Dilations, Poduct Systems and Weak Dilations*

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Abstract

We generalize Bhat's construction of product systems of Hilbert spaces from E_0–semigroups on $\mathcal{B}(H)$ for some Hilbert space H to the construction of product systems of Hilbert modules from E_0–semigroups on $\mathcal{B}^a(E)$ for some Hilbert module E. As a byproduct we find the representation theory for $\mathcal{B}^a(E)$, if E has a unit vector. We proof a necessary and sufficient criterion when the conditional expecation generated by the unit vector defines a weak dilation of a CP-semigroup in the sense of [BS00]. Finally, we show that also white noises on general von Neumann algebras in the sense of [Küm85] can be extended to white noises on a Hilbert module.

1 Introduction

Let $\mathcal{B}$ be a unital C^*–algebra. Product systems of Hilbert $\mathcal{B}$–$\mathcal{B}$–modules appeared naturally in Bhat and Skeide [BS00] in the dilation theory of completely positive (CP-) semigroups T on $\mathcal{B}$. A *product system* is a family $E^\odot = \big(E_t\big)_{t\in\mathbb{T}}$ (where $\mathbb{T}$ is $\mathbb{R}_+$ or $\mathbb{N}_0$) of Hilbert $\mathcal{B}$–$\mathcal{B}$–modules with an identification

$$E_s \odot E_t \;=\; E_{s+t} \tag{1.1}$$

such that

$$(E_r \odot E_s) \odot E_t \;=\; E_r \odot (E_s \odot E_t), \tag{1.2}$$

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and $E_0 = \mathcal{B}$ (where $\mathcal{B} \odot E_t = E_t = E_t \odot \mathcal{B}$ in the canonical way). By the *trivial* product system we mean $E_t = \mathcal{B}$ for all $t \in \mathbb{T}$. Recall that an *$\mathcal{A}$–$\mathcal{B}$–module* ($\mathcal{A}$ another C^*–algebra) is a Hilbert $\mathcal{B}$–module (right, of course) with a non-degenerate ($*$–)representation of $\mathcal{A}$ by elements in the (C^*–algebra) $\mathcal{B}^a(E)$ of adjointable (and, therefore, bounded and right linear) mappings on E. By $\mathcal{B}^{a,bil}(E)$ we denote the subspace of bilinear or *two-sided* mappings. In particular, an *isomorphism* of two-sided Hilbert modules is a two-sided unitary. The inner product on the *tensor product* is $\langle x \odot y, x' \odot y' \rangle = \big\langle y, \langle x, x' \rangle y' \big\rangle$. For a detailed introduction to Hilbert modules (adapted to our needs) we refer to Skeide [Ske01a], for a quick reference (without proofs) to Bhat and Skeide [BS00].

Formally, product systems appear as a generalization of Arveson's product systems of Hilbert spaces [Arv89] (Arveson systems for short), however, the construction of Arveson systems starting from an *E_0–semigroup* (i.e. a semigroup of unital endomorphisms) on $\mathcal{B}(H)$ for some Hilbert space is very much different from the construction in [BS00] (which starts from a CP-semigroup and yields almost trivialities for E_0–semigroups).

In [Bha96] Bhat discovered another possibility to construct the Arveson system of a (normal, strongly continuous) E_0–semigroup ϑ on $\mathcal{B}(H)$ (H an infinite-dimensional separable Hilbert space). Our first result (Section 2) shows that, contrary to Arveson's original approach [Arv89] via intertwiner spaces of ϑ_t, Bhat's approach generalizes directly to *strict* E_0–semigroups on $\mathcal{B}^a(E)$ for some Hilbert $\mathcal{B}$–module E, at least, if E has a *unit vector* ξ (i.e. $\langle \xi, \xi \rangle = \mathbf{1}$). We show that the constructed product system does not depend (up to isomorphism) on the choice of the unit vector, and that product systems classify strict E_0–semigroups ϑ on the same $\mathcal{B}^a(E)$ up to outer conjugacy. *En passant* we obtain the general form of strict representations of $\mathcal{B}^a(E)$ on another Hilbert module (not necessarily over $\mathcal{B}$). The strict topology of $\mathcal{B}^a(E)$ arises by the observation due to Kasparov [Kas80] that $\mathcal{B}^a(E)$ is the multiplier algebra of the C^*–subalgebra of *compact operators* $\mathcal{K}(E)$ which is generated by the *rank-one operators* $xy^* \colon z \mapsto x\langle y, z \rangle$. In other words, $\mathcal{B}^a(E)$ is the completion of $\mathcal{K}(E)$ in the topology generated by the two families $a \mapsto \|ak\|$ and $a \mapsto \|ka\|$ ($k \in \mathcal{K}(E)$) of seminorms. Here we follow Lance' convention [Lan95] and by the *strict topology* we mean always the restriction to the unit-ball of $\mathcal{B}^a(E)$. One can show that on the ball the strict topology coincides with the $*$–strong topology. In the case of Hilbert spaces the strict topology is the $*$–σ–strong topology. It is well-known that normal representations of $\mathcal{B}(H)$ are also $*$–σ–strong, so for Hilbert modules the strict topology on the ball is, indeed, an appropriate substitute of the normal topology.

