in h_0 where $\lambda = \sqrt{1 - x^2 - y^2}e^{i\varphi}$, $\chi = \arg(x + iy)$. Then we have

(i)

$$\lim_{q\to 1} \langle \hat{e}_{q^2}(\lambda) | \rho_\chi | \hat{e}_{q^2}(\lambda) \rangle = \delta_{\varphi xy}.$$

(ii)

$$\lim_{q\to -1} \langle c \odot \hat{e}_{q^2}(\lambda) | \rho_\chi | c \odot \hat{e}_{q^2}(\lambda) \rangle = \langle c | \hat{\delta}_{\varphi xy} | c \rangle.$$

Proof Assume, for the moment, that $\chi = 0$. Then, since ρ_0 does not distinguish between γ and γ^* and by hermiticity, it follows that it is sufficient to prove the statement for $\gamma^n \alpha^m$ with $n, m \in \mathbf{N}_0$.

We have $\rho_0(\alpha) e_{q^2}(\lambda) = \lambda e_{q^2}(\lambda)$ and $\rho_0(\gamma) e_{q^2}(\lambda) = e_{q^2}(q\lambda)$ (cf. Appendix B). Therefore, α^m gives a factor $(\pm\lambda)^m$ and γ^n gives a factor q^n in the argument of the q–coherent state. Thus, denoting the normalization factor in the second case by $|c|_\odot$, we have to compute the expressions

$$\lim_{q\to 1} \lambda^m \frac{\langle e_{q^2}(\lambda) | e_{q^2}(q^n\lambda)\rangle}{e_{q^2}^{|\lambda|^2}} \quad = \quad \lim_{q\to 1} \lambda^m \frac{e_{q^2}^{q^n|\lambda|^2}}{e_{q^2}^{|\lambda|^2}}$$

in the first case, and

$$\lim_{q\to -1} \frac{1}{|c|_\odot^2 e_{q^2}^{|\lambda|^2}} \Big(|c_1|^2 \lambda^m \langle e_{q^2}(\lambda) | e_{q^2}(q^n\lambda)\rangle \; + \; \overline{c_1} c_2 (-\lambda)^m \langle e_{q^2}(\lambda) | e_{q^2}(-q^n\lambda)\rangle \; +$$

$$+ \; \overline{c_2} c_1 \lambda^m \langle e_{q^2}(-\lambda) | e_{q^2}(q^n\lambda)\rangle \; + \; |c_2|^2 (-\lambda)^m \langle e_{q^2}(-\lambda) | e_{q^2}(-q^n\lambda)\rangle \Big)$$

$$= \lim_{q\to 1} \frac{(|c_1|^2\lambda^m + |c_2|^2(-\lambda)^m) e_{q^2}^{q^n(-1)^n|\lambda|^2} + (\overline{c_1}c_2(-\lambda)^m + \overline{c_2}c_1\lambda^m) e_{q^2}^{q^n(-1)^{n+1}|\lambda|^2}}{|c|_\odot^2 e_{q^2}^{|\lambda|^2}}$$

in the second case where we transformed the limit $q \to -1$ into a limit $q \to 1$. The recursion formula for the q–exponential reads

$$e_{q^2}^{\pm q^{n+2}|\lambda|^2} \quad = \quad (1 \mp q^n|\lambda|^2) e_{q^2}^{\pm q^n|\lambda|^2}.$$

In the first case we see that

$$\lim_{q\to 1} \frac{e_{q^2}^{q^{2n}|\lambda|^2}}{e_{q^2}^{|\lambda|^2}} \; = \; (1-|\lambda|^2)^n \quad \text{and} \quad \lim_{q\to 1} \frac{e_{q^2}^{q^{2n+1}|\lambda|^2}}{e_{q^2}^{|\lambda|^2}} \; = \; (1-|\lambda|^2)^n \lim_{q\to 1} \frac{e_{q^2}^{q|\lambda|^2}}{e_{q^2}^{|\lambda|^2}},$$

if the latter limit exists. For the second case we consider

$$e_{q^2}^{-q^n|\lambda|^2} \quad < \quad e_{q^2}^{-q^{n+1}|\lambda|^2} \tag{6.2}$$

for $q \in (0,1)$. Since $|\lambda| \neq 0$, Proposition 6.2 yields

$$\frac{e_{q^2}^{-|\lambda|^2}}{e_{q^2}^{|\lambda|^2}} \quad \to \quad 0$$

as $q \to 1$. Therefore, $|c|_\odot$ converges to 1. It follows by (6.2) and the recursion formula that

$$\frac{e_{q^2}^{\pm(-1)^n q^n|\lambda|^2}}{e_{q^2}^{|\lambda|^2}} \quad \to \quad 0 \;\; \text{for} \;\; \pm(-1)^n = -1.$$

For $\pm(-1)^n = 1$ we obtain the same numbers as in the first case. Thus, it remains to calculate

$$\lim_{q\to 1} \frac{e_{q^2}^{q|\lambda|^2}}{e_{q^2}^{|\lambda|^2}}.$$

We define

$$F_q(x) \quad = \quad \frac{e_{q^2}^{qx}}{e_{q^2}^{x}}.$$

By looking at the factors $\frac{1-q^{2k}x}{1-q^{2k+1}x}$ of the infinite product we easily find that F_q is a strongly decreasing function of $x \in (0,1)$, i.e. $F_q(x)F_q(qx) < F_q(qx)F_q(qx) < F_q(qx)F_q(q^2x)$. Since $F_q(x)F_q(qx) = 1-x$, we have

$$1 - x \; < \; F_q(qx)^2 \; < \; 1 - qx. \tag{6.3}$$

Therefore,

$$\lim_{q\to 1} F_q(x) \;=\; \lim_{q\to 1} \frac{1-x}{F_q(qx)} \;=\; \sqrt{1-x},$$

and, henceforth,

$$\lim_{q\to 1} \frac{e_{q^2}^{q^n|\lambda|^2}}{e_{q^2}^{|\lambda|^2}} \;=\; \sqrt{1-|\lambda|^2}^{\,n}.$$

We insert this and obtain

$$\lim_{q\to 1} \langle \hat{e}_{q^2}(\lambda)|\rho_0(\gamma^n\alpha^m)|\hat{e}_{q^2}(\lambda)\rangle \;=\; \lambda^m \sqrt{1-|\lambda|^2}^{\,n}$$

and

$$\lim_{q\to -1} \langle c\odot\hat{e}_{q^2}(\lambda)|\rho_0(\gamma^n\alpha^m)|c\odot\hat{e}_{q^2}(\lambda)\rangle \;=\; \lambda^m \sqrt{1-|\lambda|^2}^{\,n} \begin{cases} |c_1|^2 + |c_2|^2(-1)^m & \text{for } n \text{ even} \\ \overline{c_1}c_2(-1)^m + \overline{c_2}c_1 & \text{otherwise} \end{cases}$$

which is the claimed result for $\chi = 0$.

The general case can be obtained by multiplying with the factor $e^{i\chi}$ for each γ and $e^{-i\chi}$ for each γ^*. ■

The excluded cases $|\lambda_0| = 0, 1$ can be obtained as the limit $\lambda \to \lambda_0$ of the above expressions. In our next step we include these cases by replacing λ with a function $\lambda(q)$ which converges to λ_0 as q tends to ± 1.

Notice that the approximating expressions, in (i) and (ii) of the foregoing Proposition, are analytic functions in the variable $|\lambda|^2$ and can be continued analytically to $|\lambda|^2 < \frac{1}{1-q}$. For a given $\lambda_0 = e^{i\varphi}|\lambda_0| \in \overline{U_1(0)}$ we introduce the function

$$\lambda(q) \;=\; \begin{cases} \lambda_0 & \text{for } \left(1 - \frac{1}{2}|\lambda_0|^2\right)^{-\frac{\ln 2}{\ln q^2}} < 1 - q^2 \\ e^{i\varphi}(|\lambda_0| + \delta\lambda) & \text{otherwise} \end{cases}$$

on $\frac{1}{2} \le q^2 < 1$, where $\delta\lambda$ is a non-negative real number such that

$$\left(1 - \frac{1}{2}(|\lambda_0| + \delta\lambda)^2\right)^{-\frac{\ln 2}{\ln q^2}} \;=\; 1 - q^2.$$

We explain why this is well-defined. Notice that $\kappa = -\frac{\ln 2}{\ln q^2}$ is a positive real number which tends to infinity as q tends to ± 1. Therefore, if the first case is not true, it is always possible to find a unique $\delta\lambda$ such that the second case is fulfilled. The lower boundary for q^2 guarantees that $|\lambda(q)| \le 1$. (If $q^2 = \frac{1}{2}$, we have $\kappa = 1$ and $1 - q^2 = \frac{1}{2}$, i.e. $|\lambda_0| + \delta\lambda = 1$.) Obviously, $\delta\lambda$ converges to 0 as $q \to \pm 1$. The worst case for this convergence is $\lambda_0 = 0$, i.e. the convergence is uniformly in λ_0. We collect the properties of $\lambda(q)$.

