Michael Skeide

Hilbert Modules and Applications in Quantum Probability

Michael Skeide

Lehrstuhl für Wahrscheinlichkeitstheorie und Statistik
Brandenburgische Technische Universität Cottbus
Postfach 10 13 44, D–03013 Cottbus, Germany
E-mail: `skeide@math.tu-cottbus.de`
Homepage: `http://www.math.tu-cottbus.de/INSTITUT/lswas/_skeide.html`

Meinen Eltern

Preface

In these notes we treat problems related to that part of quantum probability dealing with the theory of dilations of semigroups of completely positive mappings on a unital C^*–algebra (CP-semigroups). Our main tool is the theory of Hilbert modules, which so far has not yet drawn so much attention in quantum probabilty. We hope that these notes demonstrate that the use of Hilbert modules allows, firstly, to rephrase known results in a simple way (for instance, a reformulation of quantum stochastic calculus on Fock spaces) furnished with new effective proofs which allow easily to generalize the range of applicability (for instance, extension of results well-known for the algebra $\mathcal{B}(G)$ of operators on a Hilbert space G to more general C^*–algebras) and, secondly, to find new useful and interesting structures leading to powerful results which cannot even be formulated without Hilbert modules (in the first place, product systems of Hilbert modules).

The choice of subjects is entirely personal and represents the major part of the author's contributions to quantum probability by using Hilbert modules. The material is arranged in an introduction, four parts, each of which with an own introduction, and five appendices.

Although in Part I we give a thorough introduction to Hilbert modules including full proofs, this is not a book on Hilbert modules. The presentation of the material (which may be considered as the author's only contribution in Chapters 1 – 4, except possibly the notion of von Neumann modules [Ske00b] which seems to be new) is adapted precisely to the quantum probabilist's needs. Basic knowledge of C^*–algebras and a few elementary properties of von Neumann algebras (like, for instance, in Murphy [Mur90]) are sufficient, and starting from this level these notes are self-contained, although some experience in quantum probability should be helpful. In Chapter 5 of Part I we introduce the notion of *completely positive definite kernels* (being a generalization of both positive definite kernels and completely positive mappings) and *completely positive definite semigroups* (*CPD-semigroups*) [BBLS00]. These notions, crucial for Part III, may be considered as the first original result.

Using Hilbert modules, in the first place, means to provide a representation space for an algebra to act on. So far, this is a property which Hilbert modules have in common with Hilbert spaces. However, often the algebra of all operators on a Hilbert space is too simple (or better too big in order that the structure we are interested in extends to it), and it is not always easy to decide whether a concrete operator belongs to a distinguished subalgebra of

$\mathcal{B}(G)$ or not. Considering algebras of (adjointable) operators on Hilbert modules has the advantage that, although still allowing for sufficiently many interesting C^*–algebras, much of the simplicity of the Hilbert space case is preserved. The most important representation modules (but by far not all) are *Fock modules* which we consider in Part II. Starting from Pimsner's [Pim97] and Speicher's [Spe98] *full* Fock module, we discuss the *time ordered* Fock module [BS00], the *symmetric* Fock module [Ske98a] (including the realization from [AS00a, AS00b] of the *square of white noise* introduced in Accardi, Lu and Volovich [ALV99]) and the relation, discovered in [AS98], to *interacting* Fock spaces as introduced in Accardi, Lu and Volovich [ALV97].

Part III about *tensor product systems of Hilbert modules* [BS00] may be considered as the heart of these notes. In some sense it is related to every other part. The most important product system consists of *time ordered Fock modules* (treated in detail in Part II) and by Section 14.1 any dilation on a Hilbert module (i.e. also the dilation constructed in Part IV) leads to a product system [Ske00a] and, thus, the classification of product systems is in some sense also a classification of dilations. In Chapter 15 we give a summary of solved and some interesting open problems related to product systems. In particular, the open problems show that these notes cannot be more than the beginning of a theory of product systems of Hilbert modules.

While Part III is "purely module" (it can not be formulated without Hilbert modules), Part IV, where we present the *quantum stochastic calculus on the full Fock module* from [Ske00d], is among the results which are generalizations of well-known quantum probabilistic methods to Hilbert modules. It demonstrates how effectively results for operators on Hilbert spaces can be rephrased and simplified by using Hilbert modules, and then generalize easily to arbitrary C^*–algebras.

The appendices serve different purposes. Appendix A (Banach spaces, pre–C^*–algebras, inductive limits), Appendix B (functions with values in Banach spaces) and Appendix E contain preliminary material which does not belong to Part I or material which is, although well-known, difficult to find in literature in a condensed form. These appendices are included for the reader's convenience and to fix notations. Appendix C (on Hilbert modules over $*$–algebras with the very flexible algebraic notion of positivity from [AS98]) and Appendix D (the construction from [Ske98a] of a full Fock module describing a concrete quantum physical model) contain original material which we did not want to omit, but which would somehow interrupt the main text too much if included there.

We would like to mention that (with exception of a few words in Section 17.2) we do not tackle problems related to notions of quantum stochastic independence. If we intend to give a "round picture" of this subject this would fill another book.

• • •

How to read this book? Some parts are more or less independent of each other and it is not necessary to read them all. The expert in Hilbert modules need not read most of Part I (although we hope that also the expert will find some suprising proofs of well-known statements). We collect all non-standard notions in Appendix E. Much of Part I is illustrated in form of examples, which point directly to our applications. In particular, there are two series of examples, one related to matrices of Hilbert modules (being crucial for product systems) and the other related to modules over $\mathcal{B}(G)$ (building the bridge to existing work in the framework of Hilbert spaces) which we refer to throughout the remaining parts. As it is sometimes tedious to follow all references, also the contents of these examples is collected in Appendix E in a condensed form.

The full Fock module in Chapter 6 is the basis for all other Fock modules in Part II. The reader who is interested only in the quantum stochastic calculus in Part IV, needs to read only Chapter 6. The reader interested only in product systems needs to read also Chapter 7 on the time ordered Fock module (the most important product system). Also Chapter 5 from Part I on CPD-semigroups is indispensable for product systems. On the other hand, if the reader is interested only in product systems for their own sake but not so much in dilations, then he may skip Chapter 12 which deals mainly with aspects coming explicitly from CP-semigroups. The remainder of Part II is independent of all what follows. Chapter 8 deals with symmetric Fock modules and the square of white noise. Chapter 9 illuminates the close relationship between interacting Fock spaces and full Fock modules. In both chapters the modules are no longer modules over C^*–algebras, and it is necessary to know Appendix C where we provide the appropriate generalizations.

In any case we assume that notions from Appendix B, where we introduce all functions spaces in these notes and some lattices related to interval partitions, are known.

• • •

Acknowledgements. Many people helped me finishing these notes with discussions, direct and indirect support, general interest and lots of patience. First of all, I want to express my gratitude to my coworkers, who ventured together with me to “try out” the language of Hilbert modules. These are Luigi Accardi (who brought me into contact with Hilbert modules in quantum probability), Stephen Barreto, Rajarama Bhat (who taught me dilations and products systems of Hilbert spaces and was the first to share my belief that consequent use of Hilbert modules is promising) and Volkmar Liebscher.

Further thanks go to those people who taught me quantum probability and whose influence on my way of mathematical thinking is invaluable: Luigi Accardi, Wolfgang Freudenberg, Michael Schürmann and Wilhelm von Waldenfels. Many colleagues helped me in discussions and brought to my attention aspects which I missed before. Among many others, I want to mention Uwe Franz, Claus Köstler, Sergej Kozyrev, Yun Gang Lu and Roland Speicher.

Finally, I acknowledge gratefully financial support from several institutions. Without the research grant by the DFG for two visits of one year each at the Centro Vito Volterra in Rome this book would not exist. In this context, it is my special wish to express my gratitude to Luigi Accardi, who did not only host me at the Centro Vito Volterra during the grants, but also continued inviting me for many visits, thus being responsible for the continuity necessary for finishing this work. The most recent results were discovered during a two week RiP-program in Oberwolfach (with Rajarama Bhat), a two week visit at the ISI in Bangalore (sponsered by DGF and ISI) and a one month stay of my Indian colleagues in Germany supported by a DAAD-DST Project based Personnel exchange Programme.

Contents

Part III

Product systems 161

Introduction

Quantum probability is a generalization of classical probability theory, designed to include also the description of quantum physical systems. The descriptive idea is simple. We replace a probability space by a commutative $*$–algebra of (measurable) functions on the probability space and the probability measure gives rise to a state on this algebra. Events, i.e. (measurable) subsets of the probability space, are represented by indicator functions, which are (more or less) the projections in our algebra. Allowing the algebra to be noncommutative (and, thus, forgetting about the underlying space) we are in the situation of quantum physics where the observable quantities are represented by (self-adjoint) elements of a $*$–algebra, and where expectation values of the observables are computed with the help of a state. For instance, we obtain the probability that in a measurement of an observable we measure an eigenvalue λ of that observable just by evaluating the state at the projection onto the eigenspace of that eigenvalue.

Apart from this purely descriptive idea (to replace a space by an algebra of functions on the space is the basis for many noncommutative generalizations like noncomutative topology, noncommutative geometry, etc.), real quantum probability starts when probabilistic ideas come into the game. A typical situation in classical probability (in some sense the starting point of modern classical probability) is the famous experiment by Brown who observed in the microscope a particle moving on the surface of a liquid under the influence of the thermal movement of the molecules of the liquid. The situation is typical in a two-fold sense.

Firstly, we are interested in the movement of the observed particle and not so much in the thermal environment. In other words, we observe a small subsystem of a bigger system, and we are satisfyied by knowing only the evolution of the small subsystem. So, there should be a prescription how to come from the big system to the small subsystem. In classical probability this prescription is *conditioning* onto the subsystem (see Example 4.4.14) and it is one of the striking features that conditioning is always possible, independently of the concrete nature of the system and its subsystem. The quantum analogue is a *conditional expectation* from the algebra describing the big system onto the subalgebra corresponding to the small subsystem (which should leave invariant the state). There are two problems with this idea. On the one hand, it is no longer true that such a (state preserving) conditional expectation from an algebra onto a subalgebra always exists. On the other hand, the properties of conditional expectations are too narrow to include all interesting (physical!) examples. For instance, existence of a conditional expectation $M_4 = M_2 \otimes M_2 \to M_2 \otimes \mathbf{1}$ preserving a state ψ on M_4 implies that $\psi = \psi_1 \otimes \psi_2$ is a tensor product of states ψ_i on M_2. Thus, many authors prefer to use *transition expectations* fulfilling weaker conditions (cf. Section 14.2).

Secondly, although in principle the evolution of the whole system is deterministic (because the microscopic laws of nature, be it classical or quantum, are deterministic), we can never get hold of a complete description of the whole system, because it is too complex. Here, classically, the strategy is to describe the big system by a probabilistic model. In other words, one may say that the system *is* in a certain state only with a certain probability. The micro physical differential equations decribing the precise evolution of the whole system transform into *stochastic differential equations* (or *Langevin equations*). These equations must be solved, if we are interested in data of the *reservoir* (for instance, the *photon statistics* of the electro magnetic field in thermal equilibrium). After conditioning onto the the small subsystem we obtain the equations for the small system (the *master equations* which takes into account only the *average influence* of the environment). These are sufficient, if we are interested only in data of the small subsystem (for instance, in the *level statistics* of an atom in the electro magnetic field).

The probabilistic models which describe the whole system are motivated by the idea that states of *thermal equilibrium* are states of *maximal dissorder* (or *entropy*). Such considerations lead to stochastic processes as integrators (representing the average effect of the *heat bath* on the particle) which have *independent increments* (i.e. the differential in the Langevin equation *driving* the particle are stochastically independent of what happened before) and the distribution of the single increment should be interpretable as a *stochastic sum* of many independent and identically distributed micro effects, i.e. the distribution should be, for instance, some *central limit distribution.* It is noteworthy that the notion of independence in quantum probabilty (and, thus, what the central limit distibutions of differentials in *quantum stochastic differential equations* can be) is, unlike the classical case, far from being unique.

It is not our intention to give a detailed account of all these aspects nor to give a complete list of references, and we content ourselves with the preceding very rough outline of some of the basic ideas. For more detailed introductions from very different points of view we recommend the monographs by Meyer [Mey93] and Parthasarathy [Par92] and the references therein. An effective way to come from concrete physical systems to the situation described before is the *stochastic limit* which has a form very much like a central limit procedure; see the monograph by Accardi, Lu and Volovich [ALV01].

Unlike in classical probability, where we have essentially one notion of independence (which determines to some extent the possible distributions of the integrators in the Langevin equation), in quantum probability there is a still increasing number of concrete independences. As we wrote in the preface we will not consider these interesting questions.

The mathematical abstraction of the preceding considerations is the setting of *dilation theory*. An introduction can be found in Chapter 10. However, before we start describing in greater detail how dilations led us directly to product systems of Hilbert modules, the central notion in these notes, we would like to mention that our investigations started from applications (the interpretation of the *QED-module* as *full Fock module* as described in Appendix D) and lead back to them (be it the calculus on the full Fock module in Part IV, be it the classification of full Fock modules by their associated product systems as in Example 14.1.4, or be it the interpretation of *interacting Fock spaces* like that of the *square of white noise* as Hilbert module in Chapters 8 and 9).

•••

Let us denote by $\mathcal{B}$ a unital C^*–algebra which stands for the observables of the small subsystem. Being only a subsystem subject to interaction with the whole system, $\mathcal{B}$ is a typical example for an *open system*. It is a general *ansatz* used by many authors (we mention only Davies [Dav76]) that the *reduced dynamics* of the subsystem is described by a *unital CP-semigroup* T on $\mathcal{B}$. (Other names are *quantum dynamical semigroup*, or *Markovian semigroup* (which sometimes is only required to be positive), and sometimes the word *conservative* is used instead of *unital*.) Often (for instance, in the example of the *QED-module* which we consider in Appendix D) it can be shown from *first principles* that the reduced dynamics has this form and also in the classical case it is possible to describe a (stationary) classical Markov process (for instance, a *diffusion*) by a unital CP-semigroup (see, for instance, the introduction of Bhat and Parthasarathy [BP94]).

The whole system is described on a unital C^*–algebra (or sometimes a pre–C^*–algebra) $\mathcal{A}$, its time evolution by an E_0*–semigroup* ϑ on $\mathcal{A}$. (Many authors require automorphism groups instead of unital endomorphisms. However, in this case $\mathcal{A}$ comes along with a *future subalgebra* which is left invariant by the automorphisms to times $t > 0$, thus, giving rise to an endomorphism semigroup on the future. We do not comment on this and refer the reader to the literature, e.g. Arveson [Arv96].) Considering $\mathcal{B}$ as a subalgebra of $\mathcal{A}$ means to embed $\mathcal{B}$ with a monomorphism $\mathfrak{i}\colon \mathcal{B} \to \mathcal{A}$. (We could directly identify $\mathcal{B}$ as a subalgebra of $\mathcal{A}$, however, we find it more convenient to have the freedom to consider different embeddings $\mathfrak{i}$.) The conditioning is provided by an *expectation* $\mathfrak{p}\colon \mathcal{A} \to \mathcal{B}$ such that $\mathfrak{i} \circ \mathfrak{p}\colon \mathcal{A} \to \mathfrak{i}(\mathcal{B})$ is a *conditional expectation* (see Section 4.4). We speak of a *dilation* of T, if $\mathfrak{p} \circ \vartheta_t \circ \mathfrak{i} = T_t$. (This means that Diagram (10.1.1) in Chapter 10, where we give an introduction to dilation, commutes.)

In this set-up there are at least two possibilities of how Hilbert modules (right modules over $\mathcal{B}$ with a $\mathcal{B}$–valued inner product; see Chapter 1) come into the game. Both are based on the generalization of the GNS-construction for a state on a C^*–algebra $\mathcal{A}$ to a GNS-construction for a *completely positive mapping* between C^*–algebras $\mathcal{A}$ and $\mathcal{B}$ (see Section 4.1). The result is a Hilbert $\mathcal{A}$–$\mathcal{B}$–module E (a Hilbert module with a representation of $\mathcal{A}$) and a *cyclic vector* $\xi \in E$ such that the completely positive mapping is recovered as a *matrix element* $a \mapsto \langle \xi, a\xi \rangle$ with the cyclic vector.

If we start with a dilation (i.e. if we know the big system), then the first possibility to obtain a Hilbert module is by GNS-construction for the expectation $\mathfrak{p}$ (being in the first place completely positive). The algebra $\mathcal{A}$ may or may not be representated faithfully on the GNS-module E and the image of $\mathcal{A}$ in the algebra $\mathcal{B}^a(E)$ of all (adjointable) operators on E may or may not be all of $\mathcal{B}^a(E)$. In these notes we will concentrate on the case where both assertions are true, in other words, where $\mathcal{A} = \mathcal{B}^a(E)$. By construction E has a *unit vector* ξ (i.e. $\langle \xi, \xi \rangle = \mathbf{1}$). The most general E_0–semigroups which we consider (not necessarily coming from a dilation) are E_0–semigroups on $\mathcal{B}^a(E)$ where E is some Hilbert module E with a unit vector ξ in Section 14.1.

More interesting is the situation where we start with a CP-semigroup T on $\mathcal{B}$ and ask whether it is possible to construct a dilation of T. In other words, we want to construct a big system with a time evolution into which we can embed the small one, giving us back the correct evolution of the small system. Here we are concerned with a whole family $\breve{E}_t$ of GNS-modules with cyclic vectors $\breve{\xi}_t$, one for each T_t. Each GNS-module is a Hilbert $\mathcal{B}$–$\mathcal{B}$–module so that we may construct the tensor product

$$\breve{E}_{t_n} \odot \ldots \odot \breve{E}_{t_1}$$

which is again a Hilbert $\mathcal{B}$–$\mathcal{B}$–module. Fixing the sum $t = t_n + \ldots + t_1$, one can show that these tensor products form an inductive system and the inductive limits E_t (over finer and finer partitions of $[0, t]$) for all t fulfill

$$E_s \odot E_t \;=\; E_{s+t}.$$

In this way we met in Bhat and Skeide [BS00] *product systems of Hilbert modules*, a key notion for these notes. We explain this fully in Section 11.3 for slightly more general *CPD-semigroups* (a multi index version of CP-semigroup introduced in [BBLS00]; see below). The unit vectors $\breve{\xi}_t$ survive the inductive limit and give rise to a family of vectors $\xi_t \in E_t$ which fulfill

$$\xi_s \odot \xi_t \;=\; \xi_{s+t}.$$

We call such a family a *unit*. We recover the CP-semigroup as $T_t(b) = \langle \xi_t, b\xi_t \rangle$.

For Hilbert spaces the notion of product systems and units was introduced by Arveson [Arv89a]. For Hilbert modules it seems to have appeared first in [BS00] and the constructions of [Arv89a] and [BS00] are, in fact, very different. We study product systems in Part III and also comment on the relation to Arveson systems in more detail.

Once the notion is established, it is natural to try a classification by analogy with Arveson's. He defines as *type I* a product system which is *generated* (i.e. spanned in a suitable sense) by its units and he shows [Arv89a] that all type I Arveson systems are symmetric Fock spaces $\Gamma(L^2([0,t],\mathfrak{H}))$ and the units $h_t = \psi(\mathbb{I}_{[0,t]f})e^{tc}$ are exponential vectors of indicator functions times a scaling. The analogue for Hilbert modules is the *time ordered Fock module* [BS00], i.e. the analogue of the Guichardet picture of the symmetric Fock space. (A direct generaliztion of the symmetric Fock space is not possible.) We discuss the time ordered Fock module in Chapter 7 and find its (continuous) units, which (as shown in [LS00b]) may again be understood as *exponential units* with a (considerably more complicated) rescaling by a semigroup $e^{t\beta}$ in $\mathcal{B}$ ($\beta \in \mathcal{B}$). However, before we can show the analogue of Arveson's result we need some more preparation.

The crucial object in Arveson's proof is the *covariance function*, a ($\mathbb{C}$–valued) kernel on the set of units sending a pair h_t, h'_t of units to the derivative of the semigroup $\langle h_t, h'_t \rangle$ at $t = 0$. It is well-known that this kernel is *conditionally positive definite* (because it is the derivative of a semigroup under *Schur product* of *positive definite* kernels. The matrix elements $\langle \xi_t, \xi'_t \rangle$ (for two units ξ_t, ξ'_t in a product system of modules) will in general not form a semigroup *in* $\mathcal{B}$. However, the definition of tensor product (see Section 4.2) and unit are born to make the mappings

$$\mathfrak{T}_t^{\xi,\xi'} : b \longmapsto \langle \xi_t, b\xi'_t \rangle$$

a semigroup *on* $\mathcal{B}$. Inspired by a (slightly weaker) definition in Accardi and Kozyrev [AK99] we define in Chapter 5 *completely positive definite kernels* with values in the bounded mappings on $\mathcal{B}$ and semigroups of such (*CPD-semigroups*). We characterize the *generators* of (uniformly continuous) CPD-semigroups as those kernels which are *conditionally completely positive definite*. Many statements about completely positive definite kernels are straightforward translations from statements about completely positive mappings on $M_n(\mathcal{B})$. Surprisingly, the concrete form of the generator of a CPD-semigroup (in other words, the substitute for Arveson's covariance function) cannot be obtained by direct generalization of the Christensen-Evans generator for CP-semigroups and the proof of the conjecture in Theorem 5.4.14 has to wait until Section 13.3. There we also show that type I product systems of Hilbert modules generated (in a suitable weak topology) by their units (in the sense of Proposition 11.2.3) are time ordered Fock modules. (To be precise we show this for von Neumann algebras and von Neumann modules. In the remaining cases we have inclusion as a strongly dense subset. See Chapter 3 for details on the involved topologies.)

The preceding results (discovvered in [BBLS00]) cannot be more than the very beginning of a classification of product systems. A detailed summary of what we have achieved and a lot of natural questions, either with partial answers (for instance, a preliminary definition of type II product systems) or still totally open, can be found in Chapter 15.

Considering product systems as the most important new idea, we mention only very briefly further results and refer the reader to the introductions of those parts where they appear. Starting from a product system with a unital unit (i.e. a unit consisting of unit vectors), we may embed E_t as $\xi_s \odot E_t$ into E_{s+t}. This gives rise to a second inductive limit E with a unit vector ξ. Also the inductive limit fulfills a factorization, namely,

$$E \odot E_t \;=\; E.$$

With this factorization we may define an E_0–semigroup $\vartheta_t\colon a \mapsto a \odot \mathsf{id}_{E_t} \in \mathcal{B}^a(E \odot E_t) = \mathcal{B}^a(E)$ on $\mathcal{B}^a(E)$. Setting $\mathfrak{i}(b) = j_0(b) = \xi b \xi^*$ and $\mathfrak{p}(a) = \langle \xi, a\xi \rangle$ we find a dilation of the unital CP-semigroup $T_t(b) = \langle \xi_t, b\xi_t \rangle = \langle \xi, \vartheta_t \circ j_0(b)\xi \rangle$. Moreover, the representations $j_t = \vartheta_t \circ j_0$ fulfil the Markov property $j_t(\mathbf{1}) j_{s+t}(b) j_t(\mathbf{1}) = j_t \circ T_s(b)$. If the product system and the unit are those constructed from a given unital CP-semigroup (see above), then the j_t form the *weak Markov flow* constructed in Bhat and Parthasarathy [BP94, BP95]. The dilation (constructed in [BS00]) is an extension (from a subalgebra of $\mathcal{B}^a(E)$ to $\mathcal{B}^a(E)$) of the dilation constructed in Bhat [Bha99]. Of course, the results in [BP94, BP95, Bha99] are formulated in the language of Hilbert spaces and, as pointed out in [BS00], in general the dilation does not extend to some $\mathcal{B}(H)$ (where H is some Hilbert space). This is only one good reason to consider operators on Hilbert modules. We explain all this in Section 11.4 and Chapter 12.

The dilation constructed so far has a sort of defect. It is non-unital (unless the CP-semigroup to be dilated is the trivial one). Instead, the unit of $\mathcal{B}$ evolves as an increasing family of projections. We call such dilations *weak dilations*. Mathematically, there is *nothing bad* in considering weak dilations, but in physical applications one is interested primarily in unital dilations. However, in Chapter 14 (where we discuss alternative constructions of product systems) we show with the help of product systems that any dilation on a Hilbert module (also a unital one) is also a weak dilation, if we replace the embedding $\mathfrak{i}$ by j_0. Thus, studying weak dilations (for instance, by their product systems) is always also a study of other dilations. The converse question, namely, whether a weak dilation can be turned into a unital dilation (by replacing the non-unital embedding j_0 by a unital one $\mathfrak{i}$ without changing the CP-semigroup) is an important open problem. This is related to the fact that we may always embed $\mathcal{B}^a(E)$ as $\mathcal{B}^a(E) \odot \mathsf{id}_F$ unitally into $\mathcal{B}^a(E \odot F)$, whereas, an anlogue embedding of $\mathcal{B}^a(F)$ fails in general. (A prominent exception of this failure are modules over $\mathcal{B}(G)$ where $E \odot F \cong F \odot E$; see Section 3.4.)