Dilations are among the most important objects in the theory of open systems. Suppose $\mathfrak{i} \colon \mathcal{B} \to \mathcal{B}^a(E)$ is a (homomorphic) embedding such that $\langle \xi, \mathfrak{i}(b)\xi \rangle = b$. (In other words, $a \mapsto \mathfrak{i}(\langle \xi, a\xi \rangle)$ is a conditional expectation from $\mathcal{B}^a(E)$ onto $\mathfrak{i}(\mathcal{B})$.) Then we may ask, whether the mappings $T_t \colon b \mapsto \langle \xi, \vartheta_t \circ \mathfrak{i}(b)\xi \rangle$ form a semigroup T (of course, completely positive and unital). In this case, we say $(E, \vartheta, \mathfrak{i}, \xi)$ is a *dilation* of T on E. (If $T_t = \mathsf{id}$ is the *trivial* semigroup, then we speak of a *white noise*.) A dilation may be *unital* or not, depending on whether $\mathfrak{i}$ is unital or not. Since ξ is a unit vector we have always the particular (usually non-unital) embedding $j_0(b) = \xi b \xi^*$. If for this embedding ϑ defines a dilation, then we say (E, ϑ, ξ) is a *weak dilation* of T on E. In this case one can show that $j = \big(j_t\big)_{t \in \mathbb{T}}$ with $j_t = \vartheta_t \circ j_0$ defines a *weak Markov flow* of T in the sense of Bhat and Parthasarathy [BP94, BP95] (i.e. $p_t j_{s+t}(b) p_t = j_t \circ T_s(b)$ where $p_t = j_t(\mathbf{1})$); see [BS00, Ske01a] for details.

In Section 3 we provide some results which allow to decide, if a triple (E, ϑ, ξ) is a weak dilation or a even a white noise. To have a weak dilation (of a necessarily unital CP-semigroup) it it is necessary and sufficient that the family of projections p_t is increasing, and for a white noise we must have $j_t(b)\xi = \xi b$ for all t.

Many dilations are obtained by a cocycle perturbation of a white noise and in Section 2 we see that the product system is invariant under cocycle perturbation (i.e. outer conjugacy). The question arises, whether there are white noises on algebras $\mathcal{A}$ different from $\mathcal{B}^a(E)$, which do not extend to some $\mathcal{B}^a(E)$. In Section 4 we show that at least in the case of automorphism white noises we may embed $\mathcal{A}$ into a suitable $\mathcal{B}^a(E)$ to which the semigroup ϑ extend and is implemented unitarily. Unfortunately, the associated product system is the trivial one.

We close with two technical remarks. We replace normality in the case of Hilbert spaces by strictness. The only place we need strictness is in the proof of Theorem 2.1 where an endomorphism should send an approximate unit for $\mathcal{K}(E)$ to a net converging strictly to $\mathbf{1}$. Passing to von Neumann modules E (whence, $\mathcal{B}^a(E)$ is a von Neumann algebra on a naturally associated Hilbert space) we may again weaken to normal endomorphisms of $\mathcal{B}^a(E)$; see Skeide [Ske01a, BS00, Ske00b] for details about von Neumann modules.

Our methods work provided there exists a unit vector. In Hilbert spaces this is a triviality. The following example shows that here it is a restriction even for von Neumann modules. Nevertheless, we are mainly interested in dilations and in the case of dilations existence of a unit vector is automatic.

1.1 Example. For a projection $p \in \mathcal{B}$ let $E = p\mathcal{B}$ be some right ideal in a von Neumann algebra $\mathcal{B}$. Then E is a Hilbert $\mathcal{B}$–module (even a von Neumann module). Let $q \in \mathcal{B}$ be the central projection generating the strong closure of the ideal $q\mathcal{B}$ of $\mathcal{B}_E = \mathsf{span}(\mathcal{B}p\mathcal{B})$. Of course, if $q \neq \mathbf{1}$, then E cannot have a unit vector. However, also if $q = \mathbf{1}$, the question for a possible unit vector $pb \in E$ has different answers, depending on the choice of $\mathcal{B}$ and p.

Let $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2$. Then $q = \mathbf{1}$ (because M_2 does not contain non-trivial ideals) and

E consists of all matrices $B = \begin{pmatrix} b & b' \\ 0 & 0 \end{pmatrix}$ $(b, b' \in \mathbb{C})$. Consequently, $\langle B, B\rangle = B^*B = \begin{pmatrix} \bar{b}b & \bar{b}b' \\ \bar{b'}b & \bar{b'}b' \end{pmatrix}$. If this is $\mathbf{1}$ then $\overline{b'}b = 0$ from which $b' = 0$ or $b = 0$ so that $\overline{b'}b' = 0$ or $\bar{b}b = 0$. Hence, $\langle B, B\rangle \neq \mathbf{1}$.