Proposition 6.4 *For the function $\lambda(q)$ on $\frac{1}{2} \le q^2 < 1$ which is assigned to any $\lambda_0 =\in \overline{U_1(0)}$ by the above definition the following holds.*

(i) $$0 < |\lambda(q)| \leq 1.$$

(ii) $$\lambda(q) \text{ is continuous.}$$

(iii) $$\lim_{q\to\pm 1} \lambda(q) = \lambda_0 \text{ uniformly in } \lambda_0.$$

(iv) *For any $\lambda_0 \neq 0$ we even have $\lambda(q) = \lambda_0$ for q sufficiently close to ± 1.*

(v) $$\left(1 - \frac{1}{2}|\lambda(q)|^2\right)^{-\frac{\ln 2}{\ln q^2}} \leq 1 - q^2 \text{ for all } \lambda_0.$$

Having done these preparations, we can prove the following

Theorem 6.5 *Let (φ, x, y) be a point in $SU(2)$ and set $\lambda_0 = \sqrt{1 - x^2 - y^2}e^{i\varphi}$, $\chi = \arg(x + iy)$. Then we have*

(i)

$$\lim_{q\to 1} \langle \hat{e}_{q^2}(\lambda(q))|\rho_\chi|\hat{e}_{q^2}(\lambda(q))\rangle \quad = \quad \delta_{\varphi xy}.$$

(ii)

$$\lim_{q\to -1} \langle c \odot \hat{e}_{q^2}(\lambda(q))|\rho_\chi|c \odot \hat{e}_{q^2}(\lambda(q))\rangle \quad = \quad \langle c|\hat{\delta}_{\varphi xy}|c\rangle.$$

uniformly *in λ_0 and in the unit vector $c = (c_1, c_2) \in \mathbf{C}^2$.*

Proof Consider the proof of Proposition 6.3. The expressions to be calculated are the same except that the fixed number λ is replaced everywhere by $\lambda(q)$. The expressions are linear combinations of the functions $\frac{1}{|c|^2_\odot}$, $\frac{F_q(|\lambda|^2)}{|c|^2_\odot}$, and $\frac{e_{q^2}^{-|\lambda|^2}}{|c|^2_\odot e_{q^2}^{|\lambda|^2}}$ where the coefficients are polynomials $P(q^2, \lambda, \overline{\lambda})$. If we insert $\lambda(q)$, these coefficients assume their limits $P(\pm 1, \lambda_0, \overline{\lambda_0})$ uniformly in λ_0 because λ_0 does so. Of course, $|c|^2_\odot$ assumes its limit 1 uniformly in λ_0, if $\frac{e_{q^2}^{-|\lambda(q)|^2}}{e_{q^2}^{|\lambda(q)|^2}}$ does so. Thus, our proof is complete if we show that $\lim_{q\to 1} F_q(|\lambda(q)|^2) = \sqrt{1 - |\lambda_0|^2}$ and $\lim_{q\to 1} \frac{e_{q^2}^{-|\lambda(q)|^2}}{e_{q^2}^{|\lambda(q)|^2}} = 0$ uniformly in λ_0.

Consider (6.3) which holds for $0 < x < 1$. We obtain

$$\frac{1-x}{\sqrt{1-qx}} \;<\; F_q(x) \;<\; \frac{1-x}{\sqrt{1-x}}.$$

This inequality also holds for $x = 1$, if we change the $<$ signs to $\leq$. Therefore,

$$\begin{aligned} |\sqrt{1-x} - F_q(x)| \quad &\leq \quad (1-x)\left(\frac{1}{\sqrt{1-x}} - \frac{1}{\sqrt{1-qx}}\right) \quad = \quad \frac{1-x}{\sqrt{1-x}\sqrt{1-qx}}(\sqrt{1-qx} - \sqrt{1-x}) \\ &= \quad \sqrt{\frac{1-x}{1-qx}}\frac{x(1-q)}{\sqrt{1-qx} + \sqrt{1-x}}. \end{aligned}$$

We easily check that the function $\frac{x(1-x)}{1-qx}$ of x has a unique maximum on $(0, 1)$ at $x_0 = \frac{1-\sqrt{1-q}}{q}$. Thus, $1 - x_0 = \sqrt{1-q}\frac{1-\sqrt{1-q}}{q}$ and $1 - qx_0 = \sqrt{1-q}$. Therefore,

$$\sqrt{\frac{x(1-x)}{1-qx}} \quad \leq \quad \frac{1-\sqrt{1-q}}{q} \quad = \quad x_0.$$

We obtain

$$|\sqrt{1-x} - F_q(x)| \;\leq\; \frac{1-\sqrt{1-q}}{q}\,\frac{\sqrt{x}(1-q)}{\sqrt{1-qx}+\sqrt{1-x}} \;\leq \frac{\sqrt{1-q}}{q}.$$

This is the uniform convergence of $F_q(x) \to \sqrt{1-x}$.

κ was given by $\kappa = -\frac{\ln 2}{\ln q^2}$. Obviously we have $q^{2\kappa} = \frac{1}{2}$. By $[\kappa]$ we denote the greatest integer less than or equal to κ. We have

$$q^{2k} \;\geq\; \frac{1}{2} \text{ for } k \leq [\kappa].$$

From (6.1) we obtain

$$\left(\frac{e_{q^2}^{-|\lambda|^2}}{e_{q^2}^{|\lambda|^2}}\right)^2 \;=\; \prod_{k=0}^{\infty}\left(1 - \frac{4q^{2k}|\lambda|^2}{(1+q^{2k}|\lambda|^2)^2}\right) \;\leq\; \prod_{k=0}^{\infty}(1-q^{2k}|\lambda|^2)$$
$$\leq\; \prod_{k=0}^{[\kappa]}(1-q^{2k}|\lambda|^2) \;\leq\; \left(1-\frac{1}{2}|\lambda|^2\right)^{[\kappa]+1} \;\leq\; \left(1-\frac{1}{2}|\lambda|^2\right)^{\kappa}$$

for all λ. We insert $\lambda(q)$ and obtain

$$\frac{e_{q^2}^{-|\lambda(q)|^2}}{e_{q^2}^{|\lambda(q)|^2}} \;\leq\; \sqrt{\left(1-\frac{1}{2}|\lambda(q)|^2\right)^{\kappa}} \;\leq\; \sqrt{1-q^2} \;=\; \sqrt{1+q}\sqrt{1-q}$$

for $q^2 > \frac{1}{2}$. This is the uniform convergence of $\frac{e_{q^2}^{-|\lambda(q)|^2}}{e_{q^2}^{|\lambda(q)|^2}} \to 0$. ■

Notice that $\langle c|\hat{\delta}_{\varphi xy}|c\rangle$ can be written in the form

$$\langle c|\hat{\delta}_{\varphi xy}|c\rangle \;=\; \mathrm{Tr}\;|c\rangle\langle c|\hat{\delta}_{\varphi xy} \;=\; \mathrm{Tr}\;\hat{c}\hat{\delta}_{\varphi xy},$$

where we introduced the matrix $\hat{c} = \begin{pmatrix} |c_1|^2 & c_1\overline{c_2} \\ c_2\overline{c_1} & |c_2|^2 \end{pmatrix}$. If we assign to any matrix $\widehat{m}$ of unit trace the operator

$$\mathcal{M}_\lambda(\widehat{m}) \;=\; \frac{1}{1+(m_{12}+m_{21})\frac{e_{q^2}^{-|\lambda|^2}}{e_{q^2}^{|\lambda|^2}}}\Big(|\hat{e}_{q^2}(\lambda)\rangle\, m_{11}\,\langle\hat{e}_{q^2}(\lambda)| + |\hat{e}_{q^2}(\lambda)\rangle\, m_{12}\,\langle\hat{e}_{q^2}(-\lambda)| +$$
$$+\,|\hat{e}_{q^2}(-\lambda)\rangle\, m_{21}\,\langle\hat{e}_{q^2}(\lambda)| + |\hat{e}_{q^2}(-\lambda)\rangle\, m_{22}\,\langle\hat{e}_{q^2}(-\lambda)|\Big)$$

in $\mathcal{B}(h_0)$ which also has trace 1, then we obtain

$$\mathrm{Tr}\;\mathcal{M}_\lambda(\hat{c})\rho_\chi \;=\; \langle c\odot\hat{e}_{q^2}(\lambda)|\rho_\chi|c\odot\hat{e}_{q^2}(\lambda)\rangle.$$

Since any positive 2×2–matrix can be decomposed into the sum of at most two dyadic products $|c\rangle\langle c|$ and the normalization factors converge to 1 (uniformly in λ_0 if λ is replaced by $\lambda(q)$), we see that for any positive matrix $\widehat{m}$ of unit trace we obtain a family $\mathrm{Tr}\;\mathcal{M}_{\lambda(q)}(\widehat{m})\rho_\chi$ of states on $\mathcal{A}_q$ such that

$$\lim_{q\to -1}\mathrm{Tr}\;\mathcal{M}_{\lambda(q)}(\widehat{m})\rho_\chi \;=\; \mathrm{Tr}\;\widehat{m}\,\hat{\delta}_{\varphi xy}$$

uniformly in (φ, x, y) and $\widehat{m}$.

Now let $\varphi_\pm$ be arbitrary states on $\mathcal{A}_{\pm 1}$. By the last two sections of the preceeding chapter we know that there is a measure $d\nu^+$ on $SU(2)$ in the first case, and a matrix measure $d\hat{\nu}^- = \hat{n}\,d\nu^-$ on

$SU(2)$, where $\hat{n}$ is a ν^-–integrable function with values in the positive matrices of unit trace, in the second case, such that

$$\varphi_+ = \int_{SU(2)} \delta_{\varphi xy}\, d\nu^+_{\varphi xy},$$

and

$$\varphi_- = \int_{SU(2)} \mathrm{Tr}\ \hat{n}(\varphi xy)\hat{\delta}_{\varphi xy}\, d\nu^-_{\varphi xy}$$

respectively. Obviously $\langle \hat{e}_{q^2}(\lambda(q))|\rho_\chi|\hat{e}_{q^2}(\lambda(q))\rangle$ and $\mathrm{Tr}\ \mathcal{M}_{\lambda(q)}(\hat{n}(\varphi xy))\rho_\chi$ are $\nu^\pm$–integrable functions which converge uniformly in (φ, x, y) if q tends to ± 1. Therefore, we obtain by an application of the theorem of *majorized convergence* that the order of integration over $SU(2)$ and the limit $q \to \pm 1$ can be exchanged. The following is a simple corollary.