In Part IV we construct a unital dilation of a unital CP-semigroup T, at least in the case, when T is uniformly continuous, with the help of the *quantum stochastic calculus* developed in [Ske00d]. In the usual way, we obtain the dilation as a *cocycle perturbation* of a *white noise* (the *time shift*) on a full Fock module. Interestingly enough, also in the module case the product system of a dilation does not change under cocycle perturbation (Theorem 14.1.5) and the product system of a white noise always contains a *central unit* (i.e. a unit *commuting* with $\mathcal{B}$).

We close with the remark that existence of a central unit in a product system was *the* crucial step in showing that type I product systems are time ordered Fock modules. In fact, we explain in Section 13.3 that this existence is equivalent to the results by Christensen and Evans [CE79] on the generator of uniformly continuous CP-semigroups. Thus, in the case that, somewhen in the future, we can show existence of central units independently, we would also find a new proof of [CE79].

• • •

Conventions and notations. By $\mathbb{R}_+$ and $\mathbb{N}_0$ we denote the sets of non-negative reals and non-negative integers, respectively, where $\mathbb{N}$ denotes the set $n = 1, 2, \ldots$ of natural numbers. We set $\mathbb{R}_- = -\mathbb{R}_+$ and $\mathbb{N}_- = -\mathbb{N}_0$. All vector spaces are vector spaces over $\mathbb{C}$, and all functionals on a vector space are linear functionals. All algebras are algebras over $\mathbb{C}$. All modules are modules over algebras and carry a vector space structure which is compatible with the module operation (i.e. left multiplication $(b, x) \mapsto bx$ and right multiplication $(x, b) \mapsto xb$ are bilinear mappings). If the algebra is unital, then we always assume that the unit acts as identity on the module. A homomorphism between $*$–algebras, is always a $*$–algebra homomorphism. If the homomorphism is not involutive, then we say algebra homomorphism. If it respects additional structures, then we express this explicitly. (For instance, for a homomorphism between pre–C^*–algebras we say contractive homomorphism.)

All (semi-)(pre-)Hilbert modules are right modules. This convention is more or less forced by the fact that we write a homomorphism on the left of the element on which it acts. Consequently, all sesquilinear mappings are linear in its right argument and conjugate linear in its first argument. A deviation from these conventions would, definitely, cause a loss of the intuitive content of many formulae.

The prefix "pre" for an inner product space means that the space is not necessarily complete in the topology induced by the inner product. The prefix "semi" means that the inner product may fail to be strictly positive, in fact, it is only a semiinner product.

By $\mathcal{L}(E,F)$ we denote the space of linear mappings between vector spaces E and F. By $\mathcal{B}(E,F)$ we denote the space of bounded linear mappings between (semi-)normed vector spaces. $\mathcal{B}(E,F)$ is itself a (semi-)normed space with $\|a\| = \sup_{\|x\|\leq 1} \|ax\|$. Without mention we will also use this notation for elements of $\mathcal{L}(E,F)$ where we assign to $a \in \mathcal{L}(E,F)\backslash\mathcal{B}(E,F)$ the value $\|a\| = \infty$. By $\mathcal{L}^r(E,F)$, $\mathcal{L}^l(E,F)$, and $\mathcal{L}^{bil}(E,F)$ we denote the spaces of right linear, left linear, and bilinear (or two-sided) mappings between right modules, left modules, and bimodules (or two-sided modules), respectivly, E and F. We use similar notations for spaces $\mathcal{B}(E,F)$ of bounded mappings on (semi-)normed modules. By $(\mathcal{B}^a(E,F))$ $\mathcal{L}^a(E,F)$ we denote the space of (bounded) adjointable mappings between (semi-)inner product spaces. Whenever $F = E$ we simplify the notation to $\mathcal{L}(E)$, $\mathcal{B}(E)$, etc.. Notice that already in the case of an infinite-dimensional pre-Hilbert space H no two of the spaces $\mathcal{L}(H)$, $\mathcal{L}^a(H)$, $\mathcal{B}(H)$, $\mathcal{B}^a(H)$ coincide. We also use double indices. For instance, $\mathcal{L}^{a,bil}(E,F)$ denotes the space of adjointable bilinear mappings between two-sided semi-Hilbert modules.

Unless stated otherwise explicitly, direct sums $\oplus$ and tensor products $\otimes, \odot$ are understood algebraically. We use $\otimes$ exclusively to denote the tensor product of vector spaces (i.e. tensor products over $\mathbb{C}$), whereas $\odot$ denotes the (interior) tensor product of pre-Hilbert modules. If we refer to the algebraic tensor product *over* an algebra $\mathcal{B}$ we write $\underline{\odot}$. Only rarely we indicate the algebra by writing $\odot_{\mathcal{B}}$ or $\underline{\odot}_{\mathcal{B}}$. By $\bar{\oplus}$, etc., we denote the completions in the natural norm. By $\bar{\oplus}^s$, etc., we denote strong closures and it should be clear from the context in which space of operators we take the closure.

Let E, F, G be vector spaces with a bilinear *operation* $E \times F \to G, (x,y) \mapsto xy$, and let $A \subset E$, $B \subset F$. By AB we always mean the set $\{xy \colon x \in A, y \in B\}$. We ***do not adopt*** the usual convention where AB means the linear span (or even its closure) of this set. In fact, in many places we will have to reduce the proof of a statement to the case where $AB = \operatorname{span} AB$.

Part I

An invitation to Hilbert modules

What are Hilbert modules? Certainly the most correct answer — and at the same time the most boring — is to say that a Hilbert module is module over a C^*–algebra $\mathcal{B}$ with a $\mathcal{B}$–valued inner product fulfilling certain axioms generalizing those of the inner product on a Hilbert space (Definitions 1.1.1 and 1.2.4). More interesting answers come from applications and will depend on them. The present introduction is a mixture. In principle, it is based on the usual axiomatic approach, but the development of the theory is governed from the point of view of applications and many examples point directly to them.

Our main applications are to completely positive mappings and, in particular, their compositions. Hilbert modules (over noncommutative C^*–algebras) were introduced more or less simultaneously by Paschke [Pas73] and Rieffel [Rie74]. Already Paschke provides us with a *GNS-construction* for a completelely positive mapping $T\colon \mathcal{A} \to \mathcal{B}$ resulting in a Hilbert $\mathcal{A}$–$\mathcal{B}$–module E and a cyclic vector $\xi \in E$ such that $T(a) = \langle \xi, a\xi \rangle$; see Section 4.1. From this basic application it is perfectly clear that we are interested in two-sided Hilbert modules whose structure is much richer. For instance, any right submodule of a Hilbert $\mathcal{B}$–module generated by a *unit vector* ξ (i.e. $\langle \xi, \xi \rangle = \mathbf{1} \in \mathcal{B}$) is isomorphic to the simplest Hilbert $\mathcal{B}$–module $\mathcal{B}$ (with inner product $\langle b, b' \rangle = b^*b'$; see Example 1.1.5), whereas two left multiplications on $\mathcal{B}$ give rise to isomorphic Hilbert $\mathcal{B}$–$\mathcal{B}$–module structures, if and only if they are related by an inner automorphism of (the C^*–algebra) $\mathcal{B}$; see Example 1.6.7.

There is the well-known *Stinespring construction* for completely positive mappings. Provided that $\mathcal{B}$ is represented on Hilbert space G the Stinespring construction provides us with another Hilbert space H on which $\mathcal{A}$ acts and a mapping $L \in \mathfrak{B}(G, H)$ such that $T(a) = L^*aL$. However, the Stinespring constructions for two completely positive mappings S, T do not help us in finding the Stinespring construction for $S \circ T$, whereas its GNS-module and cyclic vector are related to those of the single mappings simply by tensor product; see Example 4.2.8.

Without two-sided Hilbert modules and their tensor products there would not exist Fock modules in Part II (and consequently also not the calculus in Part IV) nor tensor product systems of Hilbert modules in Part III. There is another aspect of a Hilbert $\mathcal{A}$–$\mathcal{B}$–module E, namely, that of a functor which sends a representation of $\mathcal{B}$ on G to a representation of $\mathcal{A}$ on $H = E \bar{\odot} G$; see Remark 4.2.9. This does not only allow us to explain the relation between GNS- and Stinespring construction; see Remark 4.1.9. It allows us to identify E as the subset of mappings $g \mapsto x \odot g$ in $\mathcal{B}(G, H)$ (for all $x \in E$). This provides us with a natural definition of *von Neumann modules* as strongly closed submodules of $\mathcal{B}(G, H)$; see Chapter 3. In cases where we need *self-duality* (Section 1.3), for instance, for existence of projections onto (strongly) closed submodules, von Neumann modules are a suitable choice. Once the effort to check that operations among von Neumann modules preserve all necessary compatibility conditions is done, they behave almost as simply as Hilbert spaces, but still preserve all the pleasant algebraic aspects of the Hilbert module approach.

In the case when $\mathcal{B} = \mathcal{B}(G)$ it is easy to see that a von Neumann module E is all of $\mathcal{B}(G, H)$ and the algebra of (adjointable) operators on E is just $\mathcal{B}(H)$. If, additionally, E is a two-sided von Neumann module over $\mathcal{B}(G)$, then H factorizes as $G \mathbin{\bar{\otimes}} \mathfrak{H}$ ($\mathcal{B}(G)$ acting in the natural way) so that $E = \mathcal{B}(G, G \mathbin{\bar{\otimes}} \mathfrak{H}) = \mathcal{B}(G) \mathbin{\bar{\otimes}^s} \mathfrak{H}$ in an obvious way. This module is generated (as a von Neumann module) by the subset $\mathbf{1} \otimes \mathfrak{H}$ of *centered* elements (consisting of those elements which *commute* with the elements of $\mathcal{B}(G)$); see Section 3.4. In other words, the structure of a two-sided von Neumann module is determined completely by its Hilbert space $\mathfrak{H} \cong \mathbf{1} \otimes \mathfrak{H}$ of *intertwiners*. This is not only the reason why Arveson's approach to E_0–semigroups on $\mathcal{B}(G)$ by the tensor product system of the intertwining Hilbert spaces was successful. We believe that it also explains why so many results on $\mathcal{B} = \mathcal{B}(G)$ can be obtained by Hilbert space techniques, whereas the same techniques fail for more general C^*–algebras or von Neumann algebras.

Our exposition is "pedestrian" and should be readable with elementary knowledge in C^*–algebra theory and very few facts about von Neumann algebras. It is based on Skeide [Ske00b] where we introduced von Neumann modules and on a course we gave at university Roma II in 1999. As we want always to underline the algebraic ideas of a construction, usually we complete or close only if this is really necessary, and in this case we usually explain why it is necessary. In Appendix A we collect basic material about pre–C^*–algebras and other (semi-)normed spaces. The expert in Hilbert modules should be aware of the numerous use of pre-Hilbert modules but, in general need not read Part I as a whole. Only the notion of completely positive definite kernels and semigroups of such in Chapter 5 is new. A condensed summary of essential definitions and structures in the remainder of Part I can be found in Appendix E. Also the contents of two crucial series of examples spread over Part I can be found in this appendix.

Chapter 1

Basic definitions and results

1.1 Hilbert modules

1.1.1 Definition. Let $\mathcal{B}$ be a pre–C^*–algebra. A *pre-Hilbert $\mathcal{B}$–module* is a right $\mathcal{B}$–module E with a sesquilinear *inner product* $\langle \bullet, \bullet \rangle \colon E \times E \to \mathcal{B}$, fulfilling

$$\langle x, yb \rangle = \langle x, y \rangle b \qquad (\textit{right linearity}) \qquad (1.1.1a)$$

$$\langle x, x \rangle \geq 0 \qquad (\textit{positivity}) \qquad (1.1.1b)$$

$$\langle x, x \rangle = 0 \Longleftrightarrow x = 0 \qquad (\textit{strict } \text{positivity}) \qquad (1.1.1c)$$

for all $x, y \in E$ and $b \in \mathcal{B}$. If strict positivity is missing, then we speak of a *semi-inner product* and a *semi-Hilbert $\mathcal{B}$–module*.

1.1.2 Proposition. *We have*

$$\langle x, y \rangle = \langle y, x \rangle^* \qquad (\textit{symmetry}) \qquad (1.1.2a)$$

$$\langle xb, y \rangle = b^* \langle x, y \rangle. \qquad (\textit{left anti-linearity}) \qquad (1.1.2b)$$

PROOF. (1.1.2b) follows from (1.1.2a) and (1.1.2a) follows from an investigation of the self-adjoint elements $\langle x + \lambda y, x + \lambda y \rangle$ of $\mathcal{B}$ for $\lambda = 1, i, -1, -i$. ■

1.1.3 Proposition. *In a pre-Hilbert $\mathcal{B}$–module E we have*

$$\langle y, x \rangle = \langle y, x' \rangle \ \forall\, y \in E \quad \textit{implies} \quad x = x'.$$

PROOF. We have, in particular, $\langle x - x', x - x' \rangle = 0$, whence, $x - x' = 0$. ■

1.1.4 Corollary. *If E is a pre-Hilbert module over a unital pre–C^*–algebra, then $x\mathbf{1} = x$ $(x \in E)$.*

1.1.5 Example. Any pre–C^*–algebra $\mathcal{B}$ is a pre-Hilbert $\mathcal{B}$–module with inner product $\langle b, b' \rangle = b^* b'$. If $\mathcal{B}$ is unital, then $\mathbf{1}$ is a module basis for $\mathcal{B}$. We say $\mathcal{B}$ is the *one-dimensional* pre-Hilbert $\mathcal{B}$–module.

More generally, a right ideal I in $\mathcal{B}$ is a pre-Hilbert $\mathcal{B}$–module (actually, a pre-Hilbert I–module) in the same way. This shows that we have some freedom in the choice of the algebra I over which a $\mathcal{B}$–module E with a semi-inner product is a semi-Hilbert I–module. In fact, any ideal I in $\mathcal{B}$ which contains the *range ideal* $\mathcal{B}_E := \mathsf{span}\langle E, E \rangle$ is a possible choice. We say a pre-Hilbert $\mathcal{B}$–module E is *full*, if $\mathcal{B}_E$ is dense in $\mathcal{B}$. We say E is *essential*, if $\mathcal{B}_E$ is an essential ideal in $\mathcal{B}$ (cf. Appendix A.7).

1.1.6 Example. Let G and H be pre-Hilbert spaces and let $\mathcal{B} \subset \mathfrak{B}(G)$ be a $*$–algebra of bounded operators on G. Then any subspace $E \subset \mathfrak{B}^a(G, H)$, for which $E\mathcal{B} \subset E$ and $E^* E \subset \mathcal{B}$ becomes a pre-Hilbert $\mathcal{B}$–module with inner product $\langle x, y \rangle = x^* y$. Positivity of this inner product follows easily by embedding $\mathfrak{B}^a(G, H)$ into the pre–C^*–algebra $\mathfrak{B}^a(G \oplus H)$; see Appendix A.7. We will use often such an embedding argument to see positivity of inner products.

1.1.7 Definition. Let $\big(E^{(t)}\big)_{t \in \mathbb{L}}$ be a family of semi-Hilbert $\mathcal{B}$–modules (where $\mathbb{L}$ is some indexing set), then also the direct sum $E = \bigoplus\limits_{t \in \mathbb{L}} E^{(t)}$ is a right $\mathcal{B}$–module in an obvious way. By defining the semi-inner product

$$\big\langle \big(x^{(t)}\big), \big(y^{(t)}\big) \big\rangle = \sum_{t \in \mathbb{L}} \langle x^{(t)}, y^{(t)} \rangle,$$

we turn E into a semi-Hilbert $\mathcal{B}$–module. (Recall that direct sums are algebraic so that the sum is only over finitely many non-zero summands.) Clearly, the semi-inner product is inner, if and only if each of the semi-inner products on $E^{(t)}$ is inner. If $\mathbb{L}$ is a finite set, then we write elements of direct sums interchangably as column or row vectors.

1.1.8 Example. Let E be a semi-Hilbert $\mathcal{B}$–module. Then by E^n ($n \in \mathbb{N}_0$) we denote the direct sum of n copies of E (where $E^0 = \{0\}$). In particular, $\mathcal{B}^n$ comes along with a natural pre-Hilbert $\mathcal{B}$–module structure. This structure must be distinguished clearly from the n–fold pre–C^*–algebraic direct sum (also denoted by $\mathcal{B}^n$). Besides the different algebraic structure (here an inner product with values in $\mathcal{B}$ and there a product with values in $\mathcal{B}^n$), the respective norms are different for $n \geq 2$. It is, however, easy to see that the two norms are always equivalent.

We will see in Section 3.2 that any pre-Hilbert $\mathcal{B}$–module can be embedded into a certain closure of a direct sum of right ideals.

1.2 Cauchy-Schwarz inequality and quotients

One of the most fundamental properties in Hilbert space theory is *Cauchy-Schwarz inequality* which asserts $\langle f,g\rangle\langle g,f\rangle \le \langle g,g\rangle\langle f,f\rangle$ for all elements f,g in a semi-Hilbert space. It allows to divide out the kernel of a semi-inner product, it shows that a semi-Hilbert space is seminormed, it shows that the operator norm in $\mathcal{B}^a(H)$ for a pre-Hilbert space H is a C^*–norm, and so on. For semi-Hilbert modules we have the following version.

1.2.1 Proposition. *For all x,y, in a semi-Hilbert module we have*

$$\langle x,y\rangle\langle y,x\rangle \le \|\langle y,y\rangle\|\ \langle x,x\rangle. \tag{1.2.1}$$

PROOF. Suppose that $\langle y,y\rangle \neq 0$. Then (1.2.1) follows by an investigation of $\langle z,z\rangle$ where $z = x - \frac{y\langle y,x\rangle}{\|\langle y,y\rangle\|}$. Making use of (1.1.1a), (1.1.1b), (1.1.2a), (1.1.2b) and the inequality $a^*ba \le \|b\|\, a^*a$ $(a,b\in\mathcal{B}, b\ge 0)$, we find

$$0 \le \langle z,z\rangle = \langle x,x\rangle - 2\frac{\langle x,y\rangle\langle y,x\rangle}{\|\langle y,y\rangle\|} + \frac{\langle x,y\rangle\langle y,y\rangle\langle y,x\rangle}{\|\langle y,y\rangle\|^2} \le \langle x,x\rangle - \frac{\langle x,y\rangle\langle y,x\rangle}{\|\langle y,y\rangle\|}.$$

If $\langle y,y\rangle = 0$, however, $\langle x,x\rangle \neq 0$, then (1.2.1) follows similarly by exchanging x and y.

The case $\langle x,x\rangle = \langle y,y\rangle = 0$ requires additional work. Like in the proof of (1.1.2a) we investigate $\langle x+\lambda y, x+\lambda y\rangle$ for $\lambda = 1, i, -1, -i$. From $\lambda = 1,-1$ we conclude that the real part of $\langle x,y\rangle$ is positive and negative, hence 0. From $\lambda = i,-i$ we conclude that the imaginary part of $\langle x,y\rangle$ is positive and negative, hence 0. This implies $\langle x,y\rangle = 0$. ■

1.2.2 Corollary. *By, setting*

$$\|x\| = \sqrt{\|\langle x,x\rangle\|} \tag{1.2.2}$$

we define a submultiplicative (i.e. $\|xb\| \le \|x\|\,\|b\|$) seminorm on E. (1.2.2) *defines a norm, if and only if E is a pre-Hilbert module.*

1.2.3 Corollary.

$$\|x\| = \sup_{\|y\|\le 1} \|\langle y,x\rangle\|.$$

PROOF. By Cauchy-Schwarz inequality $\sup_{\|y\|\le 1} \|\langle y,x\rangle\| \le \|x\|$. On the other hand, for $\|x\| \neq 0$ we have $\sup_{\|y\|\le 1} \|\langle y,x\rangle\| \ge \frac{\|\langle x,x\rangle\|}{\|x\|} = \|x\|$, which, obviously, is true also for $\|x\| = 0$. ■

1.2.4 Definition. A *Hilbert module* is a pre-Hilbert module E which is complete in the norm (1.2.2).

1.2.5 Example. Let I be an ideal in a pre–C^*–algebra. Then the C^*–norm and the pre-Hilbert module norm as in Example 1.1.5 coincide. Therefore, I is a Hilbert module, if and only if it is a C^*–algebra.

In a direct sum of pre-Hilbert modules $\left(E^{(t)}\right)_{t\in\mathbb{L}}$ as in Definition 1.1.7 we have $\left\|x^{(t)}\right\| \leq \|x\|$ for all $t \in \mathbb{L}$. In other words, the norm on a direct sum is admissible in the sense of Appendix A.2 and the direct sum is complete, if and only if $\mathbb{L}$ is a finite set and each $E^{(t)}$ is complete.

Because right multiplication $(x, b) \mapsto xb$ is jointly continuous, the completion of any pre-Hilbert $\mathcal{B}$–module is a Hilbert $\overline{\mathcal{B}}$–module in a natural fashion. Notice, however, that completeness of E does not necessarily imply completeness of $\mathcal{B}$. On the contrary, we will see in Observation 1.7.5 that the range ideal $\mathcal{B}_E$ (to which we may reduce the module structure) is very rarely complete.

Let E be a semi-Hilbert $\mathcal{B}$–module and denote by $\mathcal{N}_E = \left\{x \in E \colon \langle x, x\rangle = 0\right\}$ the subspace consisting of *length-zero elements*. The possibility for dividing out this subspace is crucial for GNS-construction and tensor product; see Chapter 4.

1.2.6 Proposition. *$\mathcal{N}_E$ is a submodule of E so that the quotient $E/\mathcal{N}_E$ is a right $\mathcal{B}$–module. $E/\mathcal{N}_E$ inherits an inner product which turns it into a pre-Hilbert $\mathcal{B}$–module by setting*

$$\langle x + \mathcal{N}_E, y + \mathcal{N}_E\rangle = \langle x, y\rangle.$$

Proof. Clearly, $x \in \mathcal{N}_E$ implies $xb \in \mathcal{N}_E$ $(b \in \mathcal{B})$. Let $x, y \in \mathcal{N}_E$. Then by Cauchy-Schwarz inequality also $x + y \in \mathcal{N}_E$. Thus, $\mathcal{N}_E$ is, indeed, a submodule of E. Let $x + n$ and $y + m$ be arbitrary representatives of $x + \mathcal{N}_E$ and $y + \mathcal{N}_E$, respectively. Then, once again, by Cauchy-Schwarz inequality $\langle x + n, y + m\rangle = \langle x, y\rangle$ so that the value of the inner product does not depend on the choice of the representatives. ■

1.2.7 Definition. By the pre-Hilbert $\mathcal{B}$–module and the Hilbert $\overline{\mathcal{B}}$–module *associated with* E, we mean $E/\mathcal{N}_E$ and $\overline{E/\mathcal{N}_E}$, respectively.

1.3 Self-duality

We have seen that we have at hand a satisfactory substitute for Cauchy-Schwarz inequality. Another basic tool in Hilbert space space theory is the *Riesz representation theorem* which asserts that Hilbert spaces are *self-dual*. This means that for any continuous linear functional φ on a Hilbert space H there exists a (unique) element $f_\varphi \in H$ such that $\varphi f = \langle f_\varphi, f\rangle$ for all $f \in H$. In other words, there exists an isometric anti-isomorphism from H to the space H'

of all continuous linear functionals on H. Recall, however, that this result for pre-Hilbert spaces is wrong, as also for a pre-Hilbert space H the space H' is complete.

1.3.1 Definition. Let E be a semi-Hilbert $\mathcal{B}$–module. By E' we denote the space

$$E' = \{\Phi\colon E \longrightarrow \mathcal{B} | \Phi(xb) = (\Phi x)b, \|\Phi\| < \infty\}$$

of *bounded right linear $\mathcal{B}$–functionals* (or short *$\mathcal{B}$–functionals*) on E. By the *dual module* E^* we mean the subspace

$$E^* = \{x^*\colon E \longrightarrow \mathcal{B}\ (x \in E) | x^* y = \langle x, y\rangle\}$$

of E'. (Obviously, $\|x^*\| = \|x\|$. Thus, the correspondence between elements $x \in E$ and $x^* \in E^*$ is one-to-one, if and only if E is a pre-Hilbert module.)

We say a pre-Hilbert $\mathcal{B}$–module E is *self-dual*, if $E^* = E'$. Also here a self-dual pre-Hilbert module is necessarily complete.

1.3.2 Example. Let I be a non-trivial ideal in a C^*–algebra $\mathcal{B}$. Then as explained in Example 1.1.5, I is a pre-Hilbert $\mathcal{B}$–module. Any element in the multiplier algebra $M(I)$ of I (see Appendix A.8) gives rise to a $\mathcal{B}$–functional on I (cf. Lemma 1.7.10). In particular, if I is non-unital, then the unit of $M(I)$ determins a $\mathcal{B}$–functional which is not in I^*. Hence, I is not self-dual.

A concrete example is that given by Paschke [Pas73] where $\mathcal{B}$ is the C^*–algebra $C[0,1]$ of continuous functions on the interval $[0,1]$ and I is the ideal $C_0[0,1]$ of those functions vanishing at 0. Clearly, $\mathcal{B} \subset I'$ (actually $\mathcal{B} = E'$). The identity mapping on I is contained in $\mathcal{B}$, however, not in I so that I is not self-dual.