Conversely, by definition in a *purely infinite* unital C^*–algebra $\mathcal{B}$ for any $a \geq 0$ (in particular, for the projection p) there exists $b \in \mathcal{B}$ such that $b^*ab = \mathbf{1}$. Instead of exploiting this systematically, we give an example. Consider the elements $b = \ell^*\binom{1}{0}$ and $b' = \ell^*\binom{0}{1}$ in $\mathcal{B} = \mathcal{B}(\mathcal{F}(\mathbb{C}^2))$ where $\mathcal{F}(\mathbb{C}^2)$ is the *full Fock space* over $\mathbb{C}^2$ and $\ell^*(x)$ denotes the usual creator to $x \in \mathbb{C}^2$. (Observe that the von Neumann algebra generated by b, b' is $\mathcal{B}$. The C^*–algebra generated by b, b' is the *Cuntz algebra* $\mathcal{O}_2$ [Cun77].) Now the matrix $B \in M_2(\mathcal{B})$ defined as before is a unit vector in $pM_2(\mathcal{B})$ (where p acts in the obvious way).

2 E_0–Semigroups and products systems

Let (E, ϑ, ξ) be a triple consisting of a Hilbert $\mathcal{B}$–module, a strict E_0–semigroup ϑ on $\mathcal{B}^a(E)$, and a unit vector $\xi \in E$. To begin with, we do not assume that (E, ϑ, ξ) is a weak dilation of a CP-semigroup on $\mathcal{B}$.

By $j_0(b) = \xi b \xi^*$ we define a faithful representation of $\mathcal{B}$ on E. We define the representations $j_t = \vartheta_t \circ j_0$ and set $p_t = j_t(\mathbf{1})$. On the Hilbert submodule $E_t = p_t E$ of E we define a left multiplication by $bx_t = j_t(b)x_t$, thus, turning E_t into a Hilbert $\mathcal{B}$–$\mathcal{B}$–module. (Clearly, $\mathbf{1}x_t = x_t$ and $E_0 \cong \mathcal{B}$ via $\xi \mapsto \mathbf{1}$.)

2.1 Theorem. *The mapping*

$$u_t \colon x \odot x_t \;\longmapsto\; \vartheta_t(x\xi^*)x_t$$

extends as an isomorphism $E \odot E_t \to E$. Moreover, the restrictions $u_{st} = u_t \restriction (E_s \odot E_t)$ are two-sided isomorphisms $E_s \odot E_t \to E_{s+t}$, fulfilling (1.2) in the identification (1.1) so that $E^\odot = \big(E_t\big)_{t\in\mathbb{T}}$ is a product system. Using the identifications

$$E \odot E_t \;=\; E \tag{2.1}$$

and (1.1), we find

$$(E \odot E_s) \odot E_t \;=\; E \odot (E_s \odot E_t) \tag{2.2}$$

and $\vartheta_t(a) = a \odot \mathsf{id}_{E_t}$.

Proof. From

$$\langle x \odot x_t, x' \odot x'_t\rangle \;=\; \big\langle x_t, \langle x, x'\rangle x'_t\big\rangle \;=\; \big\langle x_t, \vartheta_t(\xi\langle x, x'\rangle\xi^*)x'_t\big\rangle \;=\; \big\langle \vartheta_t(x\xi^*)x_t, \vartheta_t(x'\xi^*)x'_t\big\rangle$$

we see that u_t is isometric. Let $u^\lambda = \sum\limits_{k=1}^{n_\lambda} v_k^\lambda w_k^{\lambda *}$ ($v_k^\lambda, w_k^\lambda \in E$) be a bounded approximate unit for $\mathcal{K}(E)$ which, therefore, converges strictly to $\mathbf{1} \in \mathcal{B}^a(E)$. We find

$$x \;=\; \lim_\lambda \vartheta_t(u^\lambda)x \;=\; \lim_\lambda \sum_k \vartheta_t(v_k^\lambda w_k^{\lambda *})x$$
$$=\; \lim_\lambda \sum_k \vartheta_t(v_k^\lambda \xi^*)\vartheta_t(\xi w_k^{\lambda *})x \;=\; \lim_\lambda \sum_k v_k^\lambda \odot \vartheta_t(\xi w_k^{\lambda *})x, \quad (2.3)$$

where $\vartheta_t(\xi w_k^{\lambda *})x = p_t\vartheta_t(\xi w_k^{\lambda *})x$ is in E_t. In other words, u_t is surjective, hence, unitary. Clearly, in the identification (2.1) we find

$$\vartheta_t(a)(x \odot x_t) \;=\; \vartheta_t(a)\vartheta_t(x\xi^*)x_t \;=\; \vartheta_t(ax\xi^*)x_t \;=\; ax \odot x_t.$$

Suppose $p_s x = x$. Then $p_{s+t}u_t(x \odot x_t) = \vartheta_{s+t}(\xi\xi^*)\vartheta_t(x\xi^*)x_t = \vartheta_t(p_s x\xi^*)x_t = u_t(x \odot x_t)$ so that u_{st} maps into E_{s+t}. Obviously, $j_{s+t}(b)u_t(x \odot x_t) = u_t(j_s(b)x \odot x_t)$ so that u_t is two-sided on $E_s \odot E_t$. Suppose $p_{s+t}x = x$ and apply p_{s+t} to (2.3). Then a similar computation shows that we may replace v_k^λ with $p_s v_k^\lambda$ without changing the value. Therefore, $x \in u_t(E_s \odot E_t)$. In other words, u_t restricts to a two-sided unitary $u_{st}\colon E_s \odot E_t \to E_{s+t}$. The associativity conditions (1.2) and (2.2) follow by similar computations. ■