Theorem 6.6 *Arbitrary states $\varphi_\pm$ on $\mathcal{A}_{\pm 1}$ can be approximated by states on $\mathcal{A}_q$. We have*

$$\varphi_+ = \lim_{q\to 1} \int_{SU(2)} \langle \hat{e}_{q^2}(\lambda(q))|\rho_\chi|\hat{e}_{q^2}(\lambda(q))\rangle\, d\nu^+_{\varphi xy}$$

and

$$\varphi_- = \lim_{q\to -1} \int_{SU(2)} \mathrm{Tr}\ \mathcal{M}_{\lambda(q)}(\hat{n}(\varphi xy))\rho_\chi\, d\nu^-_{\varphi xy}.$$

The approximation is uniformly in $\varphi_\pm$.

So far, we know how to approximate arbitrary states (or more generally arbitrary positive functionals on $\mathcal{A}_{\pm 1}$ by states (positive functionals of the same norm) on $\mathcal{A}_q$.

6.2 Conditionally positive functionals

First let us agree on some notation. If ψ_q is any conditionally positive functional on $\mathcal{A}_q$, we denote by $\Psi_q = \psi_q \circ \mathcal{E}_q$ the corresponding raised functional on $\mathcal{F}$. By $\mathcal{K}_i^q \subset \mathcal{F}, i = 1, 2$, we denote the sets consisting of all $a \in \mathcal{F}$ such that $\mathcal{E}_q(a) \in K_i$ for the corresponding $q \in [-1, 1]$. Notice that $\delta_\varphi \circ \mathcal{E}_q$ does not depend on q. Consequently we can define the mappings $\delta_\mathcal{F} = \delta \circ \mathcal{E}_q$ and $\delta'_\mathcal{F} = \delta' \circ \mathcal{E}_q$. Since no confusion can arise, we will omit the subscript $\mathcal{F}$. The same will be done for the projections $(Id - \delta_\mathcal{F}\mathbf{1}) = (Id - \delta\mathbf{1}) \circ \mathcal{E}_q$ and $\mathcal{P}_\mathcal{F} = \mathcal{P} \circ \mathcal{E}_q$. However, notice that indeed $\mathcal{P}_{-1} \circ \mathcal{E}_{-1} = \mathcal{P} \circ \mathcal{E}_q$, but $\mathcal{P}_1 \circ \mathcal{E}_1 < \mathcal{P} \circ \mathcal{E}_q$ and, of course, $\mathcal{P}_1 \circ \mathcal{E}_1$ does not vanish on $\mathcal{I}_q$ unless $q = 1$. Thus, we can omit the superscript q in $\mathcal{K}_1^q = \mathcal{K}_1$ for all q and in $\mathcal{K}_2^q = \mathcal{K}_2$ for $q \neq 1$.

Clearly, a hermitian functional ψ_q is conditionally positive if and only if the raised functional Ψ_q is positive on $\mathcal{K}_1$. Therefore, we obtain the analogue of Proposition 6.1.

Proposition 6.7 *Let ψ_q be a family of conditionally positive functionals on $\mathcal{A}_q$ for $q \in (-1, 1)$. If the limits $\Psi_{\pm 1} = \lim_{q\to\pm 1} \Psi_q$ exist, there are conditionally positive functionals $\psi_{\pm 1}$ on $\mathcal{A}_{\pm 1}$, such that $\psi_{\pm 1} = \lim_{q\to\pm 1} \psi_q$.*

Notice that

$$\begin{aligned}
\mathcal{F} &= \mathcal{K}_1 \oplus \mathbf{C}\mathbf{1} \\
\mathcal{K}_1 &= \mathcal{K}_2 \oplus \mathbf{C}\frac{\alpha - \alpha^*}{2i} \\
\mathcal{K}_2 &= \mathcal{K}_2^1 \oplus \mathbf{C}\frac{\gamma + \gamma^*}{2} \oplus \mathbf{C}\frac{\gamma - \gamma^*}{2i} \\
\mathcal{K}_2^1 &= \mathcal{K}^2 \oplus \mathbf{C}\frac{\beta + \beta^*}{2}
\end{aligned}$$

where we defined $\mathcal{K}^2 = \overline{\text{lin}}(\mathcal{K}_1 \cdot \mathcal{K}_1)$.

Recall Definition (1.5) of the set G. By $\mathcal{G}$ we denote the corresponding set of generators of $\mathcal{F}$. We denote by

$$\mathcal{G}_n = \bigcup_{k=2}^{n} \mathcal{G}^k$$

for $n \geq 2$ the set of all monomials having length between 2 and n. In the sequel, we will approximate general conditionally positive functionals on $\mathcal{A}_{\pm 1}$ by sequences ψ_n of conditionally positive functionals on $\mathcal{A}_{q_n}$ where $q_n \to \pm 1$. The approximaton will be such that the deviation of ψ_n to its limit is less than $C\frac{1}{n}$ for all $g \in \mathcal{G}_n$ where $C > 0$ is an appropiate constant. Then ψ_n converges for all $a \in \mathcal{K}^2$, because $a \in \overline{\text{lin}}(\mathcal{G}_{n_0})$ for some n_0. On the other hand, since the $\mathcal{G}_n$ are finite sets, a limit, which exists for any $a \in \mathcal{K}^2$, can be performed uniformly on $\mathcal{G}_n$ (for fixed n).

By $\psi^{\pm}_{\lambda_0\chi}$ we denote the conditonally positive functionals

$$\begin{aligned} \psi^+_{\lambda_0\chi} &= \langle \hat{e}_{q^2}(\lambda(q)) | \rho_\chi \circ \mathcal{P} | \hat{e}_{q^2}(\lambda(q)) \rangle \\ \psi^-_{\lambda_0\chi} &= \text{Tr } \mathcal{M}_{\lambda(q)}(\widehat{m}(\lambda_0, \chi)) \rho_\chi \circ \mathcal{P} \end{aligned}$$

on $\mathcal{A}_q$, where $\widehat{m}$ is a function on $SU(2)$ with values in the positive 2×2–matrices of unit trace. We have

$$\begin{aligned} \lim_{q \to 1} \psi^+_{\lambda_0\chi} &= \delta_{\varphi xy} \circ \mathcal{P} = \delta^+_{\varphi xy} \\ \lim_{q \to -1} \psi^-_{\lambda_0\chi} &= \text{Tr } \widehat{m}(\lambda_0, \chi) \hat{\delta}_{\varphi xy} \circ \mathcal{P} = \delta^-_{\varphi xy} \end{aligned}$$

uniformly on $SU(2)$. Notice, however, that the first expression differs from $\delta_{\varphi xy} \circ \mathcal{P}_1$ by the functional $x\delta'^x + y\delta'^y$. We will be concerned with this problem later.

Denote by M_n the set

$$M_n = \left\{ (\varphi xy) \in SU(2) \middle| x^2 + y^2 \geq \frac{1}{n} \right\}.$$

Let $\psi_\pm$ be the conditionally positive functionals on $\mathcal{A}_{\pm 1}$ given by the LÉVY-KHINTCHINE formulae

$$\begin{aligned} \psi_+ &= \int_{SU(2)} \delta_{\varphi xy} \circ \mathcal{P}_1 \, d\mu^+_{\varphi xy} \\ \psi_- &= \int_{SU(2)} \text{Tr } \widehat{m}(\lambda_0, \chi) \hat{\delta}_{\varphi xy} \circ \mathcal{P} \, d\mu^-_{\varphi xy}, \end{aligned} \tag{6.4}$$

where $d\mu^\pm$ are measures having no atom at identity and fulfilling the necessary conditions

$$M^\pm = \int_{SU(2)} (x^2 + y^2) \, d\mu^\pm_{\varphi xy} < \infty$$

and $\widehat{m}$ is μ^-–integrable.

Proposition 6.8 *There are monotone sequences $\{q^\pm_n\}_{n \geq 2}$ with $-1 < q^\pm_n < 1$ and $\lim_{n \to \infty} q^\pm_n = \pm 1$, such that*

$$\Psi_\pm(a) = \lim_{n \to \infty} \int_{M_n} \Psi^\pm_{\lambda_0\chi}(a) \, d\mu^\pm_{\varphi xy}$$

for all sequences $\{q_n\}_{n \geq 2}$ with $q^+_n < q_n < 1$ and $-1 < q_n < q^-_n$, respectively, and all $a \in \mathcal{K}^2$.

N.B.: The dependence on q_n is hidden in the raised conditionally positive functionals $\Psi^\pm_{\lambda_0\chi}$ which vanish on $\mathcal{I}_{q_n}$.

Proof Since on $\mathcal{K}^2$ both the projections $\mathcal{P}$ and $\mathcal{P}_1$ dissappear, the expressions converge by (6.4) to the stated values, if we replace $\Psi^{\pm}_{\lambda_0\chi}$ by their limits $\delta^{\pm}_{\varphi xy}$.

On the other hand, the limit of the integrands can be performed uniformly on $\mathcal{G}_n$ and $SU(2)$. We choose $q_n^{\pm}$ such that

$$|\Psi^{\pm}_{\lambda_0\chi}(g) - \delta^{\pm}_{\varphi xy}(g)| \quad < \quad \frac{1}{n^2} \;\leq\; \frac{x^2+y^2}{n}$$

for all $(\varphi, x, y) \in M_n$, $g \in \mathcal{G}_n$, and q_n closer to ± 1 than $q_n^{\pm}$. We obtain

$$\int_{M_n} |\Psi^{\pm}_{\lambda_0\chi}(g) - \delta^{\pm}_{\varphi xy}(g)|\, d\mu^{\pm}_{\varphi xy} \quad < \quad \frac{M^{\pm}}{n}.$$

This is our claimed convergence. Of course, $q_n^{\pm}$ can be chosen monotone. ∎

Obviously, both the left- and right-hand side vanish on $\mathbf{1}$ and $\frac{\alpha-\alpha^*}{2i}$. And by Relation (e) we see that also $\Psi_{\pm}\left(\frac{\beta+\beta^*}{2}\right)$ is approximated properly by the right-hand side. Thus, we immediately can extend the foregoing Proposition to

$$\mathcal{K}_{\mathcal{P}} \quad = \quad \mathcal{K}^2 \oplus \mathbf{C}\frac{\beta+\beta^*}{2} \oplus \mathbf{C}\frac{\alpha-\alpha^*}{2i} \oplus \mathbf{C}\mathbf{1}$$

which is precisely the set, on which $\mathcal{P}$ and $\mathcal{P}_1$ coincide.