We see that Hilbert modules, in general, do not enjoy the property to be self-dual. As a consequence many statements in Hilbert space theory which build on the *Riesz theorem*, like existence of adjoints for bounded operators and complementability of closed subspaces, have no counter part in Hilbert module theory. On the other hand, if we are able to solve a problem for general Hilbert modules, then very often the solution is based on a purely algebraic argument working on pre-Hilbert modules, which extends by routine arguments to the completions. This is the main reason, why we decided to stay at an algebraic level.

Frank's criterion (Theorem 2.1.13 below) shows that the lack of self-duality is caused by the fact that the unit ball of a Hilbert module is not complete in a certain locally convex topology. It is possible to complete the ball coherently. However, in this case it is not granted that the inner product still takes values in $\mathcal{B}$. We see that the problem of constructing self-dual extensions, finally, is a problem of the C^*–algebra under consideration. Frank [Fra97] has shown that for a given C^*–algebra $\mathcal{B}$ any (pre-)Hilbert $\mathcal{B}$–module allows for a self-dual extension, if and only if $\mathcal{B}$ belongs to the category of *monotone complete* C^*–algebras. Thus, dealing with C^*–algebras as ring which are not monotone complete corresponds in some respects to dealing with the field of rational numbers instead of the complete field of real numbers.

In these notes we do not tackle the involved problems appearing in the theory of Hilbert modules over general monotone complete C^*–algebras. We refer the reader to Frank [Fra97] and the huge quantity of literatur quoted therein. In cases where we need self-dual Hilbert modules we concentrate on a special subcategory of the monontone complete C^*–algebras, namely, von Neumann algebras. This leads naturally, to the notion of von Neumann modules; see Chapter 3.

1.4 Operators on Hilbert modules

We introduce several spaces of operators between Hilbert modules. Since we intend to distinguish very clearly between semi-, pre- and Hilbert modules, we are forced to do at least once the somewhat tedious effort to state all the properties by which the spaces of operators are distinguished in the several cases. In Appendix A.1 we recall the basic properties of the operator (semi-)norm on spaces of operators between (semi-)normed spaces. Here we refer more to additional properties which arise due to an exisiting (semi-)Hilbert module structure. As soon as we have convinced ourselves that we may divide out kernels of semi-inner products (that is after Corollaries 1.4.3 and 1.4.4), we turn our interest to pre-Hilbert modules.

1.4.1 Definition. Let E and F be semi-Hilbert $\mathcal{B}$–modules. A mapping $a\colon E \to F$ (*a priori* neither linear nor bounded) is ***adjointable***, if there exists a mapping $a^*\colon F \to E$ such that

$$\langle x, a^*y\rangle = \langle ax, y\rangle \tag{1.4.1}$$

for all $x \in E$ and $y \in F$. By $(\mathcal{B}^a(E,F))$ $\mathcal{L}^a(E,F)$ we denote the space of ***(bounded) adjointable mappings*** $E \to F$.

We say an element $a \in \mathcal{L}^a(E)$ is ***self-adjoint***, if it fulfills (1.4.1) with $a^* = a$.

Let $a \in \mathcal{L}(E,F)$ be an arbitrary linear mapping between semi-Hilbert $\mathcal{B}$–modules E and F. Then by Corollary 1.2.3 the operator norm of a is

$$\|a\| = \sup_{\|x\|\leq 1} \|ax\| = \sup_{\|x\|\leq 1, \|y\|\leq 1} \|\langle y, ax\rangle\|\,. \tag{1.4.2}$$

1.4.2 Corollary. *If a is adjointable, then*

$$\|a^*\| = \|a\| \qquad \textit{and} \qquad \|a^*a\| = \|a\|^2 .$$

Proof. Precisely as for operators on Hilbert spaces. The first equation follows directly from (1.4.2). For the second equation we observe that, on the one hand, $\|a^*a\| \leq \|a^*\| \, \|a\| = \|a\|^2$. On the other hand,

$$\|a\|^2 = \sup_{\|x\|\leq 1} \|ax\|^2 = \sup_{\|x\|\leq 1} \|\langle x, a^*ax\rangle\| \leq \sup_{\|x\|\leq 1, \|y\|\leq 1} \|\langle y, a^*ax\rangle\| = \|a^*a\| . \blacksquare$$

If a is adjointable, then so is a^*. If E is a pre-Hilbert module, then a^* is unique. If F is a pre-Hilbert module, then a is $\mathcal{B}$–linear, i.e. in this case we have $\mathcal{L}^a(E, F) \subset \mathcal{L}^r(E, F)$. (Both assertions follow from Proposition 1.1.3.) a respects the length-zero elements, i.e. $a\mathcal{N}_E \subset \mathcal{N}_F$. (Apply (1.4.1) to $x \in \mathcal{N}_E$ and $y = ax \in F$, and then use Cauchy-Schwarz inequality.)

1.4.3 Corollary. *A mapping $a \in \mathcal{L}^a(E, F)$ gives rise to a unique element in $\mathcal{L}^a(E/\mathcal{N}_E, F/\mathcal{N}_F)$ also denoted by a.*

The corollary tells us that we may divide out the length-zero elements, if we are interested only in adjointable mappings. There is another criterion, namely Lemma A.1.3, which tells us that also bounded mappings between semi-normed spaces respect the kernel of the semi-norms.

1.4.4 Corollary. *A mapping $a \in \mathcal{B}^r(E, F)$ gives rise to a unique element in $\mathcal{B}^r(E/\mathcal{N}_E, F/\mathcal{N}_F)$ of the same norm also denoted by a.*

With few exceptions, we are interested only in bounded right linear mappings or in adjointable mappings. Therefore, we assume always that all length-zero elements have been divided out and restrict our attention to pre-Hilbert modules.

In this case we have $\mathcal{B}^a(E, F) \subset \mathcal{B}^r(E, F)$. If F is complete, then $\mathcal{B}^r(E, F)$ ($\cong \mathcal{B}^r(\overline{E}, F)$) is a Banach space, because right multiplication by elements of $\mathcal{B}$ is continuous. If E is complete, then $\mathcal{B}^a(E, F)$ is a closed subspace of $\mathcal{B}^r(E, F)$, for if $a_n \in \mathcal{B}^a(E, F)$ converges to $a \in \mathcal{B}^r(E, F)$, then also the adjoints a_n^* converge in the Banach space $\mathcal{B}^r(F, E)$ and, clearly, this limit is the adjoint of a. If E and F are complete, then $\mathcal{B}^a(E, F)$ is a Banach subspace of $\mathcal{B}^r(E, F)$.

$\mathcal{B}^r(E)$ is a normed algebra and by Corollary 1.4.2 $\mathcal{B}^a(E)$ is pre–C^*–algebra. Consequently, $\mathcal{B}^r(\overline{E}) = \overline{\mathcal{B}^r(E)}$ is a Banach algebra and $\mathcal{B}^a(\overline{E}) = \overline{\mathcal{B}^a(E)}$ is a C^*–algebra.

As a direct consequence of (1.4.1), an adjointable mapping $a\colon E \to F$ is closeable as densely defined mapping from $\overline{E}$ to $\overline{F}$. (Indeed, if a sequence x_n in E converges to 0 and ax_n converges to $y \in F$, then (1.4.1) implies $y = 0$.) Therefore, if E is complete, then by the *closed graph theorem* we have $\mathcal{L}^a(E,F) = \mathcal{B}^a(E,F)$. The same is true, if F is complete, because in this case a^* is bounded and $a^* \mapsto a$ is an isometry.

In general, we must distinguish between $\mathcal{B}^r(E,F)$ and $\mathcal{B}^a(E,F)$.

1.4.5 Example. Let $\mathcal{B}$ have a unit. Then for the pre-Hilbert $\mathcal{B}$–module $\mathcal{B}$ (Example 1.1.5) we have $\mathcal{B}^a(\mathcal{B}) = \mathcal{B}$ (via the identification $a \mapsto a\mathbf{1}$). By definition $E' = \mathcal{B}^r(E,\mathcal{B})$. However, if an element Φ has an adjoint Φ^* in $\mathcal{B}^a(\mathcal{B},E)$, then $\Phi x = \langle \mathbf{1}, \Phi x\rangle = \langle \Phi^*\mathbf{1}, x\rangle = \langle y, x\rangle$, where $y = \Phi^*\mathbf{1} \in E$. In other words, $\Phi \in E^*$. On the other hand, an element $x^* \in E^* \subset \mathcal{B}^r(E,\mathcal{B})$ has an adjoint, namely, $x\colon b \mapsto xb$. Therefore, $E^* = \mathcal{B}^a(E,\mathcal{B})$ and $E = \mathcal{B}^a(\mathcal{B},E)$. (For non-unital $\mathcal{B}$ both equalities fail; see Remark 1.7.11.) In particular, if $E' \neq E^*$, then $\mathcal{B}^r(E,\mathcal{B}) \neq \mathcal{B}^a(E,\mathcal{B})$.

1.4.6 Example. Of course, we would like to have also an example where $\mathcal{B}^r(E) \neq \mathcal{B}^a(E)$. We follow an idea by Paschke [Pas73]. Let E and F be pre-Hilbert $\mathcal{B}$–modules and let $a \in \mathcal{B}^r(E,F)$ but $a \notin \mathcal{B}^a(E,F)$. (For instance, choose E like in the preceding example, set $F = \mathcal{B}$ and let $a = \Phi \notin E^*$.) Then the operator

$$\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \colon \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ ax \end{pmatrix}$$

in $\mathcal{B}^r(E \oplus F)$ has no adjoint. Otherwise, this adjoint would have necessarily the form $\left(\begin{smallmatrix} 0 & a^* \\ 0 & 0 \end{smallmatrix}\right)$ where a^* was an adjoint of a.

Notice that the main ingredient in the preceding counter examples is a non-self-dual pre-Hilbert module. By Example 1.3.2 there exist Hilbert modules which are not self-dual. So simple completion (as in the case of pre-Hilbert spaces) is no way out.

For self-dual Hilbert modules the situation is more pleasant.

1.4.7 Proposition. *Let E, F be pre-Hilbert $\mathcal{B}$–modules and let $A\colon E \times F \to \mathcal{B}$ be a bounded $\mathcal{B}$–sesquilinear form (i.e. $A(xb, yb') = b^*A(x,y)b'$ and $\|A\| := \sup_{\|x\|\leq 1, \|y\|\leq 1} \|A(x,y)\| < \infty$). If E is self-dual, then there exists a unique operator $a \in \mathcal{B}^r(F,E)$ such that $A(x,y) = \langle x, ay\rangle$.*

Proof. $\Phi_y\colon x \mapsto A(x,y)^*$ defines an element of $E' = E^*$, so that there is a unique vector $y_a \in E$ fulfiling $A(x,y)^* = \langle y_a, x\rangle$. Clearly, $a\colon y \mapsto y_a$ defines an element of $\mathcal{B}^r(F,E)$ with the desired properties. ■

1.4.8 Corollary. *If also F is self-dual, then a is adjointable. In particular, there is a one-to-one correspondence between bounded $\mathcal{B}$–sesquilinear forms A on $E \times F$ and operators $a \in \mathfrak{B}^a(F, E)$ such that $A(x, y) = \langle x, ay \rangle$.*

PROOF. Apply Proposition 1.4.7 to the bounded $\mathcal{B}$–sesquilinear form $A^*(y, x) := A(x, y)^*$ on $F \times E$. Then the resulting operator $a^* \in \mathfrak{B}^r(E, F)$ is the adjoint of a. ■

1.4.9 Definition. The $*$–*strong topology* on $\mathfrak{B}^a(E, F)$ is the locally convex Hausdorff topology generated by the two families $a \mapsto \|ax\|$ $(x \in E)$ and $a \mapsto \|a^*y\|$ $(y \in F)$ of semi-norms.

A net a_α converges in the $*$–strong topology, if and only if a_α and a_α^* converge strongly. Clearly, $\mathfrak{B}^a(\overline{E}, \overline{F})$ is complete in the $*$–strong topology, because a $*$–strong Cauchy net converges in $\mathfrak{L}^a(\overline{E}, \overline{F})$ and $\mathfrak{L}^a(\overline{E}, \overline{F}) = \mathfrak{B}^a(\overline{E}, \overline{F})$.

The $*$–strong topology should not be confused with the *strict* topology; see Definition 1.7.15. It coincides, however, with the strict topology on bounded subsets; see Proposition 1.7.16.

The following example is the first in a whole series of examples concering matrices with entries in a Hilbert module. It also illustrates that it is often useful to interpret a $*$–algebra as an algebra of operators on a Hilbert module.

1.4.10 Example. Let E be pre-Hilbert $\mathcal{B}$–module. It is easy to see that $\mathfrak{L}^a(E^n) = M_n(\mathfrak{L}^a(E))$ where $M_n(\mathcal{A})$ denotes the $*$–algbebra of $n \times n$–matrices with entries in a $*$–algebra $\mathcal{A}$ and a matrix in $M_n(\mathfrak{L}^a(E))$ acts on a vector $X = (x_1, \ldots, x_n) \in E^n$ in the obvious way. If E is complete, then by Example 1.2.5 so is E^n, and we find $M_n(\mathfrak{B}^a(E)) = M_n(\mathfrak{L}^a(E)) = \mathfrak{L}^a(E^n) = \mathfrak{B}^a(E^n)$. In particular, $M_n(\mathfrak{B}^a(E^n))$ is a C^*–algebra. If E is not complete, then a bounded element A in $\mathfrak{L}^a(E^n)$ extends to $\overline{E}^n$ and, of course, the operator norm of A is the same on both spaces E^n and $\overline{E}^n$. Therefore, $\mathfrak{B}^a(E^n)$ is the dense pre–C^*–subalgebra of $\mathfrak{B}^a(\overline{E}^n) = M_n(\mathfrak{B}^a(\overline{E}))$ consisting of those $A \in M_n(\mathfrak{B}^a(\overline{E}))$ which leave invariant E^n. In other words, $\mathfrak{B}^a(E^n) = M_n(\mathfrak{B}^a(E))$. We identify these algebras as pre–C^*–algebras, thus, norming $M_n(\mathfrak{B}^a(E))$.

If $E = \mathcal{B}$ ($\mathcal{B}$ unital), then $\mathfrak{B}^a(\mathcal{B}) = \mathcal{B}$ (Example 1.4.5). We recover the well-known result that for a unital C^*–algebra $\mathcal{B}$ the matrix algbra $M_n(\mathcal{B})$ is also a C^*–algebra without any reference to a faithful representation of $\mathcal{B}$ on a Hilbert space. Additionally, we identify this norm as a norm of operators on the Hilbert module $\mathcal{B}^n$. (We remark that the argument can be modified suitably, if $\mathcal{B}$ is a non-unital C^*–algebra. In this case we identify $M_n(\mathcal{B})$ as closed subalgebra of $M_n(\widetilde{\mathcal{B}})$, where $\widetilde{\mathcal{B}}$ denotes the unitization of $\mathcal{B}$; see Appendix A.8.) However, even in the case, when $\mathcal{B}$ is only a pre–C^*–algebra, one can show that a C^*–norm on $M_n(\mathcal{B})$ is already determined completely by the requirement that it is isometric on a single matrix entry $\mathcal{B}_{ij} = \mathcal{B}$ of M_n.

1.5 Positivity, projections, isometries and unitaries

1.5.1 Definition. We say a linear operator a on a pre-Hilbert $\mathcal{B}$–module E is *positive*, if $\langle x, ax\rangle \geq 0$ for all $x \in E$. In this case (by linearity and polarization) a is adjointable.

Of course, a^*a is positive, if a^* exists. The following Lemma due to Paschke [Pas73] shows that for $a \in \mathfrak{B}^a(E)$ this definition of positivity is compatible with the pre–C^*–algebraic definition in Appendix A.7. We repeat Lance' elegant proof [Lan95]. Notice that, after Definition 1.5.4, we do not need this lemma before Chapter 5.

1.5.2 Lemma. *Let E be a pre-Hilbert $\mathcal{B}$–module and let $a \in \mathfrak{B}^r(E)$. Then the following conditions are equivalent.*

1. *a is positive in the pre–C^*–algebra $\mathfrak{B}^a(E)$.*

2. *a is positive according to Definition 1.5.1.*

Proof. $1 \Rightarrow 2$. $a \geq 0$ so $a = \sqrt{a}^2$ where $\sqrt{a} \in \mathfrak{B}^a(\overline{E})$, whence $\langle x, ax\rangle = \langle \sqrt{a}x, \sqrt{a}x\rangle \geq 0$.

$2 \Rightarrow 1$. $\langle x, ax\rangle \geq 0$ so $\langle x, ax\rangle = \langle ax, x\rangle$. Therefore, $\langle x, ay\rangle = \langle ax, y\rangle$, whence $a = a^*$. There exist unique positive elements a_+, a_- in $\mathfrak{B}^a(\overline{E})$ such that $a = a_+ - a_-$ and $a_+a_- = 0$. It follows

$$0 \;\leq\; \langle x, a_-^3 x\rangle \;=\; -\langle x, a_-aa_-x\rangle \;=\; -\langle a_-x, aa_-x\rangle \;\leq\; 0.$$

Therefore, $\langle x, a_-^3 x\rangle = 0$ so $\langle x, a_-^3 y\rangle = 0$ by (1.2.1), i.e. $a_-^3 = 0$, whence by Proposition A.7.3(1) $a_- = 0$. ■

Notice that if E is complete, then it is sufficient to require $a \in \mathcal{L}^r(E)$, because a is closed. A similar argument allows to generalize a well-know criterion for contractivity to pre-Hilbert modules.

1.5.3 Lemma. *A positive operator $a \in \mathcal{L}^a(E)$ is a contraction, if and only if*

$$\langle x, ax\rangle \;\leq\; \langle x, x\rangle \tag{1.5.1}$$

for all $x \in E$.

PROOF. Of course, a positive contraction fulfills (1.5.1). Conversely, let us assume that $a \geq 0$ fulfills (1.5.1). By positivity, $(x,y)_a = \langle x, ay \rangle$ is a semiiner product. In particular, by Cauchy-Schwartz inequality we have $\|(y,x)_a(x,y)_a\| \leq \|(x,x)_a\| \, \|(y,y)_a\|$, whence,

$$\|\langle x, ay \rangle\|^2 \;\leq\; \|\langle x, ax \rangle\| \, \|\langle y, ay \rangle\| \;\leq\; \|\langle x, x \rangle\| \, \|\langle y, y \rangle\| \,,$$

i.e. $\|a\| \leq 1$. ■

1.5.4 Definition. A *projection* on a pre-Hilbert module E is a self-adjoint idempotent $p \colon E \to E$, i.e. $p^2 = p = p^*$.

By definition p is adjointable and, therefore, right linear. p is positive, because $p^*p = p$, whence, $\langle x, px \rangle = \langle px, px \rangle \geq 0$. Obviously, $E = pE \oplus (\mathbf{1} - p)E$, whence, $\langle x, px \rangle \leq \langle x, x \rangle$. So by Lemma 1.5.3 p is a contraction. Since $\|p\| = \|p^*p\| = \|p\|^2$, we have $\|p\| = 1$ or $\|p\| = 0$. Clearly, p extends as a projection on $\overline{E}$. If E is complete, then so is pE.

1.5.5 Example. If $E = \bigoplus\limits_{t \in \mathbb{L}} E^{(t)}$ is a direct sum of pre-Hilbert modules, then the canonical projection $\mathfrak{p}^{(t)}$ onto the component $E^{(t)}$ is a projection in $\mathcal{B}^a(E)$. Moreover, each sum, finite or infinite, of canonical projections defines a projection in $\mathcal{B}^a(E)$ and extends as such to $\overline{E}$. It follows that $\overline{E}$ may be considered as the space of families $\big(x^{(t)}\big)_{t \in \mathbb{L}}$ $(x^{(t)} \in E^{(t)})$ for which $\sum\limits_{t \in \mathbb{L}} \langle x^{(t)}, x^{(t)} \rangle$ converges *absolutely*, i.e. convergence of sums over finite subsets $\mathbb{L}'$ of $\mathbb{L}$. To see this define $\mathfrak{p}_{\mathbb{L}'} = \sum\limits_{t \in \mathbb{L}'} \mathfrak{p}^{(t)}$, let $x \in \overline{E}$ and let $\big(x_n\big)_{n \in \mathbb{N}}$ be a sequence in E converging to x. Then

$$\|x - \mathfrak{p}_{\mathbb{L}'} x\| \leq \|x - x_n\| + \|x_n - \mathfrak{p}_{\mathbb{L}'} x_n\| + \|\mathfrak{p}_{\mathbb{L}'}(x_n - x)\| \,.$$

Therefore, choosing n sufficiently big, the first and the last summand are small, and choosing to this n the (finite) set $\mathbb{L}_0 = \{t \in \mathbb{L} \colon x_n^{(t)} \neq 0\}$, the middle summand is 0 for all $\mathbb{L}' \supset \mathbb{L}_0$. From this the statement follows.

1.5.6 Example. There exist closed submodules of a Hilbert module which are not the range of a projection. Indeed, let I be an ideal in $\mathcal{B}$ and suppose that there exists a projection p in $\mathcal{B}^a(\mathcal{B})$ such that $p\mathcal{B} = I$. Let $\Phi \in I'$ be given by $\Phi x = b^* x$ where $b \in \mathcal{B}$. Then $\Phi x = \Phi p x = (pb)^* x = \langle pb, x \rangle$ where $pb \in I$, hence, $\Phi \in I^*$. However, we know from Example 1.3.2 that there are pairs $\mathcal{B}, I$ (even complete) with $b \in \mathcal{B}$ such that the $\mathcal{B}$–functional Φ is not in I^*.

1.5.7 Definition. For a subset S of a pre-Hilbert module E we define the *orthogonal complement* of S as

$$S^\perp = \{x \in E \colon \langle s, x \rangle = 0 \ (s \in S)\}.$$

Clearly, $S^\perp$ is a submodule of E.

A submodule F of E is called ***complementary in*** E, if $E = F \oplus F^\perp$. (In this case there exists a projection p onto F.)

A pre-Hilbert module E is called ***complementary***, if it is complementary in all pre-Hilbert modules where it appears as a submodule.

Notice that this definition in the literature, usually, refers only to Hilbert modules. However, the property of a Hilbert module to be complementary or not is not affected by allowing to check this against pre-Hilbert modules.

1.5.8 Proposition. *A Hilbert module E is complementary, if and only if it is complementary in all Hilbert modules where it appears as a submodule.*

PROOF. There is only one direction to show. So let us assume that E is complementary in all Hilbert modules and let F be a pre-Hilbert module which contains E as a submodule. Then E is a complementary submodule of $\overline{F}$ so that there exists a projection $\bar{p}$ in $\mathcal{B}^a(\overline{F})$ such that $E = \bar{p}\overline{F} \subset F$. It follows that $\bar{p}$ leaves invariant F so that we may define the restriction $p = \bar{p} \upharpoonright F$ of p. We have $\langle x, \bar{p}y \rangle = \langle \bar{p}x, y \rangle$ for all $x, y \in \overline{F}$, whence *a fortiori* $\langle x, py \rangle = \langle px, y \rangle$ for all $x, y \in F$, i.e. p is self-adjoint. Of course, $p^2 = p$ and $E = pF$. Therefore, E is complementary in F. ■

1.5.9 Proposition. *Let E be a self-dual (pre-)Hilbert $\mathcal{B}$–module. Then E is complementary.*

PROOF. Let F be a pre-Hilbert $\mathcal{B}$–module containing E as a submodule. Let $y \in F$. Then the restriction of y^* to E defines a $\mathcal{B}$–functional on E. Since E is self-dual, there exists a unique $y_p \in E$ such that $\langle y, x \rangle = \langle y_p, x \rangle$ for all $x \in E$. Of course, the mapping $p\colon y \mapsto y_p$ is an idempotent and $pF = E$. Moreover, $\langle y, px \rangle = \langle py, px \rangle = \langle py, x \rangle$ for all $x, y \in F$ (because $px, py \in E$). In other words, p is a projection and E is complementary. ■

In Section 3.2 we will see that von Neumann modules are always self-dual and, therefore, complementary. Hence, if we really need projections and are not able to construct them explicitly, then we are free to pass to von Neumann modules. We emphasize, however, that the in many of our applications we will have all projections we need already on a purely algebraic level.

1.5.10 Example. Let $\mathfrak{H}$ be a pre-Hilbert space and $\mathcal{B}$ a pre–C^*–algebra and let $\mathfrak{H}_\mathcal{B} = \mathcal{B} \otimes \mathfrak{H}$ be the ***free right $\mathcal{B}$–module*** generated by $\mathfrak{H}$ with its obvious module structure. (We explain in Examples 3.4.6 and 4.2.13 why we prefer to write $\mathcal{B}$ on the left, although, here we speak only of a right module.) Defining an inner product via

$$\langle b \otimes f, b' \otimes f' \rangle = b^* b' \langle f, f' \rangle,$$

we turn $\mathfrak{H}_\mathcal{B}$ into a pre-Hilbert module. (Positivity follows easily from the observation that if $\mathfrak{H}$ is n–dimensional, then $\mathfrak{H}_\mathcal{B}$ is isomorphic to $\mathcal{B}^n$. In Section 4.3 we see another possibility, when we identify $\mathfrak{H}_\mathcal{B}$ as exterior tensor product.)