2.2 Remark. If $E = H$ is a Hilbert space with a unit vector h, we recover Bhat's construction [Bha96] resulting in a tensor product system $H^\otimes = \bigl(H_t\bigr)_{t\in\mathbb{T}}$ of Hilbert spaces. It can be shown that it coincides with the corresponding Arveson system; see [Bha96] for a proof in terms of Hilbert spaces or [Ske01a, Example 14.1.3] for a poof using the techniques from [BS00]. See also [Ske01a, Example 14.1.4] where we apply the construction to find the product systems associated with the time shift on the *time ordered* and on the *full* Fock module.

2.3 Proposition. *The product system $E^\odot$ does not depend on the choice of the unit vector ξ. More precisely, if $\xi' \in E$ is another unit vector, then $w_t x_t = \vartheta_t(\xi'\xi^*)x_t$ defines an isomorphism $w^\odot = \bigl(w_t\bigr)_{t\in\mathbb{T}}$ from the product system $E^\odot$ to the product system $E'^\odot = \bigl(E'_t\bigr)_{t\in\mathbb{T}}$ constructed from ξ' (i.e. w_t are two-sided unitaries $E_t \to E'_t$ fulfilling $w_s \odot w_t = w_{s+t}$ and $w_0 = \mathbf{1}$).*

Proof. $p'_t\vartheta_t(\xi'\xi^*) = \vartheta_t(\xi'\xi^*)$ so that w_t maps into E'_t, and $\vartheta_t(\xi'\xi^*)^*\vartheta_t(\xi'\xi^*) = p_t$ so that w_t is an isometry. As $\vartheta_t(\xi'\xi^*)\vartheta_t(\xi'\xi^*)^* = p'_t$, it follows that w_t is surjective, hence, unitary. For $b \in \mathcal{B}$ we find

$$w_t j_t(b) \;=\; \vartheta_t(\xi'\xi^*)\vartheta_t(\xi b\xi^*) \;=\; \vartheta_t(\xi' b\xi^*) \;=\; \vartheta_t(\xi' b\xi'^*)\vartheta_t(\xi'\xi^*) \;=\; j'_t(b)w_t$$

so that w_t is two-sided. In the identification (1.1) (applied to $E^\odot$ and $E'^\odot$) we find

$$
\begin{aligned}
w_s x_s \odot w_t y_t \;=\; \vartheta_t(w_s x_s \xi'^*) w_t x_t \;=\; \vartheta_t(\vartheta_s(\xi'\xi^*) x_s \xi'^*)\vartheta_t(\xi'\xi^*) y_t \\
=\; \vartheta_{s+t}(\xi'\xi^*)\vartheta_t(x_s\xi^*) y_t \;=\; w_{s+t}(x_s \odot y_t),
\end{aligned}
$$

and $w_0 = \xi'\xi^* = \mathbf{1}\mathbf{1}^* = \mathbf{1}$. In other words, the w_t form a morphism. ■

In how far E_0–semigroups on $\mathcal{B}^a(E)$ are classified by their product systems? Of course, we expect as answer that they are classified up to outer conjugacy. First, however, we must clarify in which way we have to ask this question. In Arveson's set-up all Hilbert spaces on which he considers E_0–semigroups are isomorphic. It is this hidden assumption which makes the question for outer conjugacy possible. Nothing gets lost (up to unitary isomorphism), if we restrict Arveson's set-up to a single infinite-dimensional separable Hilbert space. Now we can ask the above question in a reasonable way.

2.4 Theorem. *Let (E, ξ) be a Hilbert $\mathcal{B}$–module E with a unit vector ξ. Furthermore, let ϑ and ϑ' be two strict E_0–semigroups on $\mathcal{B}^a(E)$. Then the product systems $E^\odot$ and $E'^\odot$ associated with ϑ and ϑ', respectively, are isomorphic, if and only if ϑ and ϑ' are* outer conjugate *(i.e. $\vartheta' = \vartheta^{\mathfrak{u}}$ with $\vartheta^{\mathfrak{u}}_t(a) = \mathfrak{u}_t\vartheta_t(a)\mathfrak{u}_t^*$ for some unitary* left cocycle *$\mathfrak{u} = \big(\mathfrak{u}_t\big)_{t\in\mathbb{T}}$ for ϑ in $\mathcal{B}^a(E)$, i.e. $\mathfrak{u}_{s+t} = \mathfrak{u}_t\vartheta_t(\mathfrak{u}_s)$ and $\mathfrak{u}_0 = \mathbf{1}$).*