In the case when $q \to -1$ we even obtain by Relations (ã) and (ã)* that $\Psi_-(\gamma^{(*)})$ is approximated by the right-hand side. Henceforth, the approximation is valid on the whole of the algebra $\mathcal{F}$.

In the case when $q \to 1$ we have to add something which converges in a sufficiently uniform way to the functional

$$\begin{aligned}\delta_{\varphi xy} \circ \mathcal{P}_1 - \delta_{\varphi xy} \circ \mathcal{P} \quad &= \quad -\left(\delta'^x \delta_{\varphi xy}\left(\frac{\gamma+\gamma^*}{2}\right) + \delta'^y \delta_{\varphi xy}\left(\frac{\gamma-\gamma^*}{2i}\right)\right) \\ &= \quad -(x\delta'^x + y\delta'^y) \quad = \quad \delta'^r\end{aligned}$$

with $r = (0, -x, -y)$. This functional will also be needed in order to write down the general Gaussian part.

Proposition 6.9 *There are positive numbers ϵ_n, and a monotone function $q_0(t)$ on $(0,1)$ with $0 < q_0(t) < 1$ and $\lim_{t\to 0} q_0(t) = 1$, such that*

$$\left| \frac{\Psi^{+}_{\left(e^{it\varphi}\sqrt{1-t^2(x^2+y^2)}\right)(\chi)}(g)}{t} - \delta'^r(g) \right| \quad < \quad \frac{1}{n^2}$$

for all $g \in \mathcal{G} \cup \mathcal{G}_n$, $r \in SU(2)$, $t < \epsilon_n$ and all functions $q(t)$ such that $q_0(t) < q(t) < 1$.

N.B.: Actually, r is a vector in $\mathbf{R}^3$. By $r \in SU(2)$ we mean that the components (φ, x, y) of r describe an element of $SU(2)$ where the parameter φ lies in $[-\pi, \pi)$.

Proof First choose ϵ_n such that

$$\left| \frac{\delta_{(t\varphi)(tx)(ty)}(g)}{t} - \delta'^r \right| < \frac{1}{2n^2}$$

for all $g \in \mathcal{G} \cup \mathcal{G}_n$, $r \in SU(2)$, and $t < \epsilon_n$. This is possible, because $\mathcal{G} \cup \mathcal{G}_n$ is finite, $g \in \mathcal{K}_1$, i.e. $\delta(g) = 0$, and $r \in [-\pi, \pi]^3 \subset \mathbf{R}^3$.

Then choose $q_0(t)$, such that

$$\left| \frac{\Psi^{+}_{\left(e^{it\varphi}\sqrt{1-t^2(x^2+y^2)}\right)(\chi)}(g) - \delta_{(t\varphi)(tx)(ty)}(g)}{t} \right| < \frac{1}{2n^2}$$

for all $g \in \mathcal{G} \cup \mathcal{G}_n$, $t \in (0,1)$, and $q \in (q_0(t), 1)$. This is possible, because the approximation of $\delta_{\varphi xy}$ is uniform on $SU(2)$. Of course, $q_0(t)$ can be chosen monotone. ■

EXAMPLE 6.1 *Notice that* $\Psi^+_{\left(e^{it\varphi}\sqrt{1-t^2(x^2+y^2)}\right)(\chi)}(g)$ *is analytical in* t^2*. Thus, the derivative with respect to t is zero at* $t = 0$*. This is, why we cannot perform the t–limit first for fixed q and the limit* $q \to 1$ *afterwards. If this was possible, we could say immediately what we understand by a derivation in the direction of* $\gamma^{(*)}$ *for* $|q| < 1$*. (Just choose* $r = (0, 1, \pm 1)$*.) In general, we would expect such a derivation to be different from 0 on* $\gamma^{(*)}$ *and to vanish at least on all monomials having more than one factor* $\gamma^{(*)}$*. However, there cannot be such a functional on* $\mathcal{A}_q$ *which is also conditionally positive. (The functionals given in [35] which replace the usual derivations are not even hermitian.)*

Consider a conditionally positive functional ψ_γ *on* $\mathcal{A}_q$ *which is 0 on* $\gamma^*\gamma = \gamma\gamma^*$*. For the corresponding cocycle* η_γ *we obtain* $\|\eta_\gamma(\gamma)\|^2 = \psi_\gamma(\gamma^*\gamma) = \|\eta_\gamma(\gamma^*)\|^2 = 0$*. On the other hand, we have* $\gamma^{(*)} = \frac{q\beta\gamma^{(*)} - \gamma^{(*)}\beta}{1-q}$*. Therefore, also*

$$\psi_\gamma(\gamma^{(*)}) = \frac{q\langle \eta_\gamma(\beta^*) | \eta_\gamma(\gamma^{(*)})\rangle - \langle \eta_\gamma(\gamma^{(*)*}) | \eta_\gamma(\beta)\rangle}{1-q}$$

must be 0. This remains true also for $q = -1$*.*

Now we use Proposition 6.9 in order to approximate $\Psi_+(\gamma^{(*)})$.

Corollary 6.10 *We have*

$$\Psi_+ \quad = \quad \lim_{n\to\infty} \int_{M_n} \left(\Psi^+_{\lambda_0\chi} + \frac{\Psi^+_{\left(\sqrt{1-t_n^2(x^2+y^2)}\right)(-\chi)}}{t_n} \right) d\mu^+_{\varphi xy}$$

for all sequences $\{t_n\}_{n\geq 2}$*, and* $\{q_n\}_{n\geq 2}$ *with* $0 < t_n < \epsilon_n$*, and* $\max(q_n^+, q_0(t_n)) < q_n < 1$*.*

PROOF On $\gamma^{(*)}$ the difference between the first summand in the integrand and its limit $x \pm iy$ can be estimated from above by $\frac{1}{n^2}$. The difference between the second summand and its limit $-(x \pm iy)$ can also be estimated by $\frac{1}{n^2}$. Therefore, the integrals over these differenes converge to 0.

On $\mathcal{K}_\mathcal{P}$ the second term converges to

$$\int_{SU(2)} \delta'^r \, d\mu^+_{\varphi xy}$$

with $r = (0, -x - y)$ which can be seen by the same estimates as for the first term. Since $\delta'^r = 0$ on $\mathcal{K}_\mathcal{P}$, this limit is 0. ■

We collect the results obtained so far.

Theorem 6.11 *There are universal sequences* $\{t_n\}_{n\geq 2}$*, and* $\{q_n^\pm\}_{n\geq 2}$*, such that any pair of conditionally positive functionals* $\psi_\pm$ *on* $\mathcal{A}_{\pm 1}$ *with integral representation (6.4) can be approximated as limits of conditionally positive functionals on* $\mathcal{A}_{q_n^\pm}$ *in the form*

$$\begin{aligned} \psi_+ \quad &= \quad \lim_{n\to\infty} \int_{M_n} \left(\psi^+_{\lambda_0\chi} + \frac{\psi^+_{\left(\sqrt{1-t_n^2(x^2+y^2)}\right)(-\chi)}}{t_n} \right) d\mu^+_{\varphi xy} \\ \psi_- \quad &= \quad \lim_{n\to\infty} \int_{M_n} \psi^-_{\lambda_0\chi} \, d\mu^-_{\varphi xy}. \end{aligned}$$

Since δ, δ', and δ'' do not depent on q, the problem of approximating a GAUSSian part on $\mathcal{A}_{-1}$ is already solved. By Proposition 6.9 we also solved the problem of approximating the functional δ'^r on $\mathcal{A}_1$. (The general case $r \in \mathbf{R}^3$ can be reduced to the case when $r \in SU(2)$.) Thus, up to this moment we are able to express conditionally positive functionals ψ_+, and ψ_- on $\mathcal{A}_1$, and $\mathcal{A}_{-1}$ having no GAUSSian and anti-GAUSSian parts, respectively, by limits of conditionally positive functionals on $\mathcal{A}_q$.

Now we come to the last yet missing building blocks δ''^r and $\underline{\psi}_r$ for $r \in \mathbf{R}^3$ and $r \in \mathbf{C} \times \mathbf{R}^2$ which are needed to express the GAUSSian and anti-GAUSSian part, respectively. On $\mathcal{K}^2$ we have

$$\underline{\psi}_r \quad = \quad \lim_{t\to 0} \left\langle \frac{\hat{e}_1 + t\varphi\hat{e}_2}{t} \middle| \hat{\delta}_{0(tx)(ty)} \middle| \frac{\hat{e}_1 + t\varphi\hat{e}_2}{t} \right\rangle$$

and, of course,

$$\frac{\delta''^r}{2} \quad = \quad \lim_{t\to 0} \frac{\delta_{(t\varphi)(tx)(ty)}}{t^2}.$$

If we set $\widehat{m} = \frac{1}{1+|\varphi|^2 t^2} \begin{pmatrix} 1 & \overline{\varphi} t \\ \varphi t & |\varphi|^2 t^2 \end{pmatrix}$ in order to express the state

$$\frac{\left\langle \frac{\hat{e}_1 + t\varphi\hat{e}_2}{t} \middle| \bullet \middle| \frac{\hat{e}_1 + t\varphi\hat{e}_2}{t} \right\rangle}{1 + |\varphi|^2 t^2}$$

in the form $\mathrm{Tr} \, \bullet \widehat{m}$, we obtain by a proof completely analogous to that of Proposition 6.9

Proposition 6.12 *There are positive numbers ϵ_n, and monotone functions $q_\pm(t)$ on $(0,1)$ with $q_\pm(t)$ between 0 and ± 1 and $\lim_{t\to 0} q_\pm(t) = \pm 1$, such that*

$$\left| \frac{\Psi^+_{\left(e^{it\varphi}\sqrt{1-t^2(x^2+y^2)}\right)(\chi)}(g)}{t^2} - \frac{\delta''^r(g)}{2} \right| \quad < \quad \frac{1}{n},$$

and

$$\left| (1 + |\varphi|^2 t^2) \frac{\Psi^-_{\left(1\,\sqrt{1-t^2(x^2+y^2)}\right)(\chi)}(g)}{t^2} - \underline{\psi}_r \right| \quad < \quad \frac{1}{n},$$

for all $g \in \mathcal{G}_n$, $r \in SU(2)$, $t < \epsilon_n$ and all functions $q(t)$, such that $q(t)$ between $q_\pm(t)$ and ± 1.