Kasparov's *absorption theorem* [Kas80] asserts that, whenever a Hilbert $\mathcal{B}$–module E is ***countably generated*** as Hilbert module (i.e. E is the closed $\mathcal{B}$–linear hull of countably many of its elements), then

$$\overline{\mathfrak{H}_\mathcal{B}} \;\cong\; \overline{\mathfrak{H}_\mathcal{B}} \oplus E$$

(cf. Definition 1.5.11) for a separable infinite-dimensional Hilbert space $\mathfrak{H}$. This result is crucial for KK–theory, however, we do not need it, so we omit the proof. (An elegant proof due to Mingo and Phillips [MP84] can be found in, literally, every text book with a chapter on Hilbert modules like e.g. [Bla86, Lan95, WO93]. Notice that the isomorphism depends heavily on the choice of the generating subset, i.e. the identification is not canonical.) A result by Frank [Fra99] asserts that a self-dual countably generated Hilbert module over a unital C^*–algebra $\mathcal{B}$ is a direct sum of a finitely generated Hilbert submodule and a countable direct sum over finite dimensional ideals in $\mathcal{B}$. This shows that self-dual Hilbert modules are only rarely countably generated, at least, not in norm topology.

1.5.11 Definition. An ***isometry*** between pre-Hilbert $\mathcal{B}$–modules E and F is a mapping $\xi\colon E \to F$ which preserves inner products, i.e. $\langle \xi x, \xi y \rangle = \langle x, y \rangle$. A ***unitary*** is a surjective isometry. We say E and F are ***isomorphic*** pre-Hilbert $\mathcal{B}$–modules, if there exists a unitary $u\colon E \to F$.

1.5.12 Observation. Also isometries and unitaries extend as isometries and unitaries, respectively, to the completions. Moreover, if an isometry has dense range, then its extension to the completions is a unitary. Clearly, the range of an isomtery is complete, if and only if its domain is complete.

A unitary u is adjointable where the adjoint is $u^* = u^{-1}$. Of course, if u is unitary, then so is u^*. An isometry ξ need not be adjointable. This follows easily from the following proposition and the fact that there are non-complementary pre-Hilbert modules. In this case the canonical embedding of a submodule F into a module E in which it is not complementary is a non-adjointable isometry.

1.5.13 Proposition. *An isometry $\xi\colon E \to F$ is adjointable, if and only if there exists a projection $p \in \mathcal{B}^a(F)$ onto the submodule ξE of F.*

Proof. If ξ is adjointable, then $\xi^*\xi = \mathsf{id}$ and $\xi\xi^*$ is a projection onto the range of ξ. So let us assume that there exists a projection p onto the range of ξ. Denote by p_ξ the mapping $F \to \xi E$ defined by $p_\xi y = py$, and denote by $i_\xi\colon \xi E \to F$ the canonical embedding. Then $p_\xi \in \mathcal{B}^a(F, \xi E)$ and $i_\xi \in \mathcal{B}^a(\xi E, F)$ where $p_\xi^* = i_\xi$ (and conversely). Denote by u_ξ the unitary $u_\xi x = \xi x$ in $\mathcal{B}^a(E, \xi E)$. Then

$$\langle x, u_\xi^{-1} py \rangle = \langle p u_\xi x, y \rangle = \langle \xi x, y \rangle$$

so that $u_\xi^{-1} p$ is the adjoint of ξ. ■

1.6 Representations and two-sided Hilbert modules

Now we come to the most important objects in these notes, two-sided Hilbert modules. They arise naturally by GNS-construction for completely positive mappings (Section 4.1) and they may be composed by the (interior) tensor product to obtain new two-sided modules (Section 4.2). Without tensor product there is no Fock module (Parts II and IV) and no tensor product system (Part III).

In principle, an $\mathcal{A}$–$\mathcal{B}$–module E is a right $\mathcal{B}$–module with a homomorphism j from $\mathcal{A}$ into the right module homomorphisms. Often two-sided Hilbert modules are defined in that way with emphasis on the homomorphism j. In fact, it is the possibility to have more than one homomorphism j on the same right module which is responsible for the flexibility in applications. We prefer, however, to put emphasis not on the homomorphism j but on the space as two-sided module. Usually, we will write ax (and not $j(a)x$) for the left action of an element $a \in \mathcal{A}$. By the *canonical homomorphism*, we mean the mapping which sends $a \in \mathcal{A}$ to the operator $x \mapsto ax$ in $\mathcal{L}^r(E)$. Sometimes (e.g. as in Example 1.6.7) a right module has a natural left action of $\mathcal{A}$ which is not that of the bimodule structure under consideration. In this case, we will denote by ax the natural left action and by $a.x$ the left action of the bimodule structure.

1.6.1 Definition. A *representation* of a pre–C^*–algebra $\mathcal{A}$ on a pre-Hilbert $\mathcal{B}$–module E is a homomorphism $j\colon \mathcal{A} \to \mathcal{L}^a(E)$ of $*$–algebras. A representation is *contractive*, if j is a contraction (i.e. $\|j\| \leq 1$). A representation is *non-degenerate*, if $\operatorname{span} j(\mathcal{A})E = E$. A representation is *total* (for a topology on E), if $\operatorname{span} j(\mathcal{A})E$ is dense in E (in that topology).

Clearly, a representation j extends to a representation $\overline{\mathcal{A}} \to \mathcal{B}^a(\overline{E})$, if and only if it is contractive. If $\mathcal{A}$ is spanned by its unitaries or quasi unitaries (for instance, if $\mathcal{A}$ is a C^*–algebra), then any homomorphism $\mathcal{A} \to \mathcal{L}^a(E)$ takes values in $\mathcal{B}^a(E)$. If $\mathcal{A}$ is a C^*–algebra or spanned by its C^*–subalgebras, then a representation is contractive automatically; see Appendix A.7. We will be concerned mainly with this case. If $\mathcal{A}$ is unital, then a representation j is non-degenerate or total, if and only if $j(\mathbf{1}) = \mathbf{1}$. If E is a semi-Hilbert module, then by Corollaries 1.4.3 and 1.4.4 a homomorphism $j\colon \mathcal{A} \to \mathcal{L}^a(E)$ and a (contractive) algebra homomorphism $j\colon \mathcal{A} \to \mathcal{B}^r(E)$ give rise to a homomorphism $j\colon \mathcal{A} \to \mathcal{L}^a(E/\mathcal{N}_E)$ and a (contractive) algebra homomorphism $j\colon \mathcal{A} \to \mathcal{B}^r(E/\mathcal{N}_E)$, respectively. Therefore, often we may define representations on $E/\mathcal{N}_E$ by defining such on E.

1.6.2 Definition. A *(pre-)Hilbert $\mathcal{A}$–$\mathcal{B}$–module* E is a (pre-)Hilbert $\mathcal{B}$–module and an $\mathcal{A}$–$\mathcal{B}$–module such that the canonical homomorphism defines a total (non-degenerate) representation of $\mathcal{A}$. We say E is *contractive*, if the canonical homomorphism is contractive.

If it is clear what $\mathcal{A}$ and $\mathcal{B}$ are, then we also say two-sided (pre-)Hilbert module. If $\mathcal{A} = \mathcal{B}$, then we say two-sided (pre-)Hilbert $\mathcal{B}$–module.

By $\mathcal{B}^{bil}(E, F)$ and $\mathcal{B}^{a,bil}(E, F)$ we denote the space of bounded and of bounded adjointable, respectively, *two-sided* (i.e. $\mathcal{A}$–$\mathcal{B}$–linear) mappings between pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–modules E and F. An *isomorphism* of pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–modules is a two-sided unitary.

1.6.3 Remark. $\mathcal{B}^{a,bil}(E)$ is the relative commutant of the image of $\mathcal{A}$ in $\mathcal{B}^a(E)$.

1.6.4 Observation. The complement of an $\mathcal{A}$–$\mathcal{B}$–submodule of E is again an $\mathcal{A}$–$\mathcal{B}$–submodule of E. The range pE of a projection is an $\mathcal{A}$–$\mathcal{B}$–submodule, if and only if $p \in \mathcal{B}^{a,bil}(E)$. Sufficiency is clear. To see necessity, assume that $apx \in pE$ for all $x \in E$, $a \in \mathcal{A}$. In other words, $ap = pap = (pa^*p)^* = (a^*p)^* = pa$.

We present some examples. Although these examples appear to be simple, they illustrate already essential features of the two-sided module structure and will appear in many contexts.

1.6.5 Example. Let $\mathcal{A}$ be a pre–C^*–algebra and let E be a pre-Hilbert $\mathcal{B}$–module. The simplest action of $\mathcal{A}$ on E is the trivial one $ax = 0$. Of course, this action is highly degenerate. In order to have a two-sided module structure, we extend the trivial left action of $\mathcal{A}$ in the only possible way to a unital left action of the *unitization* $\widetilde{\mathcal{A}}$ of $\mathcal{A}$; see Appendix A.8. Clearly, this extension turns E into a contactive pre-Hilbert $\widetilde{\mathcal{A}}$–$\mathcal{B}$–module. Because $\mathcal{B}$ is an ideal in $\widetilde{\mathcal{B}}$, we have a natural pre-Hilbert $\widetilde{\mathcal{B}}$–module structure on E turning it into a pre-Hilbert $\widetilde{\mathcal{A}}$–$\widetilde{\mathcal{B}}$–module. This module, indeed, appears in unitizations of completely positive contractions and semigroups of such; see Section 12.3.

1.6.6 Example. Let E be a pre-Hilbert $\mathcal{B}$–module. By Example 1.4.10 E^n is a pre-Hilbert $M_n(\mathcal{B}^a(E))$–$\mathcal{B}$–module. If E is a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module, then $\mathcal{A}$ has a homomorphic image in $\mathcal{B}^a(E)$. Therefore, $M_n(\mathcal{A})$ has a homomorphic image in $M_n(\mathcal{B}^a(E))$ (of course, acting non-degenerately on E^n) so that E^n is a pre-Hilbert $M_n(\mathcal{A})$–$\mathcal{B}$–module. This example (together with Examples 1.7.6, 1.7.7, and 4.2.12.) provides us with simple proofs of results on CP-semigroups on $M_n(\mathcal{B})$.

1.6.7 Example. Let $\mathcal{B}$ be a unital pre–C^*–algebra. By the two-sided pre-Hilbert module $\mathcal{B}$ we mean the one-dimensional pre-Hilbert module $\mathcal{B}$ (see Example 1.1.5) equipped with the natural left multiplication $b\colon x \mapsto bx$ $(x \in \mathcal{B})$. However, there are many other possibilities. Let ϑ be a unital endomorphism of $\mathcal{B}$. Then we define $\mathcal{B}_\vartheta$ as the pre-Hilbert $\mathcal{B}$–module $\mathcal{B}$, but equipped with the left multiplication $b.x = \vartheta(b)x$. Observe that $\mathcal{B}$ still is generated by $\mathbf{1}$ as right module and that $\mathbf{1}$ *intertwines* ϑ and id in the sense that $b.\mathbf{1} = \mathbf{1}\vartheta(b)$. Any left multiplication on $\mathcal{B}$ arises in this way from a unital endomporphism ϑ. Indeed, given a left multiplication $x \mapsto b.x$ on $\mathcal{B}$, we define $\vartheta(b) = b.\mathbf{1}$ (i.e. $b.\mathbf{1} = \mathbf{1}\vartheta(b)$). Then

$$\vartheta(bb') = (bb').\mathbf{1} = b.(b'.\mathbf{1}) = b.(\mathbf{1}\vartheta(b')) = (b.\mathbf{1})\vartheta(b) = \vartheta(b)\vartheta(b')$$

and

$$\vartheta(b^*) = b^*.\mathbf{1} = \langle \mathbf{1}, b^*.\mathbf{1}\rangle = \langle b.\mathbf{1}, \mathbf{1}\rangle = \langle \mathbf{1}\vartheta(b), \mathbf{1}\rangle = \vartheta(b)^*\langle \mathbf{1}, \mathbf{1}\rangle = \vartheta(b)^*.$$

An automorphism of the pre-Hilbert module $\mathcal{B}$ is a unitary in $\mathcal{B}^a(\mathcal{B}) = \mathcal{B}$ (cf. Example 1.4.5) acting by usual multiplication on elements of $\mathcal{B}$ and any pre-Hilbert $\mathcal{B}$–module generated by a single unit vector is isomorphic to $\mathcal{B}$. For two-sided modules the situation changes considerably. In order that an automorphism u of the pre-Hilbert $\mathcal{B}$–module $\mathcal{B}$ defines an isomorphism $\mathcal{B}_\vartheta \to \mathcal{B}_{\vartheta'}$, we must have

$$u(b.x) \;=\; b.(ux) \qquad \text{that is} \qquad u\vartheta(b)x \;=\; \vartheta'(b)ux.$$

Putting $x = \mathbf{1}$, we find $u\vartheta(b) = \vartheta'(b)u$ or $\vartheta'(b) = u\vartheta(b)u^*$. In other words, $\mathcal{B}_\vartheta$ and $\mathcal{B}_{\vartheta'}$ are isomorphic two-sided pre-Hilbert modules, if and only if ϑ and ϑ' are conjugate via an inner automorphism $u \bullet u^*$ of the pre–C^*–algebra $\mathcal{B}$. In general, there are non-inner automorphisms, so there are non-isomorphic one-dimensional two-sided pre-Hilbert $\mathcal{B}$–modules. From here it is only a simple step to the classification of E_0–semigroups (i.e. semigroups of unital endomorphisms) up to unitary cocycle conjugacy; see Example 11.1.3.

1.6.8 Example. Any pre- (or semi-)Hilbert module E over a commutative algebra $\mathcal{B}$ has a trivial left module structure over $\mathcal{B}$ where right and left multiplication are just the same.

We denote this trivial left multiplication by b^r, i.e. $b^r : x \mapsto xb$, in order to distinguish it from a possible non-trivial left multiplication. Even if $\mathcal{B}$ is noncommutative, the operation b^r is a well-defined element of $\mathcal{B}^a(E)$, whenever b is in the *center* $C_{\mathcal{B}}(\mathcal{B})$ of $\mathcal{B}$ (cf. Definition 3.4.1). Clearly, b^r commutes with all $a \in \mathcal{B}^a(E)$. In Theorem 4.2.18 we will see that if E contains a unit vector, and if $\mathcal{B}$ is unital, then $b \mapsto b^r$ is an isomorphism from $C_{\mathcal{B}}(\mathcal{B})$ onto the center of $\mathcal{B}^a(E)$.

1.6.9 Example. Let $\mathcal{B} = \mathbb{C}^2$, i.e. the subalgebra of diagonal matrices in the algebra of 2×2–matrices M_2. There are precisely four left multiplications on $\mathbb{C}^2$, comming from the four unital endomorphisms of $\mathbb{C}^2$ (in the sense of Example 1.6.7), all of them giving rise to pairwise non-isomorphic two-sided Hilbert module structures on $\mathbb{C}^2$. The first comming from the identity automorphism. We denote this Hilbert module by $\mathbb{C}^2_+$. The second comming from the flip automorphism $\alpha\binom{z_1}{z_2} = \binom{z_2}{z_1}$. Here we have $\binom{z_1}{z_2}.x = x\binom{z_2}{z_1}$ for all $x \in \mathbb{C}^2$. We denote this Hilbert module by $\mathbb{C}^2_-$. (Notice that the flip is an automorphism of the algebra $\mathbb{C}^2$ but not of the Hilbert module $\mathbb{C}^2$, because it is not right linear.) Finally, there are the two endomorphisms of $\mathbb{C}^2$ being the unital extensions of the trivial endomorphism $\mathbb{C} \to \{0\}$. The two possibilites correspond to the choice to embed the copy of $\mathbb{C}$ which is annihilated either into the first component of $\mathbb{C}^2$, or into the second.

Denote by $e_1 = \binom{1}{0}, e_2 = \binom{0}{1}$ the canonical basis of $\mathbb{C}^2$. Let E be a two-sided pre-Hilbert $\mathbb{C}^2$–module which is generated as a two-sided module by a single cyclic vector ξ. Then E decomposes into the submodules $\mathbb{C}e_i\xi e_j$ $(i, j = 1, 2)$ some of which may be $\{0\}$. If all four spaces are non-trivial, than $\mathbb{C}e_1\xi e_1 \oplus \mathbb{C}e_2\xi e_2$ is isomorphic to $\mathbb{C}^2_+$, whereas $\mathbb{C}e_1\xi e_2 \oplus \mathbb{C}e_2\xi e_1$ is isomorphic to $\mathbb{C}^2_-$. If some of the spaces $\mathbb{C}e_i\xi e_j$ are trivial, then F is at least contained in $\mathbb{C}^2_+ \oplus \mathbb{C}^2_-$. Notice that none of the submodules $\mathbb{C}e_i\xi e_j$ is isomorphic to $\mathbb{C}^2$ as right Hilbert module, because the inner products takes values in $\mathbb{C}e_j$, whereas the inner product of $\mathbb{C}^2$ takes values in all of $\mathbb{C}^2$, independently of a possible left multiplication. This example helps us to understand completely postive semigroups on $\mathbb{C}^2$.

1.6.10 Example. Let G and $\mathfrak{H}$ be Hilbert spaces. Then $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$ (cf. Example 1.1.6) is a Hilbert $\mathcal{B}(G)$–$\mathcal{B}(G)$–module in an obvious way. We will see in Example 3.3.4 that any von Neumann $\mathcal{B}(G)$–$\mathcal{B}(G)$–module E (Definitions 3.1.1 and 3.3.1) must be of this form. Example 3.3.4 is the basis to understand the relation between Arveson's tensor product systems of Hilbert spaces [Arv89a] and tensor product systems of Hilbert modules [BS00] (Example 11.1.4).

1.6.11 Example. We close with an example which comes directly from a physical problem. Let $d \geq 3$ be an integer and let $\mathcal{P} = \mathcal{C}_b(\mathbb{R}^d)$ the C^*–algebra of bounded continuous

($\mathbb{C}$–valued) functions on $\mathbb{R}^d$. We write elements of $\mathcal{P}$ as functions $h(p)$ of $p \in \mathbb{R}^d$. Let $E_0 = \mathcal{C}_c(\mathbb{R}^d, \mathcal{P})$ be the space of continuous $\mathcal{P}$–valued functions on $\mathbb{R}^d$ with compact support. We write elements of E_0 as functions $f(k)$ of $k \in \mathbb{R}^d$. (So, fixing k, $f(k)$ is still a function of p.) E_0 has already a natural two-sided pre-Hilbert $\mathcal{P}$–module structure where the module operations are pointwise multiplication from the right and from the left, respectively, and where the inner product of f and g is just the usual *Riemann-Bochner integral* $\int dk\, f^*(k)g(k)$; see Appendix B. Here we need, however, a different structure. Only the right multiplication is the natural one. The left multiplication of an element $f \in E_0$ by a function $h \in \mathcal{P}$ is defined as

$$[h.f](k) = h(p+k)f(k),$$

where, clearly, $h(p+k)$ defines for each fixed k an element of $\mathcal{P}$ which has to be multiplied with the value $f(k)$ of f at this fixed k.

$$\langle f, g\rangle = \int dk\, f^*(k)\delta\big(p \cdot k + \tfrac{|k|^2}{2} + |k|\big)g(k) \tag{1.6.1}$$

defines a semi-inner product, where we interpret the (one-dimensional) δ–function just as a formal prescription of how to evaluate the $|k|$–integration in polar coordinates; see Remark D.3.7. Finally, we divide out the kernel of this inner product and obtain a two-sided pre-Hilbert $\mathcal{P}$–module E.

This module appears in the *stochastic limit* for an electron coupled to a field in the vacuum state. The inner product has been computed by Accardi and Lu [AL96]. Also the idea to interpret it as an inner product of a Hilbert module is due to [AL96]. The interpretation as two-sided Hilbert module together with the correct left multiplication, and the result that (1.6.1), indeed, defines an element of $\mathcal{P}$ can be found in [Ske98a]. We explain the physical model, how the module E arises and, in particular, that all preceding prescriptions make sense in Appendix D. In the context of this section it is noteworthy that it is the correct left multiplication of E which made it possible to interpret the the limit module of the stochastic limit as a Fock module over E; cf. Example 6.1.7.

1.7 Finite-rank operators and compact operators

Let E and F be pre-Hilbert $\mathcal{B}$–modules. Proceeding as in Example 1.1.5 we embed $\mathcal{B}^a(F, E)$ into

$$\mathcal{B}^a(F \oplus E) = \begin{pmatrix} \mathcal{B}^a(F,F) & \mathcal{B}^a(E,F) \\ \mathcal{B}^a(F,E) & \mathcal{B}^a(E,E) \end{pmatrix}.$$

From this embedding it follows immediately that for all $a \in \mathcal{B}^a(F,E)$ the element a^*a is positive in $\mathcal{B}^a(F)$, because it is positive in $\mathcal{B}^a(F \oplus E) \supset \mathcal{B}^a(F)$. In other words, $\mathcal{B}^a(F,E)$ with inner product $\langle a, a' \rangle = a^*a'$ is a contractive pre-Hilbert $\mathcal{B}^a(E)$–$\mathcal{B}^a(F)$–module. In Chapter 2 we investigate such *generalized matrix algebra structuctures* systematically.

If we put $F = \mathcal{B}$ and take also into account that $E \subset \mathcal{B}^a(\mathcal{B}, E)$, we see that E may be embedded into the lower left corner of $\mathcal{B}^a(\mathcal{B} \oplus E)$. Clearly, this embedding is isomteric and respects all module operations. Now we ask for the $*$–subalgebra of $\mathcal{B}^a(\mathcal{B} \oplus E)$ generated by E. Clearly, the upper right corner is E^* and the upper left corner is $\mathcal{B}_E \subset \mathcal{B} \subset \mathcal{B}^a(\mathcal{B})$. The lower right corner consists of the span of all operators on E of the form $z \mapsto x\langle y, z \rangle$ $(x, y \in E)$. These are the analogues of the finite rank operators on Hilbert spaces.

1.7.1 Definition. Let E and F be pre-Hilbert $\mathcal{B}$–modules. An operator of the form $xy^* \in \mathcal{B}^a(E,F)$ $(x \in F \subset \mathcal{B}^a(\mathcal{B}, F), y \in E \subset \mathcal{B}^a(\mathcal{B}, E))$ is called *rank-one operator* with adjoint given by yx^*. The linear span $\mathcal{F}(E,F)$ of all rank-one operators is called the space of *finite rank operators*, its completion $\mathcal{K}(E,F)$ is called the Banach space of *compact operators*.

1.7.2 Remark. Our notation has the same advantages as *Dirac's bra(c)ket notation* $|x\rangle\langle y| := xy^*$. Additionally, it makes a lot of brackets dissappear, thus leading to clearer formulae. See, for instance the proof of Theorem 4.2.18 or Section 14.1.

1.7.3 Remark. Notice that the elements of $\mathcal{K}(E,F)$ can be considered as operators $E \to F$, in general, only if F is complete. Notice also that, in general, neither the finite rank operators have finite rank in the sense of operators between linear spaces, nor the compact operators are compact in the sense of operators between Banach spaces. (Consider, for instance, the identity operator on $E = \mathcal{B}$, which is rank-one, if $\mathcal{B}$ is unital, but non-compact as operator on the normed space $\mathcal{B}$ as soon as $\mathcal{B}$ is infinite-dimensional.)

1.7.4 Observation. Interestingly enough, both $\mathcal{F}(E,F)$ and $\mathcal{K}(E,F)$ are $\mathcal{B}^a(F)$–$\mathcal{B}^a(E)$–modules. In particular, $\mathcal{F}(E) := \mathcal{F}(E,E)$ is an ideal in $\mathcal{B}^a(E)$ and $\mathcal{K}(E) := \mathcal{K}(E,E)$ is a closed ideal in $\mathcal{B}^a(\overline{E})$. It follows that $\mathcal{F}(E,F)$ is a pre-Hilbert $\mathcal{B}^a(E)$–$\mathcal{F}(E)$–module (which may be considered also as a pre-Hilbert $\mathcal{B}^a(E)$–$\mathcal{B}^a(E)$–module) and that $\mathcal{K}(E,F)$ is a Hilbert $\mathcal{B}^a(\overline{E})$–$\mathcal{K}(E)$–module (which may be considered also as a Hilbert $\mathcal{B}^a(\overline{E})$–$\mathcal{B}^a(\overline{E})$–module).