Proof. Let $\mathfrak{u}$ be a unitary left cocycle for ϑ such that $\vartheta' = \vartheta^{\mathfrak{u}}$. Then $\mathfrak{u}_t p_t = p'_t\mathfrak{u}_t$. Therefore, $\mathfrak{u}_t$ restricts to a unitary $u_t\colon E_t \to E'_t$ (with inverse $u_t^* = \mathfrak{u}_t^* \upharpoonright E'_t$, of course). Moreover, identifying $E \odot E_t = E = E \odot E'_t$, we find

$$
\mathfrak{u}_t(x \odot x_t) \;=\; \mathfrak{u}_t\vartheta_t(x\xi^*)x_t \;=\; \vartheta'_t(x\xi^*)\mathfrak{u}_t x_t \;=\; x \odot u_t x_t.
$$

It follows that $(a \odot \mathsf{id}_{E'_t})\mathfrak{u}_t = \mathfrak{u}_t(a \odot \mathsf{id}_{E_t})$ for all $a \in \mathcal{B}^a(E)$. Specializing to $a = j_0(b)$ so that $a \odot \mathsf{id}_{E_t} = j_t(b)$ and $a \odot \mathsf{id}_{E'_t} = j'_t(b)$, we see that u_t is the (unique) element in $\mathcal{B}^{a,bil}(E_t, E'_t)$ such that $\mathfrak{u}_t = \mathsf{id} \odot u_t$. From

$$
\mathsf{id} \odot u_{s+t} \;=\; \mathfrak{u}_{s+t} \;=\; \mathfrak{u}_t\vartheta_t(\mathfrak{u}_s) \;=\; (\mathsf{id} \odot \mathsf{id}_{E'_s} \odot u_t)(\mathsf{id} \odot u_s \odot \mathsf{id}_{E_t}) \;=\; \mathsf{id} \odot u_s \odot u_t
$$

we see that $u^\odot = \big(u_t\big)_{t\in\mathbb{T}}$ is a morphism.

Conversely, suppose $u^\odot$ is an isomorphism from $E^\odot$ to $E'^\odot$. Then $\mathfrak{u}_t = \mathsf{id} \odot u_t \colon E = E \odot E_t \to E \odot E'_t = E$ defines a unitary on E. We find

$$
\mathfrak{u}_t\vartheta_t(a)\mathfrak{u}_t^* \;=\; (\mathsf{id} \odot u_t)(a \odot \mathsf{id}_{E_t})(\mathsf{id} \odot u_t^*) \;=\; (a \odot \mathsf{id}_{E'_t})(\mathsf{id} \odot u_t u_t^*) \;=\; a \odot \mathsf{id}_{E'_t} \;=\; \vartheta'(a)
$$

and as above

$$
\mathfrak{u}_{s+t} \;=\; \mathsf{id} \odot u_s \odot u_t \;=\; (\mathsf{id} \odot \mathsf{id}_{E'_s} \odot u_t)(\mathsf{id} \odot u_s \odot \mathsf{id}_{E_t}) \;=\; \mathfrak{u}_t\vartheta_t(\mathfrak{u}_s)
$$

In other words, $\mathfrak{u} = \bigl(\mathfrak{u}_t\bigr)_{t\in\mathbb{T}}$ is a unitary left cocycle and $\vartheta' = \vartheta^{\mathfrak{u}}$. ■

Notice that by a slight extension of the proof of Theorem 2.1 we settle completely the theory of strict representations of $\mathcal{B}^a(E)$. The uniqueness statements in the following corollary are restrictions from Proposition 2.3 and Theorem 2.4.

2.5 Corollary. *Let E be a Hilbert $\mathcal{B}$–module with a unit vector ξ, and let $\vartheta\colon \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ be a strict unital representation on a Hilbert $\mathcal{C}$–module F ($\mathcal{C}$ some unital C^*–algebra). Then the submodule $F_\vartheta = \vartheta(\xi\xi^*)F$ of F with the left action by $= \vartheta(\xi b\xi^*)$ of $b \in \mathcal{B}$ is a Hilbert $\mathcal{B}$–$\mathcal{C}$–module such that $F = E \odot F_\vartheta$ via $x \odot y \mapsto \vartheta(x\xi^*)y$ and*

$$\vartheta(a) \;=\; a \odot \mathsf{id}_{F_\vartheta}\,.$$

F_ϑ does not depend (up to two-sided isomorpism) on the choice of ξ. Moreover, if $\vartheta'\colon \mathcal{B}^a(E) \to \mathcal{B}^a(F')$ is another strict unital representation such that there exists a two-sided isomorphism $u\colon F_\vartheta \to F'_{\vartheta'}$, then $\mathfrak{u} = \mathsf{id}_E \odot u\colon F = E \odot F_\vartheta \to E \odot F'_{\vartheta'} = F'$ defines a unitary in $\mathcal{B}^a(F,F')$ such that $\vartheta'(a) = \mathfrak{u}\vartheta(a)\mathfrak{u}^*$.

3 Weak dilation and white noise

Now we want to know under which circumstances (E, ϑ, ξ) is a weak dilation, or even a white noise.