This means that the GAUSSian and anti-GAUSSian part can be approximated by $\Psi^\pm$ at least on $\mathcal{K}^2$. Again the statement remains true on $\mathcal{K}_\mathcal{P}$ and in the case when $q \to -1$ it is even true on the whole algebra $\mathcal{F}$. For $q \to 1$ consider the sequence of conditionally positive functionals

$$\psi^0_n \quad = \quad \frac{\psi^+_{\left(e^{it_n\varphi}\sqrt{1-t_n^2(x^2+y^2)}\right)(\chi)}}{t_n^2} + \frac{\psi^+_{\left(e^{is_n\varphi}\sqrt{1-s_n^2(x^2+y^2)}\right)(-\chi)}}{t_n s_n}$$

on $\mathcal{A}_{q_n}$. Choose $t_n \geq t_{n+1} \to 0$ such that

$$\left| \frac{\delta_{(t_n\varphi)(t_n x)(t_n y)}(g)}{t_n^2} - \frac{\delta''^r(g)}{2} \right| \quad < \quad \frac{1}{n},$$

and then $s_n \geq s_{n+1} \to 0$ such that

$$\left| \frac{\delta_{(s_n\varphi)(-s_n x)(-s_n y)}(g)}{t_n s_n} \right| \quad < \quad \frac{1}{n},$$

for all $g \in \mathcal{G}_n$. We have

$$\frac{\delta_{(t_n\varphi)(t_n x)(t_n y)}}{t_n^2} + \frac{\delta_{(s_n\varphi)(-s_n x)(-s_n y)}}{t_n s_n} \quad \longrightarrow \quad \frac{\delta''^r}{2}$$

on $\mathcal{K}_2^1$. On the other hand,

$$\frac{\delta_{(t_n\varphi)(t_nx)(t_ny)}(\gamma^{(*)})}{t_n^2} + \frac{\delta_{(s_n\varphi)(-s_nx)(-s_ny)}(\gamma^{(*)})}{t_ns_n} \quad = \quad 0 \quad = \quad \delta''^r(\gamma^{(*)}).$$

Thus, the convergence is also on $\mathcal{K}_2$. Now we choose $q_n \leq q_{n+1} \to 1$ such that

$$\left| \Psi^+_{\left(e^{it_n\varphi}\sqrt{1-t_n^2(x^2+y^2)}\right)(\chi)}(g) - \delta_{(t_n\varphi)(t_nx)(t_ny)}(g) \right| \quad < \quad \frac{t_n^2}{n},$$

and

$$\left| \Psi^+_{\left(e^{is_n\varphi}\sqrt{1-s_n^2(x^2+y^2)}\right)(\chi)}(g) - \delta_{(s_n\varphi)(s_nx)(s_ny)} \right| \quad < \quad \frac{t_ns_n}{n}$$

for all $g \in \{\beta+\beta^*, \gamma, \gamma^*\} \cup \mathcal{G}_n$. Since $\Psi^+_{\lambda_0\chi}$ and δ''^r are 0 on $\alpha - \alpha^*$ and $\mathbf{1}$, we obtain the following

Proposition 6.13 *We have*

$$\lim_{n\to\infty} \psi_n^0 \quad = \quad \frac{\delta''^r}{2}.$$

EXAMPLE 6.2 *We mention that it is also not possible to find a conditionally positive functional ψ_{γ^2} on $\mathcal{A}_q$ for $|q| < 1$, being different from 0 on $\gamma^*\gamma$ but vanishing on all monomials, having more than three factors $\gamma^{(*)}$. (Such a functional would be the analogue of the* GAUSS*ian functionals $\delta''^{(0,1,\pm1)}$ in the classical case.) From $\psi_{\gamma^2}(\gamma^*\gamma^*\gamma\gamma) = \psi_{\gamma^2}(\gamma\gamma^*\gamma\gamma^*) = 0$ we conclude $\|\pi(\gamma)\eta_{\gamma^2}(\gamma^{(*)})\| = 0$. On the subspace where $\pi(\gamma) = 0$ we have $\eta_{\gamma^2}(\gamma^{(*)}) = 0$. On the subspace where $\pi(\gamma)$ is injective we also obtain that $\eta_{\gamma^2}(\gamma^{(*)})$ must be 0. In other words, a maximal quadratic component of a conditionally positive functional on $\mathcal{A}_q$ for $|q| < 1$ (cf. [13]) must be a* GAUSS*ian part ψ_δ of the form written down in Corollary 2.11.*

Up to this point we are able to split up a given conditionally positive functional on $\mathcal{A}_1$ or $\mathcal{A}_{-1}$ into several parts, and to approximate any of these parts by sequences of conditionally positive functionals on $\mathcal{A}_{q_n}$, where q_n converges to ±1. We emphasized that all approximations also work if the sequence q_n is replaced by a sequence q'_n where q'_n is closer to its limit ±1 than q_n. Therefore, the sequences q_n belonging to different parts of the conditionally positive functional can be chosen to be the same. We summarize.

Theorem 6.14 *For any conditionally positive functional ψ on $\mathcal{A}_{\pm1}$ there is a sequence $\{q_n\}$ with $\lim_{n\to\infty} q_n = \pm1$ and a sequence $\{\psi_n\}$ of conditionally positive functionals on $\mathcal{A}_{q_n}$, such that*

$$\psi \quad = \quad \lim_{n\to\infty} \psi_n.$$

Due to the last remark it is also always possible to find a family ψ_q of conditionally positive functionals on $\mathcal{A}_q$, such that

$$\psi \quad = \quad \lim_{q\to\pm1} \psi_q.$$

6.3 Quantization and correspondence principle

In the previous chapters we considered the algebras $\mathcal{A}_q$. We mentioned that for $|q| < 1$ all the C^*–completions are isomorphic. This is why we stressed the interpretation of these $\mathcal{A}_q$ as subalgebras of the same C^*–algebra $\mathcal{A}$. The theory of conditionally positive functionals on $\mathcal{A}_q$ for any $|q| < 1$ could be unified to the q–independent theory of conditionally positive functionals on $\mathcal{A}_\mathcal{T}$. The cases $q = \pm1$ appeared as quite different exceptional cases. (Notice that the generators $\alpha_q, \gamma_q \in \mathcal{A}$ depend continiuously on q. However, their limit to $\alpha_{\pm1}, \gamma_{\pm1}$ where q is replaced formally by ±1 is only strong and the algebra generated by these operators is not isomorphic to $\mathcal{A}_{\pm1}$, because $\alpha_{\pm1} = 0$. Actually, it is not difficult to see that γ_{-1} is not even an element of $\mathcal{A}$.)

In this chapter we took on a complementary point of view. We considered the algebras $\mathcal{A}_q$ as quotients of the same freely generated algebra $\mathcal{F}$ and the q–dependent ideals $\mathcal{I}_q$. The description in terms of the same finitlely generated algebra $\mathcal{F}$ enabled us to limit the topological problems of an infinite dimensional vector space to those of the finite sets $\mathcal{G}_n$. In this section we want to point out that this framework provides a perfect understanding of $\mathcal{A}_q$ as a quantization of $SU(2)$ and that the results of this chapter can be interpreted as a *correspondence principle*.

Suppose we investigate a classical (physical) system whose *phase space* is $SU(2)$. In classical physics the *observabels* of such a sytem are representeted as (continuous) functions on the phase space. Therefore, we can identify the set of observables on $SU(2)$ with the commutative C^*–algebra $C(SU(2))$ of continuous functions on $SU(2)$. On the other hand, the dense subalgebra $\mathcal{A}_\mathrm{f}$ is generated by the matrix entries f_{ij} considered as functions on $SU(2)$. Therefore, these four functions can be interpreted as the basic observables of the system. All other observables can be derived as functions of these basic observables (like the canonical pair of variables (q, p) in the description of a one-particle system). The expectation value of an observable is obtained by evaluating the corresponding function in $\mathcal{A}_\mathrm{f}$ in the state in which the system is.

The quantization procedure consists in replacing the commutative algebra of observables by a non-commutative algebra $\mathcal{A}_q$, and again the expectation value of an observable is obtained by evaluating a state on this observable. In order that the elements of $\mathcal{A}_q$ can be associated with the original observables in $\mathcal{A}_\mathrm{f}$ it is postulated that f_{ij} are associated with four elements $\tilde{f}_{ij}$ of $\mathcal{A}_q$, that $\mathcal{A}_q$ is generated by these four elements, and that there is a basis of $\mathcal{A}_\mathrm{f}$ which is associated with a basis of $\mathcal{A}_q$. Of course, there is a canonical homomorphism $\mathcal{E}_q$ from the free unital $*$–algebra $\mathcal{F}$ generated by the non-commuting indeterminants F_{ij} onto $\mathcal{A}_q$. In other words, $\mathcal{A}_q$ can be considered as the qotient of $\mathcal{F}$ by the ideal $\mathcal{I}_q = \ker(\mathcal{E}_q)$ in $\mathcal{F}$.