With these notations it is clear that

$$\mathsf{alg}^*(E) = \begin{pmatrix} \mathcal{B}_E & E^* \\ E & \mathcal{F}(E) \end{pmatrix} \tag{1.7.1}$$

is the pre–C^*–subalgebra of $\mathcal{B}^a(\mathcal{B} \oplus E)$ generated by E (or E^*). The pre–C^*–subalgebra generated by E and $\mathcal{B}$ is

$$\mathsf{alg}^*(E, \mathcal{B}) = \begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{F}(E) \end{pmatrix}. \tag{1.7.2}$$

Observe that $\mathsf{alg}^*(E, \mathcal{B})$ has a natural contractive representation on $\mathcal{B}_E \oplus E$. This representation is isometric, if and only if E is essential, and, of course, the restricition of this representation to $\mathsf{alg}^*(E)$ is always isometric. Observe also that $\mathcal{F}(E)$ and $\mathcal{K}(E)$ (as pre–C^*–algebras) are independent of whether we consider E as $\mathcal{B}$– or as $\mathcal{B}_E$–module.

Algebraically, the situation in (1.7.1) is symmetric under "exchange of coordinates". In particular, E^* is a pre-Hilbert $\mathcal{F}(E)$–module in an obvious manner. The only difference is that now the pre–C^*–algebra structure arises, by considering elements of $\mathsf{alg}^*(E^*)$ as operators on $\mathcal{F}(E) \oplus E^*$. However, the two C^*–norms coincide. (To see this, close $\mathsf{alg}^*(E)$ in $\mathcal{B}^a(\overline{\mathcal{B}_E \oplus E})$ and observe that the canonical mapping $\overline{\mathsf{alg}^*(E)} \to \mathcal{B}^a(\overline{\mathcal{F}(E) \oplus E^*})$ is faithful, hence, isometric.) In this way, we identify $\mathcal{B}_E$ as $\mathcal{F}(E^*)$. Clearly, the canonical representation of $\mathcal{B}_E$ on E^* extends to a contractive repesentation of $\mathcal{B}$. If $\mathcal{B}$ is unital, then this representation is non-degenerate. (If not, then we will see in Corollary 2.1.10 that the representation is at least total.) The representation is isometric, if and only if E is essential.

Let E be a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module. We ask, under which circumstances we can consider E^* as a pre-Hilbert $\mathcal{B}$–$\mathcal{A}$–module. For this the range $\mathcal{F}(E)$ of the inner product must be an ideal in $\mathcal{A}$, and $\mathcal{B}$ should act non-degenerately on E. If these requirements are fulfilled, we say E^* is the *dual pre-Hilbert $\mathcal{B}$–$\mathcal{A}$–module* of E. Obviously, $\mathcal{A}$ acts isometrically on E, if and only if E^* is essential. Of course, $(E^*)^* = E$. By

$$\mathsf{alg}^*(\mathcal{A}, E, \mathcal{B}) = \begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{A} \end{pmatrix}$$

we denote the pre–C^*–subalgebra of $\mathcal{B}^a(\mathcal{B} \oplus E) \oplus \mathcal{B}^a(\mathcal{A} \oplus E^*)$ (pre–C^*–algebraic direct sum) generated by $\mathcal{A}$, E, and $\mathcal{B}$ where the matrices in $\left(\begin{smallmatrix} \mathcal{B} & E^* \\ E & \mathcal{A} \end{smallmatrix}\right)$ are imbeded via the representation id into $\mathcal{B}^a(\mathcal{B} \oplus E)$ and via the anti-representation id^* into $\mathcal{B}^a(\mathcal{A} \oplus E^*)$.

1.7.5 Observation. $\mathcal{B}_E$ is rarely complete. Consider, for instance, $E = H^*$ for some infinite-dimensional Hilbert space. Here $\mathcal{B}_E = \mathcal{F}(H)$ is certainly not complete.

1.7.6 Example. For the moment suppose that the pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module E has a dual pre-Hilbert $\mathcal{B}$–$\mathcal{A}$–module E^* (i.e. $\mathcal{F}(E)$ is an ideal in $\mathcal{A}$). By Example 1.6.6 $(E^*)^n$ is a pre-Hilbert $M_n(\mathcal{B})$–$\mathcal{A}$–module. The inner product on $(E^*)^n$ is $\langle (x_1^*, \ldots, x_n^*), (y_1^*, \ldots, y_n^*) \rangle = \sum_{i=1}^n x_i y_i^*$. The action of $M_n(\mathcal{B})$ is isometric, if E is essential. In particular, the ideal $M_n(\mathcal{B}_E) = \mathcal{F}((E^*)^n)$ in $M_n(\mathcal{B})$ is represented isometrically on $(E^*)^n$.

Define $E_n = ((E^*)^n)^*$. Like E^n also E_n consists of vectors $X = (x_1, \dots, x_n)$. By construction E_n is a pre-Hilbert $\mathcal{A}$–$M_n(\mathcal{B})$–module with right multiplication of $B = \big(b_{ij}\big) \in M_n(\mathcal{B})$ given by $\big(XB\big)_j = \sum\limits_{i=1}^{n} x_i b_{ij}$ and inner product

$$\big\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \big\rangle_{i,j} = \langle x_i, y_j \rangle.$$

Therefore, the matrix with entries $\langle x_i, x_j \rangle$ is a positive element of $M_n(\mathcal{B})$. Setting $E = \mathcal{B}$, we recover the (trivial) result that the matrix with entries $b_i^* b_j$ is positive. The left action of $a \in \mathcal{A}$ on $X \in E_n$ is $aX = (ax_1, \dots, ax_n)$. If we choose $\mathcal{A} = \mathcal{B}^a(E)$, we see that $\mathcal{B}^a(E) \subset \mathcal{B}^a(E_n)$. (Of course, the action of $\mathcal{B}^a(E)$ is faithful.) Less obvious is that we have equality of $\mathcal{B}^a(E)$ and $\mathcal{B}^a(E_n)$. Indeed, if $\mathcal{B}$ is unital, then the matrix unit e_{ij} is an element of $M_n(\mathcal{B})$. For $a \in \mathcal{B}^a(E_n)$ we find from $(aX)e_{ij} = a(Xe_{ij})$ applied to $X = (x, \dots, x)$ that the restriction of a to the i–th component of E_n coincides with the restriction to the j–th component of E_n. If $\mathcal{B}$ is non-unital, then we consider $\widetilde{\mathcal{B}}$ or we use an approximate unit.

Finally, if $\mathcal{F}(E)$ is not an ideal in $\mathcal{A}$ and $\mathcal{A}$ is unital, then we use the fact that $\mathcal{A}$ has an image in $\mathcal{B}^a(E)$ so that the pre-Hilbert $\mathcal{B}^a(E)$–$M_n(\mathcal{B})$–module E_n is also a pre-Hilbert $\mathcal{A}$–$M_n(\mathcal{B})$–module. (Non-degeneracy might cause problems, if $\mathcal{A}$ is non-unital, but totality is preserved in any case.)

1.7.7 Example. We extend the preceding example and set $M_{nm}(E) = (E_m)^n$. It is easy to check that also $M_{nm}(E) = (E^n)_m$. We find that the pre-Hilbert $M_n(\mathcal{A})$–$M_m(\mathcal{B})$–module $M_{nm}(E)$ consists of matrices $X = \big(x_{ki}\big)$ whose inner product is

$$\langle X, Y \rangle_{ij} = \sum_{k=1}^{n} \langle x_{ki}, y_{kj} \rangle.$$

An element of $M_m(\mathcal{B})$ acts from the right on the right index (coming from E_m) and an element of $M_n(\mathcal{A})$ acts from the left on the left index (coming from E^n) in the usual way. Conversely, if E_{nm} is a pre-Hilbert $M_n(\mathcal{A})$–$M_m(\mathcal{B})$–module, then all matrix entries are isomorphic to the same pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module E and $E_{nm} = M_{nm}(E)$. (If $\mathcal{A}$ and $\mathcal{B}$ are unital, then define Q_i as the matrix in $M_n(\mathcal{A})$ with $\mathbf{1}$ in the i–th place in the diagonal and define $P_j \in M_m(\mathcal{B})$ analogously. Each of these *matrix entries* $Q_i E_{nm} P_j$ inherits a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module structure by embedding $\mathcal{A}$ and $\mathcal{B}$ into that unique place in the diagonal of $M_n(\mathcal{A})$ and $M_m(\mathcal{B})$, respectively, where it acts non-trivially on $Q_i E_{nm} P_j$. As in Example 1.7.6, by appropriate use of matrix units we see that all $Q_i E_{nm} P_j$ are isomorphic to the same pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module E and that $E_{nm} = M_{nm}(E)$. The same shows to remains true, when $\mathcal{A}$ and $\mathcal{B}$ are not necessarily unital by appropriate use of approximate units.)

Let us set $X = \big(\delta_{ij} x_i\big) \in M_n(E) := M_{nn}(E)$ for some $x_i \in E$ $(i = 1, \dots, n)$, and Y correspondingly. Then the mapping $T \colon M_n(\mathcal{A}) \to M_n(\mathcal{B})$, defined by setting $T(A) =$

$\langle X, AY \rangle$ acts matrix-element-wise on A, i.e.

$$\big(T(A)\big)_{ij} = \langle x_i, a_{ij} y_j \rangle.$$

$T(A)$ may be considered as the *Schur product* of the matrix T of mappings $\mathcal{A} \to \mathcal{B}$ and the matrix A of elements in $\mathcal{A}$.

This example will help us to show in Chapter 5 that T is *completely positive* for $Y = X$. Here we use its first part to analyze the structure of Hilbert modules over finite-dimensional C^*–algebras.

1.7.8 Corollary. *Any pre-Hilbert M_n–M_m–module E is isomorphic to $M_{nm}(\mathfrak{H})$ for some pre-Hilbert space $\mathfrak{H}$, and E is complete, if and only if $\mathfrak{H}$ is complete.*

It is well-known that finite-dimensional C^*–algebras decompose into blocks of matrix algebras M_n. So the following corollary covers all two-sided Hilbert modules over finite-dimensional C^*–algebras.

1.7.9 Corollary. *Let $\mathcal{A} = \bigoplus_{i=1}^{k} M_{n_i}$ and $\mathcal{B} = \bigoplus_{j=1}^{\ell} M_{m_j}$ be finite-dimensional C^*–algebras. Denote by $p_i = \mathbf{1}_{M_{n_i}}$ and $q_j = \mathbf{1}_{M_{m_j}}$ the central projections generating the ideals $M_{n_i} = p_i\mathcal{A}$ in $\mathcal{A}$ and $M_{m_j} = q_j\mathcal{B}$ in $\mathcal{B}$, respectively. Then any pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module E decomposes into the direct sum*

$$E \;=\; \bigoplus_{i,j} E_{ij}$$

over the M_{n_i}–M_{m_j}–modules $E_{ij} = p_i E q_j$ (cf. Example 1.6.8) which may be considered also as $\mathcal{A}$–$\mathcal{B}$–modules.

Observe that some of the E_{ij} may be trivial. Specializing to diagonal algebras $\mathcal{A} = \mathbb{C}^n$ and $\mathcal{B} = \mathbb{C}^m$ (whence, $M_{n_i} = M_{m_j} = \mathbb{C}$) we find the generalization of Example 1.6.9. Putting $\mathcal{A} = \mathcal{B} = \mathbb{C}^n$ we see that pre-Hilbert $\mathbb{C}^n$–$\mathbb{C}^n$–modules are given by $n \times n$–matrices of pre-Hilbert spaces. On the other hand, passing to the case where the blocks may be $\mathcal{B}(G)$ where G is a Hilbert space of arbitrary dimension (in other words, the strong closures of $\mathcal{A}$ and $\mathcal{B}$ are arbitrary type I von Neumann algebras), we are in the set-up of Bhat [Bha99].

We mentioned already that for unital $\mathcal{B}$ we have $\mathcal{B}^a(\mathcal{B}) = \mathcal{B}$. Now we want to see how $\mathcal{B}^a(\mathcal{B})$ looks like, in general.

1.7.10 Lemma. *For an arbitrary pre–C^*–algebra $\mathcal{B}$ we have $\mathcal{B}^a(\mathcal{B}) = M(\mathcal{B})$ where $M(\mathcal{B})$ denotes the multiplier algebra of $\mathcal{B}$.*

PROOF. See Appendix A.8 for details about multiplier algebras and double centralizers.

Let $(L,R) \in M(\mathcal{B})$ be a double centralizer and $b,b' \in \mathcal{B}$. Then $L(bb') = (Lb)b'$ so that $L \in \mathcal{B}^r(\mathcal{B})$. Furthermore, $\langle b, Lb'\rangle = b^*Lb' = (Rb^*)b' = \langle R^*b, b'\rangle$ where $R^*b = (Rb^*)^*$. In other words, R^* is an adjoint of L so that $L \in \mathcal{B}^a(\mathcal{B})$. Therefore, there exists a natural isometric embedding $M(\mathcal{B}) \to \mathcal{B}^a(\mathcal{B})$.

Conversely, if $a \in \mathcal{B}^a(\mathcal{B})$, then (L_a, R_a) with $L_a b = ab$ and $R_a b = (a^*b^*)^*$ is a double centralizer whose image in $\mathcal{B}^a(\mathcal{B})$ is a. In other words, the natural embedding is surjective, hence, an isometric isomorphism of normed spaces. From the multiplication and involution for double centralizers it follows that it is an isomorphism of pre–C^*–algebras. ■

1.7.11 Remark. Let $a \in \mathcal{B}^a(\mathcal{B})$. Clearly, a is a $\mathcal{B}$–functional on the pre-Hilbert $\mathcal{B}$–module $E = \mathcal{B}$ which has an adjoint, namely, a^*. It follows that for non-unital $\mathcal{B}$ the set $\mathcal{B}^a(E,\mathcal{B})$ of adjointable $\mathcal{B}$–functionals on E is (usually, much) bigger than E^*. As E^* and E are anti-isomorphic, for non-unital $\mathcal{B}$ also $\mathcal{B}^a(\mathcal{B},E)$ is bigger than E.

1.7.12 Observation. Notice, however, that always $\overline{E} = \mathcal{K}(\mathcal{B},E)$ and, consequently, $\overline{E}^* = \mathcal{K}(E,\mathcal{B})$. ($\mathcal{B}$ has an approximate unit.) But, $\mathcal{F}(\mathcal{B},E)$ is, in general, different from E.

1.7.13 Lemma. *Let E be a pre-Hilbert $\mathcal{B}$–module. Then the Hilbert modules $\overline{E}$ and $\mathcal{K}(E)$ have the same C^*–algebra of adjointable operators.*

PROOF. We have $\overline{E} = \overline{\mathsf{span}}\big(\mathcal{K}(E)\mathcal{K}(E,\mathcal{B})\big)$, because $\mathcal{K}(E)$ has an approximate unit, and $\mathcal{K}(E) = \overline{\mathsf{span}}\,\mathcal{K}(E,\mathcal{B})\mathcal{K}(\mathcal{B},E)$, because $\mathcal{K}(E,\mathcal{B})\mathcal{K}(\mathcal{B},E) = \overline{E}\,\overline{E}^*$. By the first equality, an operator on $\mathcal{K}(E)$ gives rise to an operator on $\overline{E}$, and by the second, an operator on $\overline{E}$ gives rise to an operator on $\mathcal{K}(E)$. It is clear that the two correspondences are inverse to eachother. ■

1.7.14 Corollary. $$M(\mathcal{K}(E)) = \mathcal{B}^a(\mathcal{K}(E)) = \mathcal{B}^a(\overline{E}).$$

1.7.15 Definition. We equip $\mathcal{B}^a(\overline{E})$ with the ***strict*** topology from the multiplier algebra of $\mathcal{K}(E)$, i.e. the strict topology on $\mathcal{B}^a(\overline{E})$ is the $*$–strong topology on $\mathcal{B}^a(\mathcal{K}(E))$.

1.7.16 Proposition. *The strict topology and the $*$–strong topology of $\mathcal{B}^a(\overline{E})$ coincide on bounded subsets.*

PROOF. If a net a_λ is bounded, then in either of the topologies it is sufficient to check convergence on dense subsets of $\mathcal{K}(E)$ and $\overline{E}$, respectively. So if $a_\lambda \to a$ in the strict topology, then $(a_\lambda - a)x \to 0$ for all x in the dense subset $\mathsf{span}\big(\mathcal{F}(E)E\big)$ of E, and similarly for a_λ^*. Conversely, if $a_\lambda \to a$ $*$–strongly, then $(a_\lambda - a)k \to 0$ on the dense subset $\mathcal{F}(E)$ of $\mathcal{K}(E)$, and similarly for a_λ^*. ■

1.7.17 Remark. The two topologies do, in general, not coincide. For instance, if H is a Hilbert space then the strict topology on $\mathcal{B}(H)$ is just the $*$–σ–strong topology, which is known to be properly stronger than the $*$–strong topology, when H is infinite-dimensional.

Lemma 1.7.13 is a special case of a result due to Pimsner [Pim97]. After Section 4.2, where we introduce tensor products, it is clear that our proof is just a translation of Pimser's original proof. Corollary 1.7.14 is a famous theorem due to Kasparov [Kas80]. Here we prove it by Lemma 1.7.10 without making use of the strict topology.

Chapter 2

Generalized matrix algebras

We have seen that a pre-Hilbert $\mathcal{B}$–module E may be embedded into a 2×2–matrix which has a natural pre–C^*–algebra structure. A simple consequence was the positivity of elements $xx^* \in \mathfrak{F}(E)$ or $\big(\langle x_i, x_j\rangle\big) \in M_n(\mathcal{B})$.

A further advantage of such an embedding is that C^*–algebras come along with a couple of additional topologies. By restriction of these topologies to a Hilbert module contained in such a generalized matrix algebra, we obtain immediately the analogues for Hilbert modules. Usually, the toplogies are compatible with the decomposition into matrix elements of the matrix algebra, so that closure of the matrix algebra in such a topology means closure of each matrix entry independently. The closure of a C^*–algebra in such a topology is, usually, again a C^*–algebra. Consequently, if we close a Hilbert $\mathcal{B}$–module E in a certain topology and $\mathcal{B}$ was already closed in this topology, then it follows that the closure of E is again a Hilbert module over $\mathcal{B}$. This observation is crucial in Chapter 3 in the proof of the fact that any Hilbert module over a von Neumann algebra allows for a self-dual extension.

The concept of embedding a Hilbert module into a C^*–algebra is not new. The idea is probably already present in Rieffel [Rie74] and appears clearly in Blecher [Ble97] in connection with operator spaces. We comment on this aspect at the end of this chapter. Our treatment here is an extension of Skeide [Ske00b].

2.1 Basic properties

2.1.1 Definition. Let $\mathcal{M}$ be an algebra with subspaces $\mathcal{B}_{ij}$ $(i, j = 1, \ldots, n)$ such that

$$\mathcal{M} = \begin{pmatrix} \mathcal{B}_{11} & \ldots & \mathcal{B}_{1n} \\ \vdots & & \vdots \\ \mathcal{B}_{n1} & \ldots & \mathcal{B}_{nn} \end{pmatrix} \qquad \Big(\text{i.e. } \mathcal{M} = \bigoplus_{i,j=1}^{n} \mathcal{B}_{ij}\Big).$$

We say $\mathcal{M}$ is a *generalized matrix algebra* (of *order* n), if the multiplication in $\mathcal{M}$ is compatible with the usual matrix multiplication, i.e. if

$$\bigl(BB'\bigr)_{ij} = \sum_{k=1}^{n} b_{ik} b'_{kj}$$

for all elements $B = \bigl(b_{ij}\bigr)$ and $B' = \bigl(b'_{ij}\bigr)$ in $\mathcal{M}$. If $\mathcal{M}$ is also a normed and a Banach algebra, then we say $\mathcal{M}$ is a *generalized normed* and a *generalized Banach matrix algebra*, respectively.

If $\mathcal{M}$ is also a $*$–algebra fulfilling

$$\bigl(B^*\bigr)_{ij} = b^*_{ji},$$

then we say $\mathcal{M}$ is a *generalized matrix $*$–algebra*. If $\mathcal{M}$ is also a (pre–)C^*–algebra, then we call $\mathcal{M}$ a *generalized matrix (pre–)C^*–algebra*.

A *generalized matrix ($*$–)subalgebra* of $\mathcal{M}$ is a collection of subspaces $\mathcal{C}_{ij} \subset \mathcal{B}_{ij}$, such that $\mathcal{N} = \bigl(\mathcal{C}_{ij}\bigr)$ is a ($*$–)subalgebra of $\mathcal{M}$.

2.1.2 Example. If $\mathcal{B}$ is a pre–C^*–algebra and $\mathcal{B}_{ij} = \mathcal{B}$, we recover the usual matrix pre–C^*–algebra $M_n(\mathcal{B})$ normed as in Example 1.4.10. In the sequel, we omit the word 'generalized' and speak just of matrix algebras. If we refer to $M_n(\mathcal{B})$, we say a 'usual matrix algebra'.

2.1.3 Remark. Clearly, the subset $\bigl\{\bigl(b_{ij}\bigr) : b_{ij} = b\delta_{ij} \ (b \in \mathcal{B})\bigr\}$ of $M_n(\mathcal{B})$ is a subalgebra isomorphic to $\mathcal{B}$, but not a matrix subalgebra.

2.1.4 Example. Let $\mathcal{M}$ be a unital pre–C^*–algebra and let $\{p_1, \ldots, p_n\}$ be a complete set of orthogonal projections. Then $\mathcal{M} = \bigl(p_i \mathcal{M} p_j\bigr)_{ij}$ endows $\mathcal{M}$ with a matrix pre–C^*–algebra structure.

2.1.5 Example. Let $\mathcal{M} = \bigl(\mathcal{B}_{ij}\bigr)_{i,j}$ be an $n \times n$–matrix pre–C^*–algebra. Then the diagonal entries $\mathcal{B}_i := \mathcal{B}_{ii}$ are pre–C^*–subalgebras of $\mathcal{M}$. If all $\mathcal{B}_i$ are unital, then the entries $\mathcal{B}_{ij}$ are pre-Hilbert $\mathcal{B}_i$–$\mathcal{B}_j$–modules with inner product $\langle b, b' \rangle = b^* b'$ and the natural right and left multiplications. Clearly, the pre-Hilbert module norm and the norm coming from $\mathcal{M}$ by restriction coincide. Moreover, $\mathcal{B}_i$ contains $\mathfrak{F}(\mathcal{B}_{ij})$ for all j.

Conversely, we already know that a pre-Hilbert $\mathcal{B}$–module E may be embedded into the matrix pre–C^*–algebras $\mathsf{alg}^*(E)$ or $\mathsf{alg}^*(E, \mathcal{B})$ (cf. (1.7.1), (1.7.2).)

2.1.6 Proposition. *The norm on a matrix pre–C^*–algebra is admissible in the sense of Definition A.2.1, i.e. $\|b_{ij}\| \leq \|B\|$ for all $B \in \mathcal{M}; i,j = 1,\ldots,n$.*

Proof. Let $B \in \mathcal{M}$ with components $b_{ij} \in \mathcal{B}_{ij}$. Then BB^* has the components $\sum\limits_{k=1}^{n} b_{ik}b_{jk}^* \in \mathcal{B}_{ij}$. On the other hand, if $C \in \mathcal{M}$ and $b_{ij} \in \mathcal{B}_{ij} \subset \mathcal{M}$, then $b_{ij}^*Cb_{ij} = b_{ij}^*c_{ii}b_{ij} \in \mathcal{B}_j$, where c_{ii} is the component of C in $\mathcal{B}_i$. Combining both, we find

$$b_{ij}^*BB^*b_{ij} = \sum_{k=1}^{n} b_{ij}^*b_{ik}b_{ik}^*b_{ij} = \sum_{k=1}^{n} (b_{ik}^*b_{ij})^*(b_{ik}^*b_{ij}) \geq (b_{ij}^*b_{ij})^*(b_{ij}^*b_{ij}),$$

so that

$$\|B\| \; \|b_{ij}\| \geq \|B^*b_{ij}\| \geq \|b_{ij}^*b_{ij}\| = \|b_{ij}\|^2 .$$

This implies $\|B\| \geq \|b_{ij}\|$ no matter, whether $b_{ij} = 0$ or not. ■

2.1.7 Corollary. *Let $\mathcal{M}$ be a matrix pre–C^*–algebra. Then $\mathcal{M}$ is complete, if and only if each $\mathcal{B}_{ij}$ is complete with respect to the norm induced by the norm of $\mathcal{M}$. In particular, $\big(\overline{\mathcal{B}}_{ij}\big) = \overline{\mathcal{M}}$ is a matrix C^*–algebra.*

2.1.8 Corollary. *The projections $\mathfrak{p}_{ij} \colon \mathcal{M} \to \mathcal{B}_{ij}, \mathfrak{p}_{ij}(B) = b_{ij}$ have norm 1. In particular, if $\mathcal{M}$ is unital, then so are $\mathcal{B}_i$ and we are in the situation of Example 2.1.4, where $p_i = \mathbf{1}_{\mathcal{B}_i}$ and $\mathfrak{p}_{ij}(B) = p_ibp_j$.*

2.1.9 Remark. Curiously, the unitization $\widetilde{\mathcal{M}}$ is (for $n \geq 2$) not a matrix algebra, because $\mathfrak{p}_{ij}(\widetilde{\mathbf{1}})$ is not an element of $\widetilde{\mathcal{M}}$. If we want to add a unit to a matrix algebra, then we must add new units for each $\mathcal{B}_i$ separately, in order to obtain again a matrix algebra.