Recall that a ***unit*** for a product system $E^\odot$ is a family $\xi^\odot = \bigl(\xi_t\bigr)_{t\in\mathbb{T}}$ of elements $\xi_t \in E_t$ such that $\xi_s \odot \xi_t = \xi_{s+t}$ and $\xi_0 = \mathbf{1}$. For all units $T_t(b) = \langle\xi_t, b\xi_t\rangle$ defines a CP-semigroup T which is unital, if and only if $\xi^\odot$ is ***unital*** (i.e. all ξ_t are unit vectors). A unit is ***central***, if $b\xi_t = \xi_t b$ for all $t \in \mathbb{T}, b \in \mathcal{B}$.

A result from [BS00] asserts that for a unital unit $\gamma_{(s+t)t}\colon x_t \mapsto \xi_s \odot x_t$ establishes an inductive system of isometric (in general not two-sided, but cf. Proposition 3.2) embeddings $E_t \to E_{s+t}$. The corresponding inductive limit has a distinguished unit vector and carries a strict E_0–semigroup which alltogether form a weak dilation of T.

3.1 Proposition. *For the triple (E, ϑ, ξ) the following conditions are equivalent.*

1. *The family p_t of projections is increasing, i.e. $p_t \geq p_0$ for all $t \in \mathbb{T}$.*
2. *The mappings $T_t(b) = \langle\xi, j_t(b)\xi\rangle$ define a unital CP-semigroup T, i.e. (E, ϑ, ξ) is a weak dilation.*
3. *$T_t(\mathbf{1}) = \mathbf{1}$ for all $t \in \mathbb{T}$.*

Under any of these conditions the elements $\xi_t = \xi \in E_t$ form a unital unit $\xi^\odot$ such that $T_t(b) = \langle \xi_t, b\xi_t \rangle$, and the j_t form a weak Markov flow of T on E. The inductive limit for $\xi^\odot$ coincides with the submodule $E_\infty = \lim_{t\to\infty} p_t E$ of E.

PROOF. $1 \Rightarrow 2$. If p_t is increasing, then $p_t\xi = p_t p_0 \xi = p_0\xi = \xi$ so that $T_t(\mathbf{1}) = \langle \xi, p_t\xi \rangle = \mathbf{1}$ and

$$\begin{aligned} T_t \circ T_s(b) &= \Big\langle \xi, j_t\big(\langle \xi, j_s(b)\xi \rangle\big)\xi \Big\rangle = \Big\langle \xi, \vartheta_t\big(\xi \langle \xi, \vartheta_s(\xi b \xi^*)\xi \rangle \xi^*\big)\xi \Big\rangle \\ &= \big\langle \xi, p_t \vartheta_t \circ \vartheta_s(\xi b \xi^*) p_t \xi \big\rangle = \big\langle \xi, \vartheta_{s+t}(\xi b \xi^*)\xi \big\rangle = T_{s+t}(b). \end{aligned}$$

$2 \Rightarrow 3$ is clear. For $3 \Rightarrow 1$ assume that T_t is unital. We find $p_0 = \xi\xi^* = \xi T_t(\mathbf{1})\xi^* = \xi\xi^*\vartheta_t(\xi\xi^*)\xi\xi^* = p_0 p_t p_0$, hence, $p_t \geq p_0$ for all $t \in \mathbb{T}$.

If p_t is increasing then $p_t\xi = \xi$ so that ξ is, indeed, contained in all E_t. Obviously, $\xi_s \odot \xi_t = \vartheta_t(\xi_s\xi^*)\xi_t = \vartheta_t(\xi\xi^*)\xi = \xi = \xi_t$ so that $\xi^\odot$ is a unital unit. However, the identification of ξ as an element $\xi_t \in E_t$ changes left multiplication, namely, $b\xi_t = j_t(b)\xi$, i.e. $T_t(b) = \langle \xi_t, b\xi_t \rangle$. The Markov property follows as that for the inductive limit in [BS00]. As above, we have $\xi_s \odot x_t = p_t x_t = x_t$. In other words, $\gamma_{(s+t)t}$ is the canonical embedding of the subspace E_t into E_{s+t}. This identifies E_∞ as the inductive limit for $\xi^\odot$. ■

3.2 Proposition. *On E_∞ there exists a (unital!) left multiplication of $\mathcal{B}$ such that all E_t are embedded into E_∞ as two-sided submodules, if and only if the unit $\xi^\odot$ is central, i.e. if $(E_\infty, \vartheta, \mathfrak{i}, \xi)$ with $\mathfrak{i}$ being the canonical left multiplication of $\mathcal{B}$ on E is a (unital) white noise.*

PROOF. Existence of a left multiplication on E_∞ implies (in particular) that $b\xi = j_0(b)\xi = \xi b$. The converse direction is obvious. ■

This shows importance of existence of a central unit which was also crucial in Barreto, Bhat, Liebscher and Skeide [BBLS00] in the analysis of so-called type I product systems. Without central unit we may not even hope to understand a dilation as a cocycle perturbation of a white noise, because by Theorem 2.4 a cocycle perturbation does not change the product system. Meanwhile, in [Ske01b] we introduced an index theory for product systems with a central unital unit (called *spatial* product systems) paralelling that of Arveson.