Here we do not want to investigate which quantizations are possible, i.e. which ideals of $\mathcal{F}$ can be found, such that all the conditions are fulfilled. (We mention the further restriction that it is not only an algebra which is to be quantized but also a Hopf algebra.) In the appendix of [35] an exhausting investigation can be found of the motivation leading to the quantization by the ideals $\mathcal{I}_q$ which are generated by Relations (1.4) and identification of the matrix entries with the corresponding generators α and γ. As usual the quantization depends on a parameter $q \in [-1, 1]$ and the classical case is contained ($q = 1$). Furthermore, the ideals $\mathcal{I}_q$ are finitely generated and the generating relations depend weakly continuous on q. (This weak continuity enabled us to prove Proposition 6.1.)

In quantum physics by the correspondence principle one means that the quantum description of a system is close to its classical discription if only the quantization parameter is close to its classical limit. In other words, for a given *finite* set of observables and a state on $\mathcal{A}_1$ there should be a number q close to 1 and a state on $\mathcal{A}_q$, such that the difference between the expectation values of an observable in the two states becomes arbitrarily small. But this is precisely what we did by investigating the sets $\mathcal{G}_n$. Actually, we showed a correspondence principle not only for states but also for conditionally positive functionals on both $\mathcal{A}_1$ and $\mathcal{A}_{-1}$. The convolution exponential

$$\varphi_t \;=\; e_\star^{t\psi},$$

with the usual multiplication replaced by the convolution $\star$, establishes a one-to-one correspondence between conditionally positive functionals vanishing at identity and convolotion semi-groups of states (cf. [27]). Therefore, one can say that we also proved a correspondence principle for convolution semi-groups of states.

Appendix

A q–Analysis

We present the well-known results on q–analysis in a slightly modified form which is more convinient for our purposes. The proofs of formulae are omitted if they consist in simple computation.

A.1 q–Derivative and q–integral

Definition A.1 *Let $q \in (-1,1)$ be a real number and $S_0 \subset \mathbf{C}$ a star shaped area having 0 as star point. By $C_\omega(S_0)$ we denote the space of analytic functions on S_0. We introduce the two linear mappings $d_q, \int_q : C_\omega(S_0) \to C_\omega(S_0)$ by*

(i)
$$d_q(f) = \frac{df}{d_q z}, \quad \text{where} \quad \frac{df}{d_q z}(w) \;=\; \frac{f(w) - f(qw)}{w}.$$

(ii)
$$\int_q (f) = \int_q f, \quad \text{where} \quad \left(\int_q f\right)(w) \;=\; \int_0^w f(z)\, d_q z \;=\; \sum_{k=0}^{\infty} q^k w f(q^k w).$$

By expanding f into a power series, we easily see that $d_q f$ and $\int_q f$ are indeed in $C_\omega(S_0)$.

N.B.: In order to obtain the usual notions of q–derivative and q–integral we have to divide our derivative by $(1-q)$ and to multiply our integral by $(1-q)$. By looking at the corresponding expressions

$$\frac{f(w) - f(qw)}{w - qw}, \quad \text{and} \quad \sum_{k=0}^{\infty} (q^k w - q^{k+1} w) f(q^k w)$$

we immediately see that they tend to the derivative and integral, respectively, of usual analysis as q tends to 1.

Theorem A.1 *(Main Theorem)*

(i)
$$\int_q d_q f \;=\; f - f(0)$$

(ii)
$$d_q \int_q f \;=\; f.$$

Proof By replacing the infinite sum in the definition of the integral by a finite sum and then performing the limit, we see that

$$\left(\int_q d_q f\right)(w) \;=\; f(w) - \lim_{n\to\infty} f(q^{n+1} w)$$

and

$$\left(d_q \int_q f\right)(w) \;=\; f(w) - \lim_{n\to\infty} q^{n+1} f(q^{n+1} w).$$

From this the statements follow. ■

By direct computation we obtain the following rules.

Theorem A.2 *For $f, g \in C_\omega(S_0)$ we have*

$$\text{(i)} \qquad \frac{d(fg)}{d_q z}(z) = f(z)\frac{dg}{d_q z}(z) + \frac{df}{d_q z}(z)g(qz) \qquad \text{(q–Leibnitz rule)}$$

$$\text{(ii)} \qquad \frac{d(\frac{1}{f})}{d_q z}(z) = -\frac{\frac{df}{d_q z}(z)}{f(z)f(qz)}$$

$$\text{(iii)} \qquad \int_0^w f(z)\frac{dg}{d_q z}(z)\, d_q z + \int_0^w \frac{df}{d_q z}(z)g(qz)\, d_q z$$

$$= f(w)g(w) - f(0)g(0) \qquad \text{(q–partial integration)}$$

Theorem A.3 *For the non-negative powers of z we obtain*

$$\text{(i)} \qquad d_q(z^k) = (1-q^k)z^{k-1}$$

$$\text{(ii)} \qquad \int_q (z^k) = \frac{z^{k+1}}{1-q^{k+1}}.$$

N.B.: Of course, it is possible to extend the operation of q–derivation to functions which are analytic on an area S such that $qS \subset S$. In the next paragraph such functions will actually appear.

A.2 q–Exponential function and q–Eulerian integral

Theorem A.4 *The q–exponential function.*

(i) By setting

$$e_q^z = \prod_{k=0}^{\infty} \frac{1}{1-q^k z}$$

we define a meromorphic function on $\mathbf{C}\backslash\{q^{-k} | k \in \mathbf{N}_0\}$.

(ii) On $U_1(0)$ we have

$$e_q^z = \sum_{k=0}^{\infty} \frac{z^k}{(1-q)\cdots(1-q^k)}$$

(iii) e_q^z is different from 0 everywhere. By setting

$$(e_q^z)^{-1} = \prod_{k=0}^{\infty}(1-q^k z)$$

we define a transcendent function.

Proof Consider the power series in (ii). Clearly, its radius of convergence is 1. (We have $\frac{|z|}{1-q^k} \leq \frac{|z|}{1-q^K}$ for $k > K$ and for any $z \in U_1(0)$ we can find $K \in \mathbf{N}_0$, such that $\frac{|z|}{1-q^K} < 1$.) We find

$$e_q^z - e_q^{qz} = z e_q^z.$$

and consequently

$$e_q^z = \frac{1}{1-z} e_q^{qz} = \frac{1}{1-z} \cdots \frac{1}{1-q^k z} e_q^{q^{k+1} z}.$$

Since $\lim_{k\to\infty} e_q^{q^{k+1}z} = 1$, we have that $e_q^{q^{k+1}z}$ is different from zero for almost all k. Therefore, we find

$$e_q^z \;=\; \lim_{k\to\infty} \frac{e_q^z}{e_q^{q^{k+1}z}} \;=\; \prod_{k=0}^{\infty} \frac{1}{1-q^k z}.$$

Now let z be with $|z| \geq 1$ and $z \neq q^{-k}, k \in \mathbf{N}_0$. We can find $K \in \mathbf{N}$ such that $q^K|z| < 1$. Therefore, we see by using

$$e_q^z \;=\; e_q^{q^K z} \prod_{k=0}^{K-1} \frac{1}{1-q^k z}$$

that e_q^z is analytic on the given domain.

Now suppose that e_q^z is 0 for some z. By the recursion formula we see that 0 must be an accumulation point of zeros. Therefore, the function has to assume the value 0 at 0 in contradiction to $e_q^0 = 1$. Thus, we can define the reciprocal of e_q^z on the whole complex plain. Since this function assumes the value 0 for $z = q^{-k}$, this function cannot be a polynomial. It must be transcendent. ∎

Using the recursion formula and our derivation rule (ii), we obtain

Corollary A.5 *The operation of q–derivation can be performed for e_q^z and its reciprocal at any point of their domains. We obtain*

$$(i) \qquad \frac{de_q^z}{d_q z}(z) \;=\; e_q^z$$

$$(ii) \qquad \frac{d(e_q^z)^{-1}}{d_q z}(z) \;=\; (e_q^{qz})^{-1}.$$

N.B.: Suppose that f and $\widetilde{f}$ are two (non-vanishing) solutions of the q–differential equation (i). It is not difficult to see that their quotient must be a constant. Therefore, e_q^z is the only solution of (i), fulfilling $e_q^0 = 1$. A similar statement is true for (ii).

Notice that the usual form of the q–exponential is given by

$$\sum_{k=0}^{\infty} \frac{z^k}{\left(\frac{1-q}{1-q}\right)\cdots\left(\frac{1-q^k}{1-q}\right)} \;=\; e_q^{(1-q)z}.$$

This function converges pointwise to the exponential for $q \to 1$.

Now we can describe the q–factorial $[k]_q! = (1-q)\cdots(1-q^k)$ by a q–EULERian integral. (To obtain the usual definition one has to divide by $(1-q)^k$.)

Theorem A.6 *We have*

$$\int_0^1 \frac{z^k}{e_q^{qz}}\, d_q z \;=\; (1-q)\cdots(1-q^k).$$

PROOF We prove the statement by induction. Since

$$\int_0^1 \frac{d_q z}{e_q^{qz}} \;=\; -((e_q^1)^{-1} - (e_q^0)^{-1}) \;=\; 1,$$

the statement is true for $k = 0$.

Now suppose that it is true for $k \geq 0$. We obtain

$$\begin{aligned}
\int_0^1 \frac{z^{k+1}}{e_q^{qz}}\, d_q z &= -\int_0^1 z^{k+1}(d_q(e_q^z)^{-1})(z)\, d_q z \\
&= \left[-z^{k+1}(e_q^z)^{-1}\right]_0^1 + \int_0^1 (d_q z^{k+1})(e_q^{qz})^{-1}\, d_q z \\
&= 0 + (1-q^{k+1}) \int_0^1 \frac{z^k}{e_q^{qz}}\, d_q z.
\end{aligned}$$

This proves the statement for $k+1$. ■

Now we show an estimate which will be useful in the next appendix.