2.1.10 Corollary. *Let $\big(U_\lambda\big)_{\lambda\in\Lambda}$ denote an approximate unit for $\mathcal{M}$. Set $u_\lambda^{ij} = \mathfrak{p}_{ij}(U_\lambda)$. Then*

$$\lim_\lambda u_\lambda^{k\ell}b_{ij} = \delta_{k\ell}\delta_{\ell i}b_{ij} \quad \textit{and} \quad \lim_\lambda b_{ij}u_\lambda^{k\ell} = \delta_{jk}\delta_{k\ell}b_{ij} \tag{2.1.1}$$

for all $b_{ij} \in \mathcal{B}_{ij}$. Moreover, $\big(u_\lambda^{ii}\big)_{\lambda\in\Lambda}$ and $\big(u_\lambda^{11} + \ldots + u_\lambda^{nn}\big)_{\lambda\in\Lambda}$ form approximate units for $\mathcal{B}_{ii}$ and $\mathcal{M}$, respectively. These are increasing, if $\big(U_\lambda\big)_{\lambda\in\Lambda}$ is increasing.

Proof. Equation (2.1.1) follows from $\big(BC\big)_{ij} = \sum\limits_{k=1}^{n} \mathfrak{p}_{ik}(B)\mathfrak{p}_{kj}(C)$ $(B, C \in \mathcal{M})$ and continuity of $\mathfrak{p}_{ij}$. To see that also the net $\big(u_\lambda^{ii}\big)_{\lambda\in\Lambda}$ is increasing we observe that $\mathfrak{p}_{ii}(B) = \lim\limits_\lambda u_\lambda^{ii}Bu_\lambda^{ii}$ by Equation (2.1.1) and boundedness of the net $\big(u_\lambda^{ii}\big)_{\lambda\in\Lambda}$, so that $\mathfrak{p}_{ii}$ is a positive mapping. ■

2.1.11 Corollary. *Let E and F be pre-Hilbert $\mathcal{B}$–modules. Then the unit ball of $\mathfrak{F}(E,F)$ is strongly dense in $\mathfrak{B}^r(E,F)$, and $*$–strongly (hence, by Proposition 1.7.16 also strictly) dense in the unit ball of $\mathfrak{B}^a(E,F)$.*

PROOF. Let $a \in \mathfrak{B}^r(E,F)$. By Corollary 2.1.10 applied to $\mathsf{alg}^*(E)$, there exists an approximate unit $\big(u_\lambda\big)$ for $\mathfrak{F}(E)$ such that $u_\lambda x \to x$ for all $x \in E$. Therefore, $au_\lambda x \to ax$, i.e. the net au_λ in $\mathfrak{F}(E,F)$ converges strongly to a. If a is adjointable, then also $u_\lambda a^* \in \mathfrak{F}(F,E)$ converges strongly to a^*. ■

2.1.12 Definition. The *$\mathcal{B}$–weak topology* on a pre-Hilbert $\mathcal{B}$–module E is the locally convex Hausdorff topology generated by the family $\|\langle x, \bullet\rangle\|$ $(x \in E)$ of seminorms.

When interpreted as topology on E^*, the $\mathcal{B}$–weak topology is just the strong topology of $E' = \mathfrak{B}^r(E,\mathcal{B})$ restricted to the subset E^*. By Corollary 2.1.11 the unit ball of E^* is $\mathcal{B}$–weakly dense in E' and, of course, any bounded Cauchy net in E^* converges to an element of E'. We find Frank's [Fra99] characterization of self-dual Hilbert modules.

2.1.13 Theorem. *A pre-Hilbert $\mathcal{B}$–module E is self-dual, if and only if the unit ball of E is complete with respect to the $\mathcal{B}$–weak topology.*

2.2 Representations of matrix $*$–algebras

Let $H = \bigoplus_{i=1}^{n} H_i$ be a pre-Hilbert space. Like in Section 1.7, we may decompose $\mathfrak{B}^a(H)$ according to the subspaces H_i. (In fact, the following discussion works, to some extent, also for a pre-Hilbert module $E = \bigoplus_{i=1}^{n} E_i$.) Clearly,

$$\mathfrak{B}^a\Big(\bigoplus_{i=1}^{n} H_i\Big) = \begin{pmatrix} \mathfrak{B}^a(H_1,H_1) & \dots & \mathfrak{B}^a(H_n,H_1) \\ \vdots & & \vdots \\ \mathfrak{B}^a(H_1,H_n) & \dots & \mathfrak{B}^a(H_n,H_n) \end{pmatrix}$$

is a matrix pre–C^*–algebra.

On the other hand, if Π is a (non-degenerate) representation of a matrix $*$–algebra $\mathcal{M}$ by bounded adjointable operators on a pre-Hilbert space H, then it is easy to check that H decomposes into the subspaces $H_i = \mathsf{span}(\Pi(\mathcal{B}_i)H)$ and that $\Pi(\mathcal{B}_{ij}) \subset \mathfrak{B}^a(H_j,H_i)$. Clearly,

$$\Pi(\mathcal{M}) = \begin{pmatrix} \Pi(\mathcal{B}_{11}) & \dots & \Pi(\mathcal{B}_{1n}) \\ \vdots & & \vdots \\ \Pi(\mathcal{B}_{n1}) & \dots & \Pi(\mathcal{B}_{nn}) \end{pmatrix}$$

is a matrix pre–C^*–subalgebra of $\mathcal{B}^a\Big(\bigoplus_{i=1}^n H_i\Big)$. (If Π is only total, then at least $\overline{H}$ decomposes into $\overline{H}_i$.)

2.2.1 Definition. A *matrix von Neumann algebra* on a Hilbert space $H = \bigoplus_{i=1}^n H_i$ is a strongly (or weakly) closed matrix $*$–subalgebra $\mathcal{M}$ of $\mathcal{B}\Big(\bigoplus_{i=1}^n H_i\Big)$. Clearly, $\mathcal{M}$ is a von Neumann algebra. In particular, $\mathcal{M}$ is unital and the unit of $\mathcal{M}$ is the sum of the units p_i of the diagonal von Neumann subalgebras $\mathcal{B}_i$; cf. Example 2.1.4.

2.2.2 Proposition. *Let $\mathcal{M} = \big(\mathcal{B}_{ij}\big)$ be a matrix pre–C^*–subalgebra of the von Neumann algebra $\mathcal{B}\Big(\bigoplus_{i=1}^n H_i\Big)$. Then $\mathcal{M}$ is strongly (weakly) closed, if and only if each $\mathcal{B}_{ij}$ is strongly (weakly) closed in $\mathcal{B}^a(H_j, H_i)$.*

Proof. The mapping $\mathfrak{p}_{ij} = p_i \bullet p_j$ is strongly (weakly) continuous. Therefore, $\mathcal{B}(H_j, H_i)$ is strongly (weakly) closed in $\mathcal{B}\Big(\bigoplus_{i=1}^n H_i\Big)$. From this the statements follow. ■

2.2.3 Proposition. *Let $\mathcal{M} = \big(\mathcal{B}_{ij}\big)$ be a strongly dense matrix pre–C^*–subalgebra of a matrix von Neumann algebra $\mathcal{M}^{vN} = \big(\mathcal{B}_{ij}^{vN}\big)$. Then the unit-ball of $\mathcal{B}_{ij}$ is strongly dense in the unit-ball of $\mathcal{B}_{ij}^{vN}$.*

Proof. Let b be an element in the unit-ball of $\mathcal{B}_{ij}^{\mathrm{vN}}$. By the *Kaplansky density theorem*, we may approximate b strongly by a net $\big(B_n\big)_{n\in\mathcal{N}}$ of elements in the unit-ball of $\mathcal{M}$. Then $\big(\mathfrak{p}_{ij}(B_n)\big)_{n\in\mathcal{N}} = \big(p_i B_n p_j\big)_{n\in\mathcal{N}}$ is a net consisting of elements in the unit-ball of $\mathcal{B}_{ij}$ which converges strongly to b. ■

2.2.4 Proposition. *Let $\mathcal{M}$ be a matrix von Neumann algebra on $\bigoplus_{i=1}^n H_i$ and let b be an element of $\mathcal{B}_{ij}$. Denote $|b| = \sqrt{b^*b}$. There exists a unique partial isometry v in $\mathcal{B}_{ij}$ such that*

$$b = v\,|b| \qquad \text{and} \qquad \mathsf{ker}(v) = \mathsf{ker}(b).$$

Proof. By *polar decomposition* there exists a unique partial isometry V in $\mathcal{M}$ with the claimed properties. Obviously, V vanishes on $H_j^\perp$ and its range is contained is in H_i. This means $v := V = \mathfrak{p}_{ij}(V) \in \mathcal{B}_{ij}$. ■

2.3 Extensions of representations

In Section 1.7 we have embedded a pre-Hilbert $\mathcal{B}$–module E into $\mathsf{alg}^*(E)$ and $\mathsf{alg}^*(E, \mathcal{B})$ which, of course, are 2×2–matrix pre–C^*–subalgebras of $\mathcal{B}^a(\mathcal{B} \oplus E)$. In these matrix algebras the 22–corner $\mathcal{F}(E)$ is the smallest possible. In this section we are interested in the 2×2–matrix pre–C^*–subalgebra

$$\mathcal{M}(E) := \begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{pmatrix}$$

of $\mathcal{B}^a(\mathcal{B} \oplus E)$ and its representations. ($\mathcal{M}(E)$ is maximal in the sense that for unital $\mathcal{B}$ we have $\mathcal{M}(E) = \mathcal{B}^a(\mathcal{B} \oplus E)$.) Like any pre–$C^*$–algebra, $\mathcal{M}(E)$ admits an isometric total representation on a pre-Hilbert space. By the preceding section the representation decomposes into subspaces H_1 and H_2 such that the representation maps the 21–corner to a subset of $\mathcal{B}^a(H_1, H_2)$. Therefore, any pre-Hilbert module can be considered as a space of operators not on, but between two pre-Hilbert spaces. However, also

$$\mathcal{M}^r(E) := \begin{pmatrix} \mathcal{B} & E' \\ E & \mathcal{B}^r(E) \end{pmatrix}$$

is a matrix algebra in an obvious manner, and also the elements of $\mathcal{M}^r(E)$ act as operators on $\mathcal{B} \oplus E$. These operators are still bounded, but not necessarily adjointable. Therefore, we turn $\mathcal{M}^r(E)$ into a normed algebra by identifying it as a matrix subalgebra of $\mathcal{B}^r(\mathcal{B} \oplus E)$. (Also here we have that $\mathcal{M}^r(E) = \mathcal{B}^r(\mathcal{B} \oplus E)$, if $\mathcal{B}$ is unital.)

The goal of this section is two-fold. Firstly, we want to construct a representation of $\mathcal{M}(E)$ in a canonical fashion. More precisely, we associate with any representation π of $\mathcal{B}$ on a pre-Hilbert G a pre-Hilbert space H and a representation Π of $\mathcal{M}(E)$ on $G \oplus H$. Secondly, we extend Π to a representation of $\mathcal{M}^r(E)$ also on $G \oplus H$. Moreover, Π turns out to be contractive, if π is contractive, and to be isometric, if π is isometric. For reasons which we explain in Remark 4.1.9, we call the restriction η of Π to E the *Stinespring representation* of E associated with the representation π, and the restriction ρ of Π to $\mathcal{B}^a(E)$ the *Stinespring representation* of $\mathcal{B}^a(E)$ associated with E and the representation π.

We will recover the pre-Hilbert space H in Example 4.2.3 just as the tensor product of E and the pre-Hilbert $\mathcal{B}$–$\mathbb{C}$–module G. This and other parts of the constructions for $\mathcal{M}(E)$ can be shown by purely C^*–algebraic methods, like uniqueness of complete C^*–norms. However, dealing with $\mathcal{M}^r(E)$, we leave the C^*–framework for a moment, and it is necessary to go back directly to the level of Hilbert spaces. The basic tool is the cyclic decompositon of the representation π. Of course, this elementary technique can be applied also directly to the case $\mathcal{M}(E)$.

2.3.1 Definition. Let π be a representation of a pre–C^*–algebra $\mathcal{B}$ on a Hilbert space G. A *cyclic decomposition* of (π, G) is a pair $\big(G_0, (G_\alpha, g_\alpha)_{\alpha \in A}\big)$ consisting of a subspace G_0 such that $\pi(\mathcal{B})G_0 = \{0\}$ and a family $(G_\alpha, g_\alpha)_{\alpha \in A}$ where the G_α are subspaces of G invariant for

π such that $G = \overline{G_0 \oplus \bigoplus_{\alpha \in A} G_\alpha}$ and vectors $g_\alpha \in \overline{G}_\alpha$ such that $\pi(\mathcal{B})g_\alpha = G_\alpha$. We say G_α is a *cyclic subspace* and g_α is a *cyclic vector* for G_α.

Any representation on a Hilbert space admits a cyclic decomposition. (Set $G_0 = (\pi(\mathcal{B})G)^\perp$ and restrict to $G_0^\perp$. Then apply *Zorn's lemma* to the partially ordered set of families of mutually orthogonal closed subspaces invariant for π where each subspace is generated by $\pi(\mathcal{B})$ and one of its vectors.) If $\mathcal{B}$ is unital, then the restriction of π to $\bigoplus_{\alpha \in A} G_\alpha$ is non-degenerate, because $g_\alpha \in G_\alpha$.

Let π be a representation of $\mathcal{B}$ on a pre-Hilbert space G. Denote by $\bigl(G_0, (G_\alpha, g_\alpha)_{\alpha \in A}\bigr)$ a cyclic decomposition of the extension $\overline{\pi}$ of π to $\overline{G}$. Then any vector in G may be approximated by vectors in $G_0 \oplus \bigoplus_{\alpha \in A} G_\alpha$. We define a sesquilinear form on $E \otimes G$ by setting

$$\langle x \otimes g, x' \otimes g' \rangle = \bigl\langle g, \pi(\langle x, x' \rangle) g' \bigr\rangle. \tag{2.3.1}$$

2.3.2 Proposition. *The sesquilinear form defined by* (2.3.1) *is positive. Henceforth, $E \otimes G$ is a semi-Hilbert space.*

PROOF. We have to show that $\sum_{i,j=1}^{n} \bigl\langle g_i, \pi(\langle x_i, x_j \rangle) g_j \bigr\rangle \geq 0$ for all choices of $n \in \mathbb{N}$, $x_i \in E$ and $g_i \in G$ $(i = 1, \ldots, n)$. Each g_i is the (norm) limit of a sequence $\bigl(g_i^m\bigr)$ whoose members have the form $g_i^m = g_0^m + \sum_\alpha \pi(b_i^{m\alpha}) g_\alpha$ where the sum runs (for fixed i and m) only over finitely many $\alpha \in A$. Of course, the $g_0^m \in G_0$ do not contribute to (2.3.1). We find

$$\begin{aligned} \sum_{i,j=1}^{n} \bigl\langle g_i, \pi(\langle x_i, x_j \rangle) g_j \bigr\rangle &= \lim_{m \to \infty} \sum_{i,j=1}^{n} \sum_{\alpha, \alpha'} \bigl\langle g_\alpha, \pi(\langle x_i b_i^{m\alpha}, x_j b_j^{m\alpha'} \rangle) g_{\alpha'} \bigr\rangle \\ &= \lim_{m \to \infty} \sum_\alpha \Bigl\langle g_\alpha, \pi\Bigl(\langle \sum_{i=1}^{n} x_i b_i^{m\alpha}, \sum_{j=1}^{n} x_j b_j^{m\alpha} \rangle\Bigr) g_\alpha \Bigr\rangle \geq 0. \blacksquare \end{aligned}$$

By $H = E \odot G := E \otimes G / \mathcal{N}_{E \otimes G}$ (and in agreement with Definition 4.2.1) we denote the pre-Hilbert space associated with $E \otimes G$. We set

$$x \odot g = x \otimes g + \mathcal{N}_{E \otimes G}.$$

With each $x \in E$ we associate a mapping

$$L_x \colon g \mapsto x \odot g.$$

This mapping is bounded, because

$$\|L_x g\|^2 = \langle g, \pi(\langle x, x \rangle) g \rangle \leq \|g\|^2 \quad \|\pi(\langle x, x \rangle)\|, \tag{2.3.2}$$

and it has an adjoint

$$L_x^*\colon y \odot g \mapsto \pi(\langle x, y\rangle)g$$

(which, therefore, is well-defined, because it has an adjoint L_x). In other words, $L_x \in \mathcal{B}^a(G, H)$.

2.3.3 Definition. We define mappings $\eta\colon E \to \mathcal{B}^a(G, H)$, $\eta(x) = L_x$ and $\eta^*\colon E^* \to \mathcal{B}^a(H, G)$, $\eta^*(x^*) = L_x^*$. Of course, $H = \mathsf{span}(\eta(E)G)$. We refer to the pair (H, η) as the *Stinespring representation* of E *associated* with π.

2.3.4 Proposition. *1. If π is a contraction (an isometry), then so is η.*

2. We have $\pi(\langle x, x'\rangle) = \eta^*(x^*)\eta(x')$ and $\eta(xb) = \eta(x)\pi(b)$.

PROOF. 1. follows from (2.3.2) and 2. follows checking it with inner products, for instance,

$$\langle g, L_x^* L_{x'} g'\rangle = \langle L_x g, L_{x'} g'\rangle = \langle x \odot g, x' \odot g'\rangle = \langle g, \pi(\langle x, x'\rangle)g'\rangle. \blacksquare$$

Let a be an element of $\mathcal{B}^a(E)$. We associate with a a mapping $\underline{\rho}(a)\colon x \otimes g \mapsto ax \otimes g$ on $E \otimes G$. From $\langle x \otimes g, ax' \otimes g'\rangle = \langle a^* x \otimes g, x' \otimes g'\rangle$ we see that $\underline{\rho}(a)$ has an adjoint. By Corollary 1.4.3, it induces a mapping $\rho(a)$ on H with adjoint $\rho(a^*)$. Clearly, the mapping $\rho\colon a \mapsto \rho(a)$ defines a non-degenerate unital representation of $\mathcal{B}^a(E)$ by possibly unbounded operators in $\mathcal{L}^a(H)$. Moreover,

$$\Pi = \begin{pmatrix} \pi & \eta^* \\ \eta & \rho \end{pmatrix}$$

(acting matrix element-wise) defines a (non-degenerate, if π is) representation of $\mathcal{M}(E)$ by possibly unbounded operators in $\mathcal{L}^a(G \oplus H)$.

2.3.5 Proposition. *If π is contractive (isometric), then so are ρ and Π.*

PROOF. We need to show only the statement for ρ (in particular, that ρ maps into $\mathcal{B}^a(H)$), because for η (and, consequently, for η^*) we know it already from Proposition 2.3.4, and the norm of an element in $\mathcal{B}^a(H_j, H_i)$ is the same when considered as element of $\mathcal{B}^a(H_1 \oplus H_2)$.

If π is a contraction, then we may do the same construction, however, starting from a representation of $\overline{\mathcal{B}}$ on $\overline{G}$ and $\overline{E}$. Then $\overline{E} \odot \overline{G}$ contains H as a dense subspace and the new ρ coincides with the old one on the subspace H. In other words, ρ extends to a representation of the C^*–algebra $\mathcal{B}^a(\overline{E})$ on $\overline{E} \odot \overline{G}$. Therefore, it must be a contraction. If π is isometric, then the representation of $\mathcal{B}^a(\overline{E})$ is faithful and, therefore, isometric. $\blacksquare$

2.3.6 Definition. We refer to the pair (H, ρ) as the *Stinespring representation* of $\mathcal{B}^a(E)$ *associated* with π. If E is a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module, then by $\rho_{\mathcal{A}}$ we mean the representation $\mathcal{A} \to \mathcal{B}^a(E) \to \mathcal{B}^a(H)$ of $\mathcal{A}$ on H. We refer to the pair $(H, \rho_{\mathcal{A}})$ as the *Stinespring representation* of $\mathcal{A}$ *associated* with E and π. If we are interested in both η and ρ, then we refer also to the triple (H, η, ρ) as the *Stinespring representations*.

This is the first of our goals. As we mentioned, we used heavily C^*–arguments. Now let a be in $\mathcal{B}^r(E)$. As before, we can define the mapping $\underline{\rho}(a)x \otimes g \mapsto ax \otimes g$ on $E \otimes G$. However, because we do not necessarily have an adjoint, we cannot easily conclude that this mapping respects the kernel $\mathcal{N}_{E\otimes G}$. Appealing to Corollary 1.4.4, we must show contractivity first.

2.3.7 Lemma. *If π is contractive (isometric), then so is $\underline{\rho}\colon a \mapsto \underline{\rho}(a)$.*

Proof. We use the notations as introduced before Proposition 2.3.2. For simplicity, we assume that $\pi = \overline{\pi}$ (i.e. $G = \overline{G}$) and that $\mathcal{B}$ is unital (if necessary, by adding a unit). Then $G_0 = \{0\}$ and $g_\alpha \in G_\alpha$. Observe that $\langle x \otimes \pi(b)g_\alpha, x' \otimes \pi(b')g_{\alpha'}\rangle$ is 0, if $\alpha \neq \alpha'$. We conclude that also $E \otimes G$ decomposes into orthogonal subspaces $\underline{H}_\alpha = \mathsf{span}(E \otimes G_\alpha)$. Clearly, $\underline{H}_\alpha$ is invariant for $\underline{\rho}$ so that

$$\left\|\underline{\rho}(a)\right\| = \sup_{\alpha \in A} \left\|\underline{\rho}(a) \upharpoonright \underline{H}_\alpha\right\|.$$

Observe that $x \otimes \pi(b)g - xb \otimes g \in \mathcal{N}_{E\otimes G}$ and that, by right linearity of a, $\underline{\rho}$ respects this relation. Since

$$\underline{H}_\alpha = \mathsf{span}(E \otimes G_\alpha) = \mathsf{span}(E \otimes \pi(\mathcal{B})g_\alpha) = E \otimes g_\alpha + \mathcal{N}_{E\otimes G_\alpha},$$

we have

$$\left\|\underline{\rho}(a) \upharpoonright \underline{H}_\alpha\right\| = \sup_{x \in E,\, \|x \otimes g_\alpha\| \le 1} \|ax \otimes g_\alpha\|$$

Let $h = L_x g_\alpha \in H$. Let $L_x = v\sqrt{\pi(\langle x, x\rangle)}$ be the polar decomposition in $\mathcal{B}(G, \overline{H})$ of L_x according to Proposition 2.2.4. Set $g = \sqrt{\pi(\langle x, x\rangle)}g_\alpha \in \overline{G_\alpha}$. Then $\|h\| = \|g\|$ and $h = vg$. By Proposition 2.2.3 v may be approximated by operators L_y where y is in the unit-ball of E and, of course, g may be approximated by elements in G_α with norm not greater than $\|g\|$. We find

$$\left\|\underline{\rho}(a) \upharpoonright \underline{H}_\alpha\right\| = \sup_{\substack{h = x\otimes g,\, g \in G_\alpha \\ \|x\| \le 1,\, \|g\| \le 1}} \left\|\underline{\rho}(a)h\right\|$$

so that

$$\left\|\underline{\rho}(a)\right\| = \sup_{\substack{\alpha \in A \\ h = x\otimes g,\, g \in G_\alpha \\ \|x\| \le 1,\, \|g\| \le 1}} \left\|\underline{\rho}(a)h\right\| = \sup_{\substack{\alpha \in A,\, g \in G_\alpha \\ \|x\| \le 1,\, \|g\| \le 1}} \|ax \otimes g\| \le \sup_{\|x\| \le 1,\, \|g\| \le 1} \|ax \otimes g\| = \|a\|.$$

Now it is clear that we may divide out $\mathcal{N}_{E\otimes G}$ and obtain a contractive representation ρ^r of $\mathcal{B}^r(E)$ by operators on H. If π is isometric, then

$$\|a\| = \sup_{\|x\|\leq 1} \|ax\| = \sup_{\|x\|\leq 1} \|\rho^r(a)L_x\| = \sup_{\substack{h=L_x g\\ \|x\|\leq 1,\,\|g\|\leq 1}} \|\rho(a)^r h\| \leq \|\rho^r(a)\|$$

shows that ρ^r (and, therefore, also $\underline{\rho}$) is an isometry, too. ∎

2.3.8 Remark. Notice that it was the cyclic decomposition which enabled us in the preceding proof to reduce the supremum over linear combinations of tensors $x\otimes g$ to the supremum over elementary tensors. Hence, also the following results, which are rather corollaries of Lemma 2.3.7, may be considered as consequences of the cyclic decomposition.

Henceforth, we assume all representations of $\mathcal{B}$ to be contractions. Recall that this is automatic, if $\mathcal{B}$ is a C^*–algebra.

2.3.9 Theorem. *Let $E_1, E_2, \ldots$ be pre-Hilbert $\mathcal{B}$–modules, and let π be a contractive representation of $\mathcal{B}$. Denote $H_i = E_i \odot G$. Then the there exist unique contractive mappings*

$$\rho^r_{ji}\colon \mathcal{B}^r(E_i, E_j) \longrightarrow \mathcal{B}(H_i, H_j),$$

fulfilling $\rho^r_{ji}(a)x\odot g = ax\odot g$. If π is isometric, then so are the ρ^r_{ji}.