4 Unital dilations and automorphism white noises

Is our set-up of dilation to $\mathcal{B}^a(E)$ very special? This section is devoted to demonstrate that the answer is a clear 'no'. First of all, let us mention that so far the large majority of concrete dilations was constructed on $\mathcal{B}(H)$, so our case $\mathcal{B}^a(E)$ is certainly more general than all these cases. The explicit construction of dilations of uniformly continuous CP-semigroups on a symmetric or full Fock module ([GS99] or [Ske00a]) shows that our set-up is not too special to obtain interesting results.

There is few literature on dilations to more general C^*–algebras or von Neumann algebras $\mathcal{A}$. Sauvageot [Sau86] constructs a unital dilation fulfilling all conditions required by other authors except that even starting from a σ–weakly continuous CP-semigroup, the dilating E_0–semigroup is not σ–weakly continuous. This is a serious obstacle which, for instance, our weak dilation from [BS00] does not have.

Another approach to (unital) dilation, to which we concentrate our attention, is that by Kümmerer [Küm85] (being the basis for the abstract quantum stochastic calculus developed by Hellmich, Köstler and Kümmerer[HKK98, Hel01, Kös00]). As these dilations are cocycle perturbations of white noises and cocycle perturbations do not change product systems, we describe only the white noise. The E_0–semigroup $\vartheta = \big(\vartheta_t\big)_{t\in\mathbb{R}_+}$ consists of normal automorphisms on a von Neumann algebra $\mathcal{A}$ ([Küm85]) and, therefore, may be extended to an automorphism group. $\mathcal{A}$ comes along with a *future subalgebra* $\mathcal{A}_+$ invariant for ϑ_t $(t \geq 0)$ and a *past subalgebra* $\mathcal{A}_-$ invariant for ϑ_t $(t \leq 0)$, such that $\mathcal{A}_+ \cap \mathcal{A}_- = \mathcal{A}_0 = \mathfrak{i}(\mathcal{B})$. As we want a white noise, ϑ must *leave invariant* the expectation $\mathfrak{p}$, i.e. $\mathfrak{p}\circ\vartheta_t = \mathfrak{p}$. In [Küm85] $\mathfrak{p}$ should also be *faithful* (i.e. $\mathfrak{p}(a^*a) = 0$ implies $a = 0$), but we need only (occasionally) the weaker requirement that the following GNS-representation of $\mathcal{A}$ is faithful.

The *GNS-module* E of $\mathfrak{p}$ is the completion of the $\mathcal{A}$–$\mathcal{B}$–module $\mathcal{A}\xi$ with left multiplication $a(a'\xi) = (aa')\xi$, with right multiplication $a\xi b = (a\mathfrak{i}(b))\xi$ and with inner product $\langle a\xi, a'\xi\rangle = \mathfrak{p}(a^*a')$. (If $\mathfrak{p}$ is not faithful, then length-zero elements must be divided out.) Then ξ is a unit vector and $\mathfrak{p}(a) = \langle \xi, a\xi\rangle$. The canonical mapping $\mathcal{A} \to \mathcal{B}^a(E)$ is the *GNS-representation* of $\mathcal{A}$.

Any automorphism α of $\mathcal{A}$ which leaves invariant $\mathfrak{p}$ gives rise to an isometry $u\colon a\xi \mapsto \alpha(a)\xi$. (Indeed, $\langle\alpha(a)\xi, \alpha(a')\xi\rangle = \mathfrak{p}\circ\alpha(a^*a') = \mathfrak{p}(a^*a') = \langle a\xi, a'\xi\rangle$.) Doing the same for α^{-1} we find that u is invertible, whence, unitary. Moreover, we find

$$\alpha(a)a'\xi \;=\; \alpha(a\alpha^{-1}(a'))\xi \;=\; ua\alpha^{-1}(a')\xi \;=\; uau^*a'\xi$$

for all $a' \in \mathcal{A}$ so that the action of $\alpha(a)$ on E is uau^*. If the GNS-representation is faithful, then $\mathcal{A} \subset \mathcal{B}^a(E)$ and $\alpha(a) = uau^*$. If the GNS-representation is not faithful, we see that α respects the kernel of the GNS-representation. Hereafter, we assume that this kernel has been divided out and, therefore, that $\mathcal{A} \subset \mathcal{B}^a(E)$.

Applying this to all ϑ_t, we find a unitary group $\big(u_t\big)_{t\in\mathbb{R}}$ in $\mathcal{B}^a(E)$ implementing ϑ_t as $u_t \bullet u_t^*$. Clearly, this group extends from $\mathcal{A}$ to an automorphism group (also denoted by

ϑ) on all of $\mathcal{B}^a(E)$. Of course, ϑ is strict. We see that automorphism white noises may be extended (possibly after having divided out the kernel of the GNS-representation) to a unitarily implemented (and, therefore, strict) automorphism white noise on $\mathcal{B}^a(E)$.

Restricting to $t \geq 0$ we are in the setting of Section 2. Unfortunately, we have $u\xi = \xi$ so that ϑ leaves invariant $p_0 = \xi\xi^*$. Consequently, the product system of Hilbert $\mathcal{B}$–$\mathcal{B}$–modules as constructed in Theorem 2.1 is the trivial one.