Proposition A.7 *For all $w \in [0,1]$ and $k \in \mathbf{N}_0$ we have*

$$\frac{w^{k+1}}{e_q^q} \leq \frac{w^{k+1}}{(1-q^{k+1})e_q^{qw}} \leq \int_0^w \frac{z^k}{e_q^{qz}}\, d_q z \leq \frac{w^{k+1}}{(1-q^{k+1})} \leq \frac{w^{k+1}}{1-q}.$$

Proof By the power series representation we see that e_q^z is a strictly increasing function on $[0,1)$. Notice also that $\left(\int_q \bullet\right)(w)$ is a monotone functional for positive w. This yields immediately the two inner estimates. The outer estimates are obvious. ■

B The representation ρ_0 as a representation on a Hilbert space of analytic functions

In this appendix we generalize the representation of the relation

$$aa^* - qa^*a = \mathbf{1}$$

given in [5] to a representation of $\mathcal{A}_q$, unitarily equivalent to ρ_0 (cf. also [6]). We will show that the scalar product stated in [5] turns indeed out to be the scalar product of the underlying Hilbert space. (The authors of [5] only showed that their scalar product yields the correct values on a special orthonormal basis. The proof of the well-definedness was left out.) However, notice that α and α^* fulfill the slightly modified commutation rule

$$\frac{\alpha\alpha^* - q^2\alpha^*\alpha}{1-q^2} = \mathbf{1}.$$

Now consider the representation ρ_0 on the Hilbert space h_0 with orthonormal basis $\{e_k\}_{k\in\mathbf{N}_0}$. We introduce the q–coherent states as eigenvectors of $\rho_0(\alpha)$. It is easy to check that they must be of the form

$$e_{q^2}(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\sqrt{1-q^2}\cdots\sqrt{1-q^{2k}}} e_k \quad \text{with} \quad |\lambda| < 1,$$

where λ is the eigenvalue. Under $\rho_0(\alpha^*)$ these vectors behave like

$$\begin{aligned}
\rho_0(\alpha^*)e_{q^2}(\lambda) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\sqrt{1-q^2}\cdots\sqrt{1-q^{2k}}} \sqrt{1-q^{2(k+1)}}\, e_{k+1} \\
&= \frac{1}{\lambda}\sum_{k=0}^{\infty} \frac{\lambda^k}{\sqrt{1-q^2}\cdots\sqrt{1-q^{2k}}}(1-q^{2k})e_k = \frac{de_{q^2}(\lambda)}{d_{q^2}\lambda}.
\end{aligned}$$

The scalar product of two such vectors is given by

$$\langle e_{q^2}(\mu) | e_{q^2}(\lambda)\rangle = e_{q^2}^{\overline{\mu}\lambda}.$$

Now we ask how to build up the identity operator out of terms of the form $|e_{q^2}(\lambda)\rangle\langle e_{q^2}(\lambda)|$.

We write $\lambda = |\lambda|e^{i\varphi}$ in polar coordinates and intergate over φ. For fixed $|\lambda|$ the double sum is absolutely convergent and no problem can arise. We obtain

$$\int_0^{2\pi} |e_{q^2}(|\lambda|e^{i\varphi})\rangle\langle e_{q^2}(|\lambda|e^{i\varphi})| \frac{d\varphi}{2\pi} = \sum_{k=0}^{\infty} |e_k\rangle \frac{|\lambda|^{2k}}{(1-q^2)\cdots(1-q^{2k})} \langle e_k|.$$

If we now could perform the q^2–EULERian integral (see Appendix A) with respect to the variable $|\lambda|^2$ in order to eliminate q^2–factorial in the denominator, we would obtaine a representation of the identity. However, in order to perform the q^2–integral the integrand has to be evaluated at $|\lambda|^2 = 1$ where the sum is no longer norm convergent. On the other hand, due to Proposition A.7 we have for $w \in [0, 1)$

$$\int_0^w \int_0^{2\pi} |e_{q^2}(|\lambda|e^{i\varphi})\rangle\langle e_{q^2}(|\lambda|e^{i\varphi})| \frac{d\varphi}{2\pi} \frac{d_{q^2}|\lambda|^2}{e_{q^2}^{q^2|\lambda|^2}} = \sum_{k=0}^{\infty} |e_k\rangle \frac{\displaystyle\int_0^w \frac{|\lambda|^{2k}}{e_{q^2}^{q^2|\lambda|^2}} d_{q^2}|\lambda|^2}{(1-q^2)\cdots(1-q^{2k})} \langle e_k|.$$

It is easy to see that

$$\lim_{w\to 1} \int_0^w \int_0^{2\pi} |e_{q^2}(|\lambda|e^{i\varphi})\rangle\langle e_{q^2}(|\lambda|e^{i\varphi})| \frac{d\varphi}{2\pi} \frac{d_{q^2}|\lambda|^2}{e_{q^2}^{q^2|\lambda|^2}} = \mathbf{1}$$

in the strong topology. Notice that the order of integrals does not matter. Using the notation

$$\int f(\lambda)\, d^2_{q^2}\lambda = \lim_{w\to 1} \frac{1}{2\pi} \int_0^w \int_0^{2\pi} f(|\lambda|e^{i\varphi})\, d\varphi\, d_{q^2}|\lambda|^2,$$

we obtain

$$\int \frac{|e_{q^2}(\lambda)\rangle\langle e_{q^2}(\lambda)|}{e_{q^2}^{q^2|\lambda|^2}} d^2_{q^2}\lambda = \mathbf{1}.$$

Obviously, this remains true if in the integrand λ is replaced by $\overline{\lambda}$.

Consider the HILBERT space H_{f} of analytic functions which is defined by assuming that

$$\left\{ \frac{z^k}{\sqrt{1-q^{2k}}} \right\}_{k\in\mathbf{N}_0}$$

forms an orthonormal basis. Notice that the scalar products for different $q \in (-1, 1)$ induce the same topology. Therefore, H_{f} consists of all power series with square summable coefficients. The scalar product of two elements $f, g \in H_{\mathrm{f}}$ can be obtained as

$$\langle f|g\rangle = \int \frac{\overline{f(\lambda)}g(\lambda)}{e_{q^2}^{q^2|\lambda|^2}} d^2_{q^2}\lambda.$$

By

$$f \in h_0 \longmapsto f(z) = \langle e_{q^2}(\overline{z})|f\rangle \in H_{\mathrm{f}}$$

we obviously define a HILBERT space isomorphism. For the representation operators in this image we obtain

$$\begin{aligned}
\rho_0(\alpha^*)f(z) &= \langle e_{q^2}(\overline{z})|\rho_0(\alpha^*)f\rangle = \langle \rho_0(\alpha)e_{q^2}(\overline{z})|f\rangle \\
&= zf(z) \\
\rho_0(\alpha)f(z) &= \langle \rho_0(\alpha^*)e_{q^2}(\overline{z})|f\rangle \\
&= \frac{df}{d_{q^2}z}(z) \\
\rho_0(\gamma^{(*)})f(z) &= f(qz).
\end{aligned}$$

The q–coherent state belonging to the eigenvalue λ is given by the q–exponential

$$\langle e_{q^2}(\overline{z})|e_{q^2}(\lambda)\rangle \;=\; e_{q^2}^{z\lambda}.$$

Let $\varphi = \langle f|\rho_0|f\rangle$ be a state with GNS–representation ρ_0. (Notice that by irreducibilty any non-vanishing vector in h_0 is cyclic for ρ_0.) Since φ is hermitian and ρ_0 does not distinguish between γ and γ^*, it is sufficient to know its values on ${\alpha^*}^n\gamma^m$ for $n, m \in \mathbf{N}_0$. We obtain

$$\varphi({\alpha^*}^n\gamma^m) \;=\; \int \overline{f(z)}z^n f(q^m z)\, d_{q^2}^2 z.$$

C Proof of Lemma 5.22

Let α and γ be two bounded operators on H fulfilling Relations (a)-(d) both having 0 not as an eigen value which determine the representation π. There are two spectral measures dE_z^α and dE_w^γ on $\mathbf{C}_0$ such that

$$\alpha \;=\; \int z\, dE_z^\alpha \quad \text{and} \quad \gamma \;=\; \int w\, dE_w^\gamma.$$

For any polynomial P we have $\gamma P(\alpha, \alpha^*) = P(-\alpha, -\alpha^*)\gamma$. Therefore, we must have

$$\gamma\, dE_z^\alpha \;=\; dE_z^{-\alpha}\gamma \;=\; dE_{-z}^\alpha \gamma, \tag{C.1}$$

because dE_z^α is the limit of polynomials in α and α^*.

Since γ is injective and normal, we can define the unitary operator

$$U_\gamma \;=\; \frac{\gamma}{|\gamma|}.$$

For any Borel set $S \subset \mathbf{C}_0$ we define the projection $E_S = \int_S dE_z^\alpha$. Since $|\gamma| = \sqrt{\gamma^*\gamma}$ commutes with dE_z^α, we see that the restriction $U_\gamma : E_S H \to E_{-S}H$ defines an isomorphism. Therefore, if (S_1, S_2) is any partition of $\mathbf{C}_0$ into Borel sets, such that $z \in S_1$ implies $-z \in S_2$, with corresponding projections E_1, E_2, we see that U_γ induces a unitary equivalence transform mapping dE_z^α to dE_{-z}^α. Let us summarize.