The correspondence is functorial in the sense that $\rho^r_{kj}(a)\rho^r_{ji}(a') = \rho^r_{ki}(aa')$. Moreover, if a is adjointable, then so is $\rho^r_{ji}(a)$ and $\rho^r_{ji}(a)^* = \rho^r_{ij}(a^*)$.

Proof. Construct the Stinespring representation of $E_i \oplus E_j$ (and $E_i\oplus E_j\oplus E_k$, respectively) associated with π, and apply Lemma 2.3.7 to $\mathcal{B}^r(E_i\oplus E_j)$ (and $\mathcal{B}^r(E_i\oplus E_j\oplus E_k)$, respectively). Then restrict to the respective matrix entries of these matrix algebras. ∎

2.3.10 Corollary. *For any $a\in\mathcal{B}^r(E_1,E_2)$ we have*

$$\langle ax, ax\rangle \leq \|a\|^2 \langle x, x\rangle.$$

Proof. This follows by considering both x and a as elements of $\mathcal{B}(G\oplus H_1\oplus H_2)$ via Stinespring representation of $E_1\oplus E_2$ for some faithful representation π of $\mathcal{B}$. ∎

2.3.11 Theorem. *Let E be a pre-Hilbert $\mathcal{B}$–module and and π a representation of $\mathcal{B}$ on a pre-Hilbert space G. Then the representation Π of $\mathcal{M}$ on $G\oplus H$ extends to a unique representation*

$$\Pi^r = \begin{pmatrix} \pi & \eta' \\ \eta & \rho^r \end{pmatrix}$$

of $\mathcal{M}^r(E)$ on $G\oplus H$.

PROOF. We have only to extend η^*. This follows as in the proof of Theorem 2.3.9 in the case $E_1 = \mathcal{B}$ and $E_2 = E$, together with the observation that $\mathcal{B} \odot G \subset G$ ($= G$, if π is non-degenerate) via the isometry $b \odot g \mapsto \pi(b)g$. ■

2.3.12 Observation. If π is total, then η' is the unique mapping $E' \to \mathcal{B}^a(H,G)$, (extending η^*) and fulfilling

$$\pi(\Phi x) = \eta'(\Phi)L_x$$

for all $\Phi \in E'$ and $x \in E$. In particular,

$$(\Phi x)^*(\Phi x) \leq \|\Phi\|^2 \langle x, x\rangle.$$

In other words, if a pre-Hilbert module is represented as a submodule of $\mathcal{B}^a(G,H)$, then E' may be identified as a subset of $\mathcal{B}^a(H,G)$. This observation is crucial in the next chapter, when we show that von Neumann modules are self-dual.

• • •

Representations of Hilbert modules. So far, we showed how to extend a representation π of the 11–corner $\mathcal{B}$ to a representation Π of $\mathcal{M}(E)$ or, more generally, of $\mathsf{alg}^*(\mathcal{B}, E, \mathcal{A})$, if E is a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module. By duality, most of the results apply also to an extension of a representation ρ of $\mathcal{A}$. The question remains, what could be a representation η of E (defined intrinsically, without reference to $\mathcal{B}$ or $\mathcal{A}$), and how does it extend to a representation of $\mathsf{alg}^*(\mathcal{B}, E, \mathcal{A})$? Without proofs we recall the results from Skeide [Ske00b].

A *representation* of a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module E (if $\mathcal{A}$ is not specified, then we put $\mathcal{A} = \mathcal{B}^a(E)$) from a pre-Hilbert space G to a pre-Hilbert space H is a linear mapping $\eta\colon E \to \mathcal{B}^a(G,H)$, fulfilling

$$\eta(xy^*z) \;=\; \eta(x)\eta(y)^*\eta(x).$$

It turns out that a representation η extends (as a representation on $G \oplus H$) to the ideal in $\mathcal{M}(E)$ generated by E. (We just send $\langle x, y\rangle$ to $\eta(x)^*\eta(y)$ and xy^* to $\eta(x)\eta(y)^*$, and show that the linear extensions of these mappings are well-definded.) A result from [Ske00b] asserts the representation extends further to $\overline{\mathsf{alg}^*}(\mathcal{B}, E, \mathcal{A})$, if and only if η is *completely bounded.*

Complete boundedness is a notion in the theory of operator spaces and, in fact, Blecher [Ble97] showed that Hilbert modules form a particularly well-behaved subclass of operator spaces. Operator spaces are Banach spaces not characterized by a single norm, but by a whole family of norms. We do not give more details. We only mention that both tensor products of Hilbert modules, which we discuss in Chapter 4, are closely related to operator spaces. *The* tensor product (often, refered to as interior tensor product) is the *Haagerup tensor product* of operator spaces, and the exterior tensor product of E with M_n provides us with the characterizing family of norms, turning E into an operator space.

Chapter 3

Von Neumann modules and centered modules

Von Neumann algebras are strongly closed $*$–subalgebras of the algebra $\mathcal{B}(G)$ of all bounded operators on a Hilbert space G. In the preceding chapter we have learned that any pre-Hilbert module can be represented as a submodule of the space $\mathcal{B}(G,H)$ of all bounded operators *between* two Hilbert spaces G and H. It is natural to introduce *von Neumann modules* as strongly closed submodules of $\mathcal{B}(G,H)$.

By Propositions 2.2.2 and 2.2.4, fundamental properties of von Neumann algebras, like the *Kaplanski density theorem* or *polar decompostion*, turn directly over to von Neumann modules. In von Neumann modules we have a substitute for orthonormal bases (Theorem 3.2.5) and von Neumann modules are self-dual (Theorem 3.2.11). The algebra of bounded right module homomorphisms on a von Neumann module is itself a von Neumann algebra (Proposition 3.1.3).

Particularly simple are von Neumann modules over $\mathcal{B}(G)$. In this case the module is necessarily the whole space $\mathcal{B}(G,H)$; see Example 3.1.2. If we have, additionally, a (normal) left action of $\mathcal{B}(G)$, then H can be written as $G \bar{\otimes} \mathfrak{H}$ and the module is $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$ with natural left action; see Example 3.3.4. This module is generated by all elements which *commute* with $\mathcal{B}(G)$. This observation will help us to understand why for the study of CP-semigroups on $\mathcal{B}(G)$ it sufficient to consider Arveson's tensor product systems of Hilbert spaces [Arv89a], instead of tensor product systems of $\mathcal{B}(G)$–modules (Example 11.1.4). We are lead to the notion of *centered* module. In Example 4.2.13 we see that centered modules are particularly well-behaved under tensor product.

A W^*–algebra is a C^*–algebra with a *pre-dual* Banach space; see Sakai [Sak71]. Like von Neumann algebras, which may be considered as *concrete* W^*–algebras, also von Neumann modules may be considered as concrete W^*–modules, in the sense that they have a pre-dual, and that any W^*–module has a representation as von Neumann module. The abstract approach is used already in the first paper by Paschke [Pas73] and exploited systematically, for instance, in [Sch96]. The concrete operator approach from Skeide [Ske00b], as we use it here, seems to be slightly more direct and elementary. It suits also better to see the relation to existing works on dilation theory in Part III.

3.1 Basic properties

In this chapter $\mathcal{B} \subset \mathcal{B}(G)$ is always a von Neumann algebra acting non-degenerately on a Hilbert space G, unless stated otherwise explicitly. For a Hilbert module E over $\mathcal{B}$ we use the notations of Section 2.3 with the exception that we denote by H the Hilbert space $E \mathbin{\bar{\odot}} G$. We always identify $x \in E$ with $L_x \in \mathcal{B}(G, H)$ and we always identify $a \in \mathcal{B}^r(E)$ with the element in $\rho^r(a) \in \mathcal{B}(H)$.

3.1.1 Definition. A *von Neumann $\mathcal{B}$–module* is a pre-Hilbert $\mathcal{B}$–module E for which $\mathcal{M}(E)$ is a matrix von Neumann algebra on $G \oplus H$. The *strong topology* on E is the relative strong topology of $\mathcal{M}(E)$.

3.1.2 Example. Let $\mathcal{B} = \mathcal{B}(G)$. Then E is necessarily all of $\mathcal{B}(G, H)$. Indeed, $\mathcal{B}(G)$ contains all rank-one operators. Since E is a right $\mathcal{B}$–module, this implies that $\mathcal{F}(G, E \odot G) \subset E$ which, clearly, is a strongly dense subset of $\mathcal{B}(G, H)$. Moreover, as $\mathcal{B}^r(E) \subset \mathcal{B}(H)$ and, on the other hand, each $a \in \mathcal{B}(H)$ gives rise to an element in $\mathcal{B}^a(E)$, we conclude that $\mathcal{B}^r(E) = \mathcal{B}^a(E) = \mathcal{B}(H)$.

3.1.3 Proposition. *E is a von Neumann module, if and only if E is strongly closed in $\mathcal{B}(G, H)$. In particular, if E is strongly closed, then $\mathcal{B}^a(E)$ is a von Neumann algebra.*

Proof. We need only to show one direction. So assume that E is strongly closed in $\mathcal{B}(G, H)$. By Proposition 2.2.2 we see that closing $\mathcal{M}(E)$, actually, means closing $\mathcal{B}^a(E)$ in $\mathcal{B}(H)$, because all other 'matrix entries' already are strongly closed. On the other hand, the strong closure $\overline{\mathcal{M}(E)}^s$ of $\mathcal{M}(E)$ is a matrix $*$–algebra. Therefore, an element a in the von Naumann algebra $\overline{\mathcal{B}^a(E)}^s$ on H acts as a right linear mapping on E, and a^* is its adjoint. We conclude that $a \in \mathcal{B}^a(E)$ and, henceforth, $\mathcal{B}^a(E)$ is strongly closed. ■

Notice that the strong topology is the locally convex Hausdorff topology on E which is generated by the family $x \mapsto \sqrt{\langle g, \langle x, x \rangle g \rangle}$ $(g \in G)$ of seminorms. However, the knowledge of this more intrinsic description does not help us to decide whether E is a von Neumann module. For that it is necessary to identify the space in which E should be closed. Theorem 3.2.17 provides us with the intrinsic criterion of self-duality, a purely algebraic property.

3.1.4 Example. Let $\mathfrak{H}$ be a Hilbert space, and let $\mathcal{B}$ be a $*$–subalgebra of M_n acting non-degenerately on $\mathbb{C}^n$. Then the Stinespring representation of the pre-Hilbert $\mathcal{B}$–module $\mathfrak{H}_{\mathcal{B}} = \mathcal{B} \otimes \mathfrak{H}$ (as in Example 1.5.10) is given by $\eta(b \otimes f)\colon v \mapsto bv \otimes f$ $(v \in \mathbb{C}^n)$. Clearly, $\eta(\mathfrak{H}_{\mathcal{B}})$ is a strongly closed subset of $\mathcal{B}(\mathbb{C}^n, \mathbb{C}^n \otimes \mathfrak{H})$. Therefore, $\mathfrak{H}_{\mathcal{B}}$ is a von Neumann $\mathcal{B}$–module. We easily see that generally a pre-Hilbert module over a finite-dimensional C^*–algebra is a von Neumann module, if and only if it is a Hilbert module.

3.1.5 Proposition. *The $\mathcal{B}$–functionals are strongly continuous mappings $E \to \mathcal{B}$. For all $x \in E$ the mapping $\mathcal{B}^a(E) \to E, a \mapsto ax$ is strongly continuous. For all $a \in \mathcal{B}^r(E)$ the mapping $E \to E, x \mapsto ax$ is strongly continuous.*

Proof. All assertions follow from the fact that multiplication in $\mathcal{B}(G \oplus H)$ is separately strongly continuous. ■

3.1.6 Proposition. *A bounded net $\left(a_\alpha\right)_{\alpha \in A}$ of elements in $\mathcal{B}^r(E)$ converges srongly in $\mathcal{B}(H)$, if and only if $a_\alpha x g$ is a Cauchy net in H for all $x \in E$ and $g \in G$, or equivalently, if $a_\alpha x$ is a strong Cauchy net in E for all $x \in E$.*

Proof. For a bounded net it is sufficient to check strong convergence on the dense subset $\mathsf{span}(EG)$ of H. ■

3.1.7 Theorem. *Let E be a von Neumann $\mathcal{B}$–module and a a self-adjoint element of $\mathcal{B}^a(E)$. There exists a projection-valued function $E_\lambda\colon \mathbb{R} \to \mathcal{B}^a(E)$ fulfilling $\lambda \leq \mu \Rightarrow E_\lambda \leq E_\mu$, $E_{\lambda+0} = E_\lambda$ (strongly), $E_{-\|a\|-0} = 0$, $E_{\|a\|} = \mathbf{1}$ and*

$$\int \lambda \, dE_\lambda = a.$$

The integral is the norm limit of Riemann sums. E_λ is called the spectral resolution of identity *associated with a.*

Proof. This is a direct translation of the corresponding statement for an operator in $\mathcal{B}(H)$; see e.g. [RSN82]. We only have to recognize that E_λ is a strong limit of polynomials in a. This guarantees that $E_\lambda \in \mathcal{B}(H)$ may be interpreted as an element of $\mathcal{B}^a(E)$. ■

3.1.8 Corollary. *Let E be a von Neumann $\mathcal{B}$–module and a a self-adjoint element of $\mathcal{B}^a(E)$ with spectral resolution of identity E_λ. Let $\Omega\colon \mathcal{B}^a(E) \to \mathcal{B}$ be a normal bounded mapping. Then with the operator-valued measure $\mu(d\lambda) = \Omega(E_{\lambda+d\lambda} - E_\lambda)$ the moments $\Omega(a^n)$ of a may be be computed by*

$$\Omega(a^n) = \int \lambda^n \mu(d\lambda).$$

3.2 Quasi orthonormal systems and self-duality

3.2.1 Definition. A *quasi orthonormal system* is a family $\bigl(e_\beta, p_\beta\bigr)_{\beta\in B}$ of pairs consisting of an element $e_\beta \in E$ and a projection $p_\beta \in \mathcal{B}$ such that

$$\langle e_\beta, e_{\beta'}\rangle = p_\beta \delta_{\beta\beta'}.$$

We say the family is *orthonormal*, if $p_\beta = \mathbf{1}$ for all $\beta \in B$.

3.2.2 Proposition. *Let $\bigl(e_\beta, p_\beta\bigr)_{\beta\in B}$ be a quasi orthonormal system. Then the increasing net*

$$\Bigl(\sum_{\beta\in B'} e_\beta e_\beta^*\Bigr)_{B'\subset B, \# B'<\infty}$$

of projections converges strongly to a projection p_B in $\mathcal{B}^a(E)$. We call p_B the projection associated with $\bigl(e_\beta, p_\beta\bigr)_{\beta\in B}$.

Proof. Clear, since $\mathcal{B}^a(E)$ is a von Neumann algebra. ■

3.2.3 Definition. A quasi orthonormal system $\bigl(e_\beta, p_\beta\bigr)_{\beta\in B}$ is called *complete*, if $p_B = \mathbf{1}$.

3.2.4 Example. Notice that the cardinality of a complete quasi orthonormal system is not unique. For instance, for the von Neumann module $\mathcal{B}$ we may choose $(\mathbf{1}, \mathbf{1})$ as well as $\bigl(p_\beta, p_\beta\bigr)_{\beta\in B}$ for an arbitrary decomposition of $\mathbf{1}$ into orthogonal projections p_β. This example also shows that the number of coefficients with respect to a quasi orthonormal system (for overcountable B) need not be countable.

As another example consider $E = \mathcal{B}(G, G)$ where G is an infinite-dimensional separable Hilbert space. Because $G \cong G^n$ for all $n \in \mathbb{N}$ we find $E \cong E^n$. Clearly, $\bigl(\mathbf{1}, \mathbf{1}\bigr)$ is a complete orthonormal system for E and $\bigl(\mathbf{1}_i, \mathbf{1}\bigr)_{i=1,\dots,n}$, where $\mathbf{1}_i$ denotes the $\mathbf{1}$ in the i–th component E in E^n, is a complete orthonormal system for E^n. This shows that even the cardinality of a complete orthonormal system is not unique. In particular, as E has complete orthonormal systems of any order $n \in \mathbb{N}$, it is, in general, not possible to guaranty that an orthonormal system of E_n of order $m < n$ can be extended to a complete orthonormal system of order n.

3.2.5 Theorem. *Any von Neumann $\mathcal{B}$–module E admits a complete quasi orthonormal system.*

Proof. An application of *Zorn's lemma* tells us that the partially oredered set consisting of all quasi orthonormal systems has a maximal element. Let $\bigl(e_\beta, p_\beta\bigr)_{\beta\in B}$ be a maximal quasi orthonormal system. If $\big(e_\beta, p_\beta\big)_{\beta\in B}$ is not complete, then $E_B^\perp = (\mathbf{1} - p_B)E$ is non-trivial. We choose $x \in E_B^\perp$ different from 0. Since $p_B e_\beta = e_\beta$ and $(\mathbf{1} - p_B)x = x$, we have $\langle x, e_\beta\rangle = (\langle e_\beta, x\rangle)^* = 0$ for all $\beta \in B$. By Proposition 2.2.4 $x = v\,|x|$ where $v \in E$ is a partial isometry. Then also $\big(e_\beta, p_\beta\big)_{\beta\in B}$ enlarged by $(v, |v|)$ is a quasi orthonormal system. This contradicts maximality of $\big(e_\beta, p_\beta\big)_{\beta\in B}$. ■

3.2.6 Corollary. *Let $\big(e_\beta, p_\beta\big)_{\beta\in B}$ be a complete quasi orthonormal system for E. Let $x \in E$. Then $b_\beta = \langle e_\beta, x\rangle$ are unique elements in $p_\beta\mathcal{B}$ such that*

$$x = \sum_{\beta\in B} e_\beta b_\beta \quad \textit{and} \quad \sum_{\beta\in B} b_\beta^* b_\beta = \sum_{\beta\in B} b_\beta^* p_\beta b_\beta = \langle x, x\rangle.$$

Conversely, if $b_\beta \in \mathcal{B}$ and $M > 0$ such that

$$\sum_{\beta\in B'} b_\beta^* p_\beta b_\beta < M$$

for all finite subsets B' of B, then

$$\sum_{\beta\in B} e_\beta b_\beta$$

exists and is an element of E.

Proof. This is an immediate consequence of Proposition 3.1.5 and of the order completeness of the von Neumann algebra $\mathcal{B}$. ■

3.2.7 Corollary. *The unit-ball of $\mathcal{F}(E)$ is strongly dense in the unit-ball of $\mathcal{B}^a(E)$.*

3.2.8 Definition. Let $\big(E_\beta\big)_{\beta\in B}$ be a family of von Neumann modules over a von Neumann algebra $\mathcal{B} \subset \mathcal{B}(G)$ and denote $E = \bigoplus_{\beta\in B} E_\beta$. Then setting $H_\beta = E_\beta \,\bar{\odot}\, G$ and $H = E \,\bar{\odot}\, G$, we have $H = \overline{\bigoplus_{\beta\in B}} H_\beta$ in an obvious manner. By the *von Neumann module direct sum* $\overline{E}^s = \overline{\bigoplus_{\beta\in B}}^s E_\beta$ we mean the strong closure of E in $\mathcal{B}(G, H)$.

Of course, by Proposition 3.1.5 the canonical projections $E \to E_\beta$ extend to projections $\mathfrak{p}_\beta \in \mathcal{B}^a(\overline{E}^s)$ onto E_β and $\sum_{\beta\in B} \mathfrak{p}_\beta = \mathbf{1}$ in the strong topology. By a proof very similar to that of Corollary 3.2.6 we obtain a characterization of elements of $\overline{E}^s$ by their components in E_β.

3.2.9 Proposition. *Let $x \in \overline{E}^s$. Then $x_\beta = \mathfrak{p}_\beta x$ are unique elements in E_β such that*

$$x = \sum_{\beta\in B} x_\beta \quad \textit{and} \quad \sum_{\beta\in B} \langle x_\beta, x_\beta\rangle = \langle x, x\rangle.$$

Conversely, if $x_\beta \in E_\beta$ *and* $M > 0$ *such that*

$$\sum_{\beta \in B'} \langle x_\beta, x_\beta \rangle < M$$

for all finite subsets B' *of* B*, then*

$$\sum_{\beta \in B} x_\beta$$

exists and is an element of $\overline{E}^s$.

3.2.10 Theorem. *Let* E *be a von Neumann* $\mathcal{B}$*–module with a complete quasi orthonormal system* $\big(e_\beta, p_\beta\big)_{\beta \in B}$*. Denote by* H_B *a Hilbert space with an orthonormal basis* $\big(e'_\beta\big)_{\beta \in B}$*. For* $h \in H_B$ *and* $b \in \mathcal{B}$ *identify* $b \otimes h$ *with the mapping* $g \mapsto bg \otimes h$ *in* $\mathcal{B}(G, G \otimes H_B)$*. Then* E *is a complemented submodule of the strong closure of* $\mathcal{B} \otimes H_B$.

Proof. E is the closure of the direct sum over all $p_\beta \mathcal{B}$. (Notice that this is the von Neumann module direct sum and not the von Neumann algebra direct sum. The former is a subset of $\mathcal{B}\Big(G, \overline{\bigoplus_{\beta \in B}} G\Big)$, whereas the latter is a subset of $\mathcal{B}\Big(\overline{\bigoplus_{\beta \in B}} G, \overline{\bigoplus_{\beta \in B}} G\Big)$.) We consider the right ideal $p_\beta \mathcal{B}$ as a subset of $\mathcal{B}$ so that $\overline{\bigoplus_{\beta \in B}}^s p_\beta \mathcal{B}$ is contained in $\overline{\bigoplus_{\beta \in B}}^s \mathcal{B}$ and $\overline{\bigoplus_{\beta \in B}}^s (\mathbf{1} - p_\beta)\mathcal{B}$ is its complement. We establish the claimed isomorphism by sending the β–th summand to $e'_\beta \otimes \mathcal{B}$. ■

3.2.11 Theorem. *Any von Neumann* $\mathcal{B}$*–module* E *is self-dual.*

Proof. Recall from Corollary 2.3.12 that E' may be identified as a subspace of $\mathcal{B}(H, G)$ containing E^*. The matrix element $\mathcal{B}_{21}$ of the von Neumann matrix subalgebra of $\mathcal{B}(G \oplus H)$ generated by E' is a von Neumann module (not necessarily over $\mathcal{B}$) containing $E'^* \supset E$. Clearly, a complete quasi orthonormal system $\big(e_\beta, p_\beta\big)_{\beta \in B}$ for E is a quasi orthonormal system also for $\mathcal{B}_{21}$. This implies

$$\sum_{\beta \in B} (\Phi e_\beta)(\Phi e_\beta)^* < \infty$$

for all $\Phi \in E'$. In particular, if we set $b_\beta = (\Phi e_\beta)^*$, then $x_\Phi = \sum_{\beta \in B} e_\beta b_\beta$ is an element of E.

Taking into account Proposition 3.1.5, we find

$$\langle x_\Phi, x \rangle = \Big(\sum_{\beta \in B} \langle x, e_\beta \rangle b_\beta\Big)^* = \sum_{\beta \in B} \Phi e_\beta \langle e_\beta, x \rangle = \Phi x$$

for all $x \in E$. (The equation is to be understood weakly, because the $*$ is only weakly continuous.) Henceforth, $\Phi = x_\Phi^* \in E^*$. ■

3.2.12 Corollary. *A subset S of a von Neumann module E is strongly total, if and only if $\langle s, x\rangle = 0$ for all $s \in S$ implies $x = 0$.*

PROOF. The von Neumann submodule of E generated by S is self-dual and, therefore, complementary, by Proposition 1.5.9. So either S is strongly total so that, of course, $\langle s, x\rangle = 0$ for all $s \in S$ implies $x = 0$ by (1.1.1c), or S is not strongly total from which we conclude that there exists a non- zero element x in the complement of this submodule for which $\langle s, x\rangle = 0$ for all $s \in S$. ■

Besides the general results on self-dual Hilbert modules like Propositions 1.4.7, 1.5.9, the following *Hahn-Banach type* extensions are of some interest.