Can we also obtain a non-trivial product system? The answer to this question depends on whether we are able or not to extend also the restriction of the automorphism group α on $\mathcal{A}$ to a proper E_0–semigroup ϑ on $\mathcal{A}_+$ to all of $\mathcal{B}^a(E_+)$ where $E_+ = \overline{\mathcal{A}_+\xi}$ is the closure of what $\mathcal{A}_+$ generates from the cyclic vector ξ. (Then we may consider the product system associated with the extension of ϑ.) This is possible, for instance, if E factorizes in the form $E = E_+ \odot E'$ for some suitable Hilbert $\mathcal{B}$–$\mathcal{B}$–module E' in such a way that an element $a \in \mathcal{A}_+$ acts as $a_+ \odot \mathsf{id}_{E'}$. Notice that in this case the extension of α to $\mathcal{B}^a(E)$ leaves invariant the subalgebra $\mathcal{B}^a(E_+) \odot \mathsf{id}_{E'} \cong \mathcal{B}^a(E')$ which contains $\mathcal{A}_+$ as a subalgebra. We know from [Ske01a] that, for instance, the white noises constructed on the full or the time ordered Fock module have this factorization property (and we know also the corresponding product systems). The fact that presently there do apparently not exist other explicit examples of white noises (over $\mathcal{B}$!) in the sense of [Küm85] supports the belief that this might remain true for all such white noises. Presently, we cannot prove it, but we think it is an interesting question.

References

[Arv89] W. Arveson. *Continuous analogues of Fock space.* Number 409 in Memoires of the American Mathematical Society. American Mathematical Society, 1989.

[Arv90] W. Arveson. Continuous analogues of Fock space IV: essential states. *Acta Math.*, 164:265–300, 1990.

[BBLS00] S.D. Barreto, B.V.R. Bhat, V. Liebscher, and M. Skeide. Type I product systems of Hilbert modules. Preprint, Cottbus, 2000.

[Bha96] B.V.R. Bhat. An index theory for quantum dynamical semigroups. *Trans. Amer. Math. Soc.*, 348:561–583, 1996.

[Bha01] B.V.R. Bhat. *Cocycles of CCR-flows.* Number 709 in Memoires of the American Mathematical Society. American Mathematical Society, 2001.

[BP94] B.V.R. Bhat and K.R. Parthasarathy. Kolmogorov's existence theorem for Markov processes in C^*–algebras. *Proc. Indian Acad. Sci. (Math. Sci.)*, 104:253–262, 1994.

[BP95] B.V.R. Bhat and K.R. Parthasarathy. Markov dilations of nonconservative dynamical semigroups and a quantum boundary theory. *Ann. I.H.P. Prob. Stat.*, 31:601–651, 1995.

[BS00] B.V.R. Bhat and M. Skeide. Tensor product systems of Hilbert modules and dilations of completely positive semigroups. *Infinite Dimensional Analysis, Quantum Probability & Related Topics*, 3:519–575, 2000.

[Cun77] J. Cuntz. Simple C^*–algebras generated by isometries. *Commun. Math. Phys.*, 57:173–185, 1977.

[GS99] D. Goswami and K.B. Sinha. Hilbert modules and stochastic dilation of a quantum dynamical semigroup on a von Neumann algebra. *Commun. Math. Phys.*, 205:377–403, 1999.

[Hel01] J. Hellmich. *Quantenstochastische Integration in Hilbertmoduln.* PhD thesis, Tübingen, 2001.

[HKK98] J. Hellmich, C. Köstler, and B. Kümmerer. Stationary quantum Markov processes as solutions of stochastic differential equations. In R. Alicki, M. Bozejko, and W.A. Majewski, editors, *Quantum probability*, volume 43 of *Banach Center Publications*, pages 217–229. Polish Academy of Sciences — Institute of Mathematics, 1998.

[Kas80] G.G. Kasparov. Hilbert C^*–modules, theorems of Stinespring & Voiculescu. *J. Operator Theory*, 4:133–150, 1980.

[Kös00] C. Köstler. *Quanten-Markoff-Prozesse und Quanten- Brownsche Bewegungen.* PhD thesis, Tübingen, 2000.

[Küm85] B. Kümmerer. Markov dilations on W^*–algebras. *J. Funct. Anal.*, 63:139–177, 1985.

[Lan95] E.C. Lance. *Hilbert C^*–modules.* Cambridge University Press, 1995.

[Sau86] J.-L. Sauvageot. Markov quantum semigroups admit covariant Markov C^*–dilations. *Commun. Math. Phys.*, 106:91–103, 1986.

[Ske00a] M. Skeide. Quantum stochastic calculus on full Fock modules. *J. Funct. Anal.*, 173:401–452, 2000.

[Ske00b] M. Skeide. Generalized matrix C^*–algebras and representations of Hilbert modules. *Mathematical Proceedings of the Royal Irish Academy*, 100A:11–38, 2000.

[Ske01a] M. Skeide. *Hilbert modules and applications in quantum probability.* Habilitationsschrift, Cottbus, 2001. Available at `http://www.math.tu-cottbus.de/INSTITUT/lswas/_skeide.html`.

[Ske01b] M. Skeide. The index of white noises and their product systems. Preprint, Rome, 2001.