Proposition C.1 *There is a* Hilbert *space H_0 with a spectral measure dE_z^0 on S_1 and a pair of isomorphisms*

$$\Phi_i : E_i H \overset{\sim}{\longrightarrow} H_0, \quad i = 1, 2,$$

such that

$$\Phi_1^{-1}\, dE_z^0 \Phi_1 \;=\; dE_z^\alpha \quad \textit{and} \quad \Phi_2^{-1}\, dE_z^0 \Phi_2 \;=\; dE_{-z}^\alpha.$$

The action of Φ_i is written multiplicatively.

We identify $H = E_1 H \oplus E_2 H$, i.e. $\oplus$ means just $+$. Obviously, we have $E_i \Phi_j^{-1} = \delta_{ij}\Phi_j^{-1}$. We define an isomorphism $\Phi : H \overset{\sim}{\to} H_0 \oplus H_0$ by $\Phi = \Phi_1 \oplus \Phi_2$. A vector $(\eta_1, \eta_2) \in H_0 \oplus H_0$ with components $\eta_i \in H_0$ is the image of

$$\eta \;=\; \Phi_1^{-1}\eta_1 + \Phi_2^{-1}\eta_2 \;=\; E_1\eta + E_2\eta \in H.$$

We find

$$\Phi\alpha\Phi^{-1} \;=\; \begin{pmatrix} \Phi_1\alpha\Phi_1^{-1} & 0 \\ 0 & \Phi_2\alpha\Phi_2^{-1} \end{pmatrix} \quad \text{and} \quad \Phi\gamma\Phi^{-1} \;=\; \begin{pmatrix} 0 & \Phi_1\gamma\Phi_2^{-1} \\ \Phi_2\gamma\Phi_1^{-1} & 0 \end{pmatrix}.$$

We easily check, that $\Phi_1\alpha\Phi_1^{-1} = -\Phi_2\alpha\Phi_2^{-1} = \int_{S_1} z\, dE_z^0 = \alpha_0$ with $\alpha_0 \in \mathcal{B}(H_0)$.

In order to have $\Phi_1\gamma\Phi_2^{-1} = \Phi_2\gamma\Phi_1^{-1}$ we will now fix on a special choice for Φ_1 and Φ_2. Notice that $\frac{\gamma^2}{|\gamma|^2}$ is a unitary operator, commuting with everything. By means of its spectral measure we can define its 4th root, also commuting with everything, such that the restriction of $\sqrt[4]{\frac{\gamma^2}{|\gamma|^2}}$ to E_1H is a unitary operator on E_1H. By $\Phi_0 = \Phi_1 \sqrt[4]{\frac{\gamma^{*2}}{|\gamma|^2}}$ we define another isomorphism $E_1H \xrightarrow{\sim} H_0$ which also fulfills $\Phi_0^{-1}\, dE_z^0 \Phi_0 \;=\; dE_z^\alpha$, i.e. $\alpha_0 = \Phi_0 \alpha \Phi_0^{-1}$. It is easily checked that the pair

$$\begin{aligned} \Phi_1 &= \Phi_0 \sqrt[4]{\frac{\gamma^2}{|\gamma|^2}} \\ \Phi_2 &= \Phi_0 \sqrt[4]{\frac{\gamma^{*2}}{|\gamma|^2}} \frac{\gamma}{|\gamma|} \end{aligned}$$

has all the properties stated in the proposition. We find $\Phi_1 \gamma \Phi_2^{-1} = \Phi_2 \gamma \Phi_1^{-1} = \Phi_0 \sqrt{\gamma^2} \Phi_0^{-1} = \gamma_0$ with $\gamma_0 \in \mathcal{B}(H_0)$. We obtain

$$\Phi\alpha\Phi^{-1} \;=\; \begin{pmatrix} \alpha_0 & 0 \\ 0 & -\alpha_0 \end{pmatrix} \quad \text{and} \quad \Phi\gamma\Phi^{-1} \;=\; \begin{pmatrix} 0 & \gamma_0 \\ \gamma_0 & 0 \end{pmatrix}.$$

Obviously α_0 and γ_0 commute, i.e. there is a spectral measure dE_{zw}^0 on $S_1 \times \mathbf{C}_0$, such that $\alpha_0 = \int_{S_1 \times \mathbf{C}_0} z\, dE_{zw}^0$ and $\gamma_0 = \int_{S_1 \times \mathbf{C}_0} w\, dE_{zw}^0$.

Now we add another representation on $H_0 \oplus H_0$, which coincides with the original one with the exception that α_0 is replaced by $-\alpha_0 \;=\; \int_{S_2 \times \mathbf{C}_0} z\, dE_{-zw}^0$. (In other words, what we have done is to extend the integration area from $S_1 \times \mathbf{C}_0$ to $\mathbf{C}_0^2$ in an anti-symmetric way.) Of course the two summands are unitarily equivalent. The direct sum of both is given by

$$\alpha \longmapsto \begin{pmatrix} \alpha_0 & 0 & 0 & 0 \\ 0 & -\alpha_0 & 0 & 0 \\ 0 & 0 & -\alpha_0 & 0 \\ 0 & 0 & 0 & \alpha_0 \end{pmatrix} \quad \text{and} \quad \gamma \longmapsto \begin{pmatrix} 0 & \gamma_0 & 0 & 0 \\ \gamma_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_0 \\ 0 & 0 & \gamma_0 & 0 \end{pmatrix}.$$

By exchanging the second and third component, we obtain a unititarily equivalent representation on $H_0 \oplus H_0 \oplus H_0 \oplus H_0$, defined by

$$\alpha \longmapsto \begin{pmatrix} \alpha_0 & 0 & 0 & 0 \\ 0 & -\alpha_0 & 0 & 0 \\ 0 & 0 & -\alpha_0 & 0 \\ 0 & 0 & 0 & \alpha_0 \end{pmatrix} \quad \text{and} \quad \gamma \longmapsto \begin{pmatrix} 0 & 0 & \gamma_0 & 0 \\ 0 & 0 & 0 & \gamma_0 \\ \gamma_0 & 0 & 0 & 0 \\ 0 & \gamma_0 & 0 & 0 \end{pmatrix}.$$

We recognize a block-diagonal form and, thus, obtain by the identification $(H_0 \oplus H_0) \oplus (H_0 \oplus H_0) = (\Phi \oplus \Phi)(H \oplus H) = (\mathbf{1}_{M_{2\times 2}} \otimes \Phi)(\mathbf{C}^2 \otimes H)$ a representation $\pi^\oplus$ by

$$\pi^\oplus(\alpha) \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \alpha_0^\oplus \quad \text{and} \quad \pi^\oplus(\gamma) \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \gamma_0^\oplus,$$

with

$$\alpha_0^\oplus \;=\; \Phi^{-1} \begin{pmatrix} \alpha_0 & 0 \\ 0 & -\alpha_0 \end{pmatrix} \Phi \;=\; \alpha \quad \text{and} \quad \gamma_0^\oplus \;=\; \Phi^{-1} \begin{pmatrix} \gamma_0 & 0 \\ 0 & \gamma_0 \end{pmatrix} \Phi \;=\; \sqrt{\gamma^2}.$$

So far we obtained for injective α and γ that $\pi^\oplus$ is a representation unitarily equivalent to the direct sum of π and the representation arising from π by changing the sign of α. Moreover, these two summands themselves turned out to be unitarily equivalent to each other. By symmetry in α and γ, also a sign change in γ must be equal to a unitary equivalence transform. Therefore, we indeed obtain $\pi \asymp \underline{\pi}$ and $\pi^\oplus \asymp \pi \oplus \underline{\pi}$.

So far we proved the statements when α and γ are injective. If now $\gamma = 0$, then obviously $\pi^\oplus(\alpha) = \alpha \oplus -\alpha$ extends to $\pi \oplus \underline{\pi}$. The case when $\alpha = 0$ is more difficult. Since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is unitarily equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we again obtain $\pi^{\oplus}(\gamma) \asymp \sqrt{\gamma^2} \oplus -\sqrt{\gamma^2}$. Notice that

$$\gamma \;=\; \int w \, dE_w^\gamma \quad \text{and} \quad \sqrt{\gamma^2} \;=\; \int \sigma(w) w \, dE_w^\gamma,$$

where $\sigma(w)$ is an appropiate sign function, depending on wether $w = \sqrt{w^2}$ or not, i.e. $\sigma(re^{i\varphi}) = 1$ for $\varphi \in [0, \pi)$ and $\sigma(re^{i\varphi}) = -1$ for $\varphi \in [\pi, 2\pi)$. Now consider the operator $\sqrt{\gamma^2} \oplus -\sqrt{\gamma^2}$. If $\sigma(w)w = -w$ in the decomposition of $\sqrt{\gamma^2}$, then $-\sigma(w)w = w$ in the decomposition of $-\sqrt{\gamma^2}$, and conversely. Therefore, $\sqrt{\gamma^2} \oplus -\sqrt{\gamma^2} \asymp \gamma \oplus -\gamma$ and again $\pi^{\oplus}(\gamma)$ extends to $\pi \oplus \underline{\pi}$. This completes the proof of Lemma 5.22.

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Lebenslauf

Am 3.6.65 wurde ich im Bürgerhospital in Frankfurt als Sohn des Postbeamten Heinrich Skeide und seiner Frau Ursula Skeide geboren. Nach der Grundschule (Uhlandschule, Frankfurt) besuchte ich 1975 - 84 das H.-v.-Gagern-Gymnasium in Frankfurt, und erlangte dort am 13.6.84 das Abitur. Nach der Schule begann ich 1984 in Darmstadt das Physikstudium. Ich setzte es 1985 in Frankfurt fort, wo ich 1986 auch die Vordiplomprüfung ablegte. Seitdem studierte ich in Heidelberg, wo ich mit einer Arbeit über den ‚stochastisch getrieben harmonischen Oszillator' am 3.4.90 mein Physik-Diplom erlangte.

Jetzt bin ich seit dem 1.10.90 Doktorand bei Herrn Prof. Dr. von Waldenfels.

Meine Staatsangehörigkeit ist deutsch.