3.2.13 Proposition. *Let $\mathcal{B}$ be a von Neumann algebra of operators on a Hilbert space G. Let F be a strongly dense $\mathcal{B}$–submodule of a von Neumann $\mathcal{B}$–module E. Then any $\mathcal{B}$–functional Φ on F extends to a (unique) $\mathcal{B}$–functional $\bar{\Phi}$ on E. Moreover, $\|\bar{\Phi}\| = \|\Phi\|$.*

PROOF. The closed subspace of H generated by FG is H. (Otherwise, F was not strongly dense in E.) By Corollary 2.3.12 Φ may be identified with an element of $\mathcal{B}(H, G)$. Of course, Φ acts stronlgy continuously on F (see proof of Proposition 3.1.5) so that also the range of the strong extension $\bar{\Phi}$ of Φ to E is $\mathcal{B}$. Clearly, $\|\bar{\Phi}\| = \|\Phi\|$. ■

3.2.14 Theorem. *Any $\mathcal{B}$–functional Φ on a $\mathcal{B}$–submodule F of a von Neumann $\mathcal{B}$–module E may be extended norm preservingly and uniquely to a $\mathcal{B}$–functional on E vanishing on $F^\perp$.*

PROOF. The strong closure $\overline{F}^s$ of F is a von Neumann module, and by Proposition 3.2.13 Φ extends uniquely to a $\mathcal{B}$–functional $\bar{\Phi}$ on $\overline{F}^s$. Since $\overline{F}^s$ is self-dual, hence, complementary, there exists a projection in $\mathcal{B}^a(E)$ onto $\overline{F}^s$. The $\mathcal{B}$–functional $\bar{\Phi}p$ has all the claimed properties and is, of course, determined uniquely. ■

3.2.15 Corollary. *Let E_1, E_2 be von Neumann $\mathcal{B}$–modules and F a $\mathcal{B}$–submodule of E_1. An arbitrary mapping a in $\mathcal{B}^r(F, E_2)$ extends uniquely to a mapping in $\mathcal{B}^a(E_1, E_2)$ having the same norm and vanishing on $F^\perp$.*

We close with some results on the relation to W^*–modules.

3.2.16 Proposition. *A von Neumann module has a pre-dual.*

PROOF. Like for von Neumann algebras. The predual of $\mathcal{B}^a(G \oplus H)$ is the space $L^1(G \oplus H)$ of *trace class operators* on $G \oplus H$. The pre-dual of a strongly closed subspace E is the Banach space $L^1(G \oplus H)/\mathcal{N}$ where $\mathcal{N}$ is the Banach subspace $\bigl\{f \in L^1(G \oplus H) \colon f(E) = \{0\}\bigr\}$ of $L^1(G \oplus H)$. ■

3.2.17 Theorem. *Let E be a pre-Hilbert module over a W^*–algebra $\mathcal{B}$. For any normal representation π of $\mathcal{B}$ on G denote by η_π the Stinespring representation associated with π. Then the following conditions are equivalent.*

1. *$\eta_\pi(E)$ is a von Neumann $\pi(\mathcal{B})$–module for some faithful normal representation π of $\mathcal{B}$.*

2. *$\eta_\pi(E)$ is a von Neumann $\pi(\mathcal{B})$–module for every faithful normal representation π of $\mathcal{B}$.*

3. *E is self-dual.*

Proof. Clearly, $2 \Rightarrow 1$.

$1 \Rightarrow 3$. Suppose $\eta_\pi(E)$ is a von Neumann $\pi(\mathcal{B})$–module and, therefore, self-dual. Let Φ be a $\mathcal{B}$–functional on E. Then $\varphi = \pi \circ \Phi \circ \eta_\pi^{-1}$ is a $\pi(\mathcal{B})$–functional on $\eta_\pi(E)$. Since $\eta_\pi(E)$ is self-dual, we find a unique $x \in E$, such that $\varphi = \eta_\pi^*(x^*)$. Clearly, $\Phi = \pi^{-1} \circ \varphi \circ \eta_\pi = x^*$ so that also E is self-dual.

$3 \Rightarrow 2$. We conclude indirectly. If $\eta_\pi(E)$ is not a von Neumann $\pi(\mathcal{B})$–module, then it is not strongly closed. Therefore, there exists an element φ^* in the strong closure of $\eta_\pi(E)$ which is not an element of $\eta_\pi(E)$. Clearly, $\varphi = (\varphi^*)^*$ is an element of $\eta_\pi(E)'$ which gives rise to a $\mathcal{B}$–functional $\Phi = \pi^{-1} \circ \varphi \circ \eta_\pi$ on E. If Φ is in E^*, then $\varphi = \pi \circ \Phi \circ \eta_\pi^{-1}$ is in $\eta_\pi(E)^*$ which contradicts our assumption. Consequently, E is not self-dual. ■

3.2.18 Corollary. *Let E be Hilbert module over a W^*–algebra $\mathcal{B}$. Then E' is a self-dual Hilbert $\mathcal{B}$–module.*

The W^*–version of Theorem 3.2.5, Proposition 3.2.16, and Corollary 3.2.18 are already due to Paschke [Pas73]. We remark, however, that Paschke proceeds somehow conversely. First, he shows that E' is a self-dual Hilbert $\mathcal{B}$–module, and then that it has a complete quasi orthonormal system.

3.3 Two-sided von Neumann modules

If E is a von Neumann $\mathcal{B}$–module with a left action of a $*$–algebra $\mathcal{A}$, then we know from Proposition 3.1.5 that the action of any operator $a \in \mathcal{A}$ on E is strongly continuous. In particular, there is no problem to extend the action of a from a strongly dense subset of E

to all of E; cf. Proposition 3.2.13. This does, however, tell us nothing about compatibility with an existing topology on $\mathcal{A}$.

If $\mathcal{A}$ is a von Neumann algebra, then we are interested in whether the inner products $\langle x, ax'\rangle$ are *compatible* with the normal topologies of $\mathcal{A}$ and $\mathcal{B}$. Choosing the normal topology instead of the strong topology, has the advantage that, in contrast with the latter, in the former continuity can be checked on bounded subsets. This suits much better to preserve normality in constructions like tensor product. See Appendix A.9 for basics about normal mappings.

3.3.1 Definition. Let $\mathcal{A}, \mathcal{B}$ be von Neumann algebras, and let E be both a von Neumann $\mathcal{B}$–module and a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module. We say E is a ***von Neumann $\mathcal{A}$–$\mathcal{B}$–module*** (or a ***two-sided*** von Neumann module), if for all $x \in E$

$$a \longmapsto \langle x, ax\rangle \tag{3.3.1}$$

is a normal mapping $\mathcal{A} \to \mathcal{B}$.

3.3.2 Lemma. *Let $\mathcal{B}$ be a von Neumann algebra on a Hilbert space G, let E be a von Neumann $\mathcal{B}$–module, and let $\mathcal{A}$ be a von Neumann algebra with a non-degenerate representation on E. Then the following conditions are equivalent.*

1. *E is a von Neumann $\mathcal{A}$–$\mathcal{B}$–module.*

2. *All mappings $a \mapsto \langle x, ay\rangle$ are σ–weakly continuous.*

3. *The canonical represenration ρ of $\mathcal{A}$ on $H = E \bar{\odot} G$ is normal.*

PROOF. $1 \Rightarrow 2$. Each mapping $a \mapsto \langle x, ay\rangle$ can be written as a linear combination of not more than four mappings of the form (3.3.1), which are normal by the assumption in 1, and each σ–weak functional on $\mathcal{B}$ (i.e. each element in the pre-dual $\mathcal{B}_*$ of $\mathcal{B}$) can be written as a linear combination of not more than four normal states. Therefore, $a \mapsto \langle x, ay\rangle$ is σ–weakly continuous.

$2 \Rightarrow 3$. For each $h \in E \odot G$ the positive functional $a \mapsto \langle h, \rho(a)h\rangle$ is σ–weak by the assumption in 2 and, therefore, normal (see Appendix A.9). It follows that also ρ is normal.

$3 \Rightarrow 1$. Each functional $a \mapsto \langle h, \rho(a)h\rangle$ is normal by the assumption in 3. In particular, the functionals $a \mapsto \varphi_g(\langle x, ax\rangle)$ where φ_g ranges over normal functionals $b \mapsto \langle g, bg\rangle$ on $\mathcal{B}$ are normal. It follows (by Appendix A.9) that the mapping (3.3.1) is normal. ■

3.3.3 Corollary. *Let E be a von Neumann $\mathcal{A}$–$\mathcal{B}$–module, and let π be an arbitrary (not necessarily faithful) normal representation of $\mathcal{B}$ on a Hilbert space G. Then the canonical representation ρ of $\mathcal{A}$ on $E \bar{\odot} G$ is normal.*

PROOF. Precisely as $2 \Rightarrow 3$ in the preceding proof. ■

In the remainder of these notes we construct a couple of two-sided Hilbert modules. The preceding properties contain everything related to the normal or σ–weak topology which we need to know, in order to show that these constructions extend to von Neumann modules in a normality preserving way.

3.3.4 Example. We know from Example 3.1.2 that von Neumann $\mathcal{B}(G)$–modules are necessarily of the form $\mathcal{B}(G, H)$. Now suppose that E is a two-sided von Neumann $\mathcal{B}(G)$–module, i.e. H carries a normal non-degenerate representation ρ of $\mathcal{B}(G)$. Therefore, there exists another Hilbert space $\mathfrak{H}$ such that ρ is unitarily equivalent to the representation $\mathsf{id} \otimes \mathbf{1}$ on $G \bar{\otimes} \mathfrak{H}$. In other words, E is isomorphic as a two-sided von Neumann module to $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$ equipped with its natural structure.

Clearly, $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$ is the strong closure of its two-sided $\mathcal{B}(G)$–submodule $\mathcal{B}(G) \otimes \mathfrak{H}$, where an element $b \otimes h \in \mathcal{B}(G) \otimes \mathfrak{H}$ is identified with the mapping $g \mapsto bg \otimes h$ in $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$. This module is generated by its subset $\mathbf{1} \otimes \mathfrak{H}$. More precisely, if $\left(e_\beta\right)_{\beta \in B}$ is an orthonormal basis for $\mathfrak{H}$, then $\left(\mathbf{1} \otimes e_\beta, \mathbf{1}\right)_{\beta \in B}$ is a complete (quasi) orthonormal system for E and, therefore, generating.

Of course, all $x \in \mathbf{1} \otimes \mathfrak{H}$ *commute* with all elements $b \in \mathcal{B}(G)$, i.e. $bx = xb$. But also the converse is true. To see this, we follow an observation by Arveson. In accordance with Definition 3.4.1, we denote by $C_{\mathcal{B}(G)}(E)$ the set of all elements in E which commute with all $b \in \mathcal{B}(G)$. Let x, y be two elements in $C_{\mathcal{B}(G)}(E)$. Then their inner product $\langle x, y \rangle$ is an element of the commutant of $\mathcal{B}(G)$ in $\mathcal{B}(G)$ (cf. Proposition 3.4.2) and, therefore, a scalar multiple $c(x, y)\mathbf{1}$ of the identity. Let $x \in C_{\mathcal{B}(G)}(E)$. Then

$$x \;=\; \sum_{\beta \in B} (\mathbf{1} \otimes e_\beta)\langle \mathbf{1} \otimes e_\beta, x \rangle \;=\; \sum_{\beta \in B} (\mathbf{1} \otimes e_\beta) c(\mathbf{1} \otimes e_\beta, x) \;=\; \mathbf{1} \otimes \sum_{\beta \in B} e_\beta c(\mathbf{1} \otimes e_\beta, x)$$

is an element of $\mathbf{1} \otimes \mathfrak{H}$. It follows that $C_{\mathcal{B}(G)}(E)$ with inner product $c(\bullet, \bullet)$ is a Hilbert space and that $\mathfrak{H}$ and $C_{\mathcal{B}(G)}(E)$ are canonically isomorphic via $h \mapsto \mathbf{1} \otimes h$.

The fact that E is (even freely) generated by its Hilbert subspace $C_{\mathcal{B}(G)}(E)$ together with the following result is responsible for the possibilty to study CP-semigroups on $\mathcal{B}(G)$ with the help of tensor product systems of Hilbert spaces (the centers of the GNS-modules of each member of the semigroup; see Example 4.2.13). It also explains why a calculus on a symmetric or full Fock space tensorized with an initial space G is successful in dilation theory of such semigroups; cf. the discussion in Section 17.1. For instance, $G \bar{\otimes} \Gamma(L^2(\mathbb{R}_+))$ is nothing but the space H of the Stinespring representation for the *symmetric* Fock module $\overline{\mathcal{B}(G) \bar{\otimes} \Gamma(L^2(\mathbb{R}_+))}^s$.

3.3.5 Proposition. *Let $E_i = \mathcal{B}(G, G \bar{\otimes} \mathfrak{H}_i)$ $(i = 1, 2)$ be arbitrary two-sided von Neumann $\mathcal{B}(G)$–modules. Then the mapping*

$$a \longmapsto a \upharpoonright \mathfrak{H}_1$$

is a linear isometric bijection from $\mathcal{B}^{a,bil}(E_1, E_2)$ to $\mathcal{B}(\mathfrak{H}_1, \mathfrak{H}_2)$. The mapping is functorial and respects adjoints, isometry and, hence, unitarity. For $E_1 = E_2$ this mapping is an isomomorphism of von Neumann algebras.

Proof. If a is two-sided, then it sends elements in $C_{\mathcal{B}(G)}(E_1)$ to elements in $C_{\mathcal{B}(G)}(E_2)$. Therefore, the restriction to $\mathfrak{H}_1 = C_{\mathcal{B}(G)}(E_1)$, indeed, defines a mapping in $\mathcal{B}(\mathfrak{H}_1, \mathfrak{H}_2)$. Conversely, an operator on $\mathcal{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ extends via ampliation to an operator in $\mathcal{B}(G \bar{\otimes} \mathfrak{H}_1, G \bar{\otimes} \mathfrak{H}_2) = \mathcal{B}^a(E_1, E_2)$ which, clearly, is two-sided. Of course, these indentifications are inverses of each other. The remaining statements are obvious. ■

3.3.6 Remark. Interpreted in terms of $\mathcal{B}^a(E) = \mathcal{B}(G \bar{\otimes} \mathfrak{H})$, the preceding proposition is nothing but the well-known result that the commutant of $\mathcal{B}(G) \otimes \mathbf{1}$ in $\mathcal{B}(G \bar{\otimes} \mathfrak{H})$ is $\mathbf{1} \otimes \mathcal{B}(\mathfrak{H})$. Of course, $\mathcal{B}(G \bar{\otimes} \mathfrak{H}) = \mathcal{B}(G) \bar{\otimes}^s \mathcal{B}(\mathfrak{H})$. ($\mathcal{B}(G) \otimes \mathcal{B}(\mathfrak{H})$ contains the strongly dense subset $\mathcal{F}(G \otimes \mathfrak{H})$.) In other words, $\mathcal{B}(G \bar{\otimes} \mathfrak{H}) = \overline{\mathsf{span}}^s\big(\mathcal{B}(G)\mathcal{B}^{a,bil}(E)\big)$. Nothing like this is true for two-sided von Neumann $\mathcal{B}$–modules over more general von Neumann algebras $\mathcal{B} \subset \mathcal{B}(G)$.

3.4 Centered Hilbert modules

Motivated by Example 3.3.4, which shows that two-sided von Neumann $\mathcal{B}(G)$–modules are generated by those elements which *commute* with $\mathcal{B}(G)$, we use this property to define *centered* modules. Centered modules were introduced in [Ske98a], because they behave particularly well with respect to tensor products; see Proposition 4.2.15. Centered von Neumann modules admit centered complete quasi orthonormal systems and normality of left multiplication is automatic.

3.4.1 Definition. The *$\mathcal{B}$–center* of a $\mathcal{B}$–$\mathcal{B}$–module E is the linear subspace

$$C_{\mathcal{B}}(E) = \big\{x \in E \colon xb = bx \ (b \in \mathcal{B})\big\}.$$

of E. In particular, $C_{\mathcal{B}}(\mathcal{B})$ is the *center* of $\mathcal{B}$.

3.4.2 Proposition. *Let E be a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module. Then*

$$\big\langle C_{\mathcal{B}}(E), C_{\mathcal{B}}(E)\big\rangle \subset C_{\mathcal{B}}(\mathcal{B}).$$

PROOF. Let $x, y \in C_{\mathcal{B}}(E)$ and $b \in \mathcal{B}$. Then $\langle x, y\rangle b = \langle x, yb\rangle = \langle x, by\rangle = \langle b^*x, y\rangle = \langle xb^*, y\rangle = b\langle x, y\rangle$. ■

3.4.3 Corollary. *If E is a Hilbert module and a von Neumann module, respectively, then $C_{\mathcal{B}}(E)$ is a Hilbert $C_{\mathcal{B}}(\mathcal{B})$–module and a von Neumann $C_{\mathcal{B}}(\mathcal{B})$–module, respectively.*

3.4.4 Corollary. *Each element in the $\mathcal{B}$–linear span of $C_{\mathcal{B}}(E)$ commutes with each element in $C_{\mathcal{B}}(\mathcal{B})$.*

3.4.5 Definition. We say a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module E is a *centered* pre-Hilbert $\mathcal{B}$–module, Hilbert $\mathcal{B}$–module and von Neumann $\mathcal{B}$–module E, if E is generated by $C_{\mathcal{B}}(E)$ as a pre-Hilbert $\mathcal{B}$–module, a Hilbert $\mathcal{B}$–module and a von Neumann $\mathcal{B}$–module, respectively.

3.4.6 Example. Generalizing Example 3.3.4, $\mathfrak{H}_{\mathcal{B}} = \mathcal{B} \otimes \mathfrak{H}$ (for some pre-Hilbert space $\mathfrak{H}$; cf. also Example 1.5.10) with its obvious left multiplication is a centered pre-Hilbert $\mathcal{B}$–module (again with module basis $\big(\mathbf{1} \otimes e_\beta\big)_{\beta \in B}$). Clearly, if $\mathcal{B} \subset \mathfrak{B}^a(G)$ acts non-degenerately on the pre-Hilbert space G, then $\mathfrak{H}_{\mathcal{B}} \odot G = G \otimes \mathfrak{H}$. The centered structure is one of the reasons, why we put $\mathcal{B}$ (and G, respectively) as left factor in the tensor product. (If we put this factor on the right, then the left action has first to "commute" with elements in the factor $\mathfrak{H}$. Centeredness means that this commutation is possible in a controllable way.) That our choice of the order is the correct one becomes explicit in Example 4.2.13.

3.4.7 Proposition. *Let j be a $\mathcal{B}$–$\mathcal{B}$–linear mapping on a $\mathcal{B}$–$\mathcal{B}$–module E. Then the $\mathcal{B}$–center of E is mapped to the $\mathcal{B}$–center of the range of j. Consequently, if E is generated by its $\mathcal{B}$–center, then so is the range of j.*

PROOF. Obvious. ■

3.4.8 Corollary. *Let $\big(e_\beta, p_\beta\big)_{\beta \in B}$ be a quasi orthonormal system in a centered von Neumann $\mathcal{B}$–module E. Furthermore, suppose that all e_β are in $C_{\mathcal{B}}(E)$ (whence, $p_\beta \in C_{\mathcal{B}}(\mathcal{B})$). Then both the range of p_B and its complement are centered von Neumann $\mathcal{B}$–modules.*

PROOF. $p_B = \sum\limits_{\beta \in B} e_\beta e_\beta^*$ and, consequently, $\mathbf{1} - p_B$ are $\mathcal{B}$–$\mathcal{B}$–linear mappings. ■

3.4.9 Theorem. *Let E be a centered von Neumann $\mathcal{B}$–module. Then E admits a complete quasi orthonormal system $\big(e_\beta, p_\beta\big)_{\beta \in B}$ consisting of elements $e_\beta \in C_{\mathcal{B}}(E)$ and central projections p_β.*

PROOF. Suppose $(e_\beta, p_\beta)_{\beta\in B}$ is not complete. By Corollary 3.4.8 we may choose a non-zero x in $C_{\mathcal{B}}(E)$ which is orthogonal to all e_β. Then $|x|$ is in the center of $\mathcal{B}$. Let $v\,|x|$ be the polar decomposition of x. Then also v is in $C_{\mathcal{B}}(E)$. (Indeed, let $|x|\,g$ be an element in the range of $|x|$ and $b \in \mathcal{B}$. Then $vb\,|x|\,g = v\,|x|\,bg = bv\,|x|\,g$. If $g \in (|x|\,G)^\perp$, then $g = (\mathbf{1} - |v|)g$, so that $vbg = vb(\mathbf{1} - |v|)g = v(\mathbf{1} - |v|)bg = 0 = bvg$. We conclude that $vb = bv$.) The pair $(v, |v|)$ extends $(e_\beta, p_\beta)_{\beta\in B}$ to a bigger quasi orthonormal system so that we are ready for an application of *Zorn's lemma*. ■

3.4.10 Theorem. *Let E be a centered von Neumann $\mathcal{B}$–module. Then E may be identified as a complemented von Neumann $\mathcal{B}$–submodule of the strong completion of $\mathcal{B} \otimes \mathfrak{H}$ where $\mathfrak{H}$ is a suitable Hilbert space in such a way that left multiplication is preserved.*

PROOF. We choose a complete orthonormal system for E which consists of elements of $C_{\mathcal{B}}(E)$ and perform the construction according to Theorem 3.2.10. On the $\mathcal{B}$–center left multiplication, clearly, is preserved. By Proposition 3.1.5 left multiplication is strongly continuous on E so that any extension is determined uniquely by its values on the $\mathcal{B}$–center. ■

3.4.11 Corollary. *E is the strong closure of the pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module direct sum of ideals $p_\beta\mathcal{B}$ of $\mathcal{B}$.*

3.4.12 Remark. Any (pre-)Hilbert $\mathcal{B}$–module E may be considered as a submodule of a von Neumann module. (For instance, we may embed E into the strong completion of any faithful representation of $\mathcal{M}(E)$). Similarly, if E is a centered (pre-)Hilbert $\mathcal{B}$–module, then E may be considered as a $\mathcal{B}$–$\mathcal{B}$–submodule of a suitable completion of $\mathcal{B} \otimes \mathfrak{H}$ for a suitable Hilbert space $\mathfrak{H}$.

3.4.13 Theorem. *Any centered von Neumann module is a two-sided von Neumann module.*

PROOF. E may be identified as a subset of $\mathcal{B}(G, G \mathbin{\bar\otimes} \mathfrak{H})$ where b acts from the left as an operator on $G \otimes \mathfrak{H}$. This is nothing but the normal $*$–representation $\mathbf{1} \otimes \mathsf{id}$ on $G \otimes \mathfrak{H}$. Therefore, also the compression of $\mathbf{1} \otimes \mathsf{id}$ to ρ on $E \mathbin{\bar\odot} G$ is normal. ■

Chapter 4

GNS-construction and tensor products

One central object in these notes are two-sided Hilbert modules, the other completely positive mappings. In Section 4.1 the two are linked together with the help of Paschke's GNS-construction for completely positive mappings [Pas73]. For a single completely positive mapping there is also the well-known *Stinespring construction*. We recover the Stinespring construction by doing the Stinespring representation (see Definition 2.3.3) for the GNS-module. Although the GNS-construction is certainly more elegant and algebraically more satisfactory, for a single mapping there is not much what cannot be achieved also with the help of the Stinespring construction.

The situation changes completely, if we consider compositions of two completely positive mappings T and S to a third one $S \circ T$. In Section 4.2 we recover the GNS-module of $S \circ T$ just as (a submodule of) the tensor product of the GNS-modules of T and S. Starting with the Stinespring construction for T and S, a similar procedure is impossible. We consider this as the key observation which is responsible for the elegant power of the module approach to dilation theory in Part III, where we try to realize the GNS-modules of a whole CP-semigroup simultaneously. This leads directly to tensor product systems of Hilbert modules.

In Section 4.3 we consider the other tensor product of Hilbert modules, the so-called *exterior tensor product*. As example we study the L^2–spaces of "functions" with values in a Hilbert module.

Conditional expectations are special completely positive mappings. In Section 4.4 we study their GNS-modules and present examples of how they typically appear in the remainder of these notes.

4.1 GNS-construction

4.1.1 Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be pre–C^*–algebras. We call a linear mapping $T\colon \mathcal{A} \to \mathcal{B}$ *completely positive*, if

$$\sum_{i,j} b_i^* T(a_i^* a_j) b_j \geq 0 \tag{4.1.1}$$

for all choices of finitely many $a_i \in \mathcal{A}, b_i \in \mathcal{B}$. We assume always that T is bounded.

Because T is bounded, it extends to a completely postive mapping $\overline{\mathcal{A}} \to \overline{\mathcal{B}}$. Therefore, we may write any positive element in $\mathcal{A}$ as a^*a for suitable $a \in \overline{\mathcal{A}}$, so that complete positivity implies positivity. If $\mathcal{A}$ is a C^*–algebra, then positivity in turn implies boundedness. (This follows by the following standard argument taken from [Lan95]. Consider the series $c = \sum\limits_{n=1}^{\infty} \frac{a_n}{n^2}$, where a_n are positive elements of $\mathcal{A}$ fulfilling $\|a_n\| = 1$ and $\|T(a_n)\| > n^3$. If such a_n existed, we had $\|T(c)\| > \left\|T\left(\frac{a_n}{n^2}\right)\right\| > n \to \infty$, so that $c \in \mathcal{A}$ could not be in the domain of T. We conclude that T must be bounded on positive elements, hence, on all elements of the unit-ball of $\mathcal{A}$.)

4.1.2 Example. The axioms of a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module E are modelled such that the mapping T, defined by setting

$$T(a) = \langle \xi, a\xi \rangle \tag{4.1.2}$$

for some element ξ in E, is completely positive (and bounded, if E is contractive).

This example is reversed by the following GNS-construction due to Paschke [Pas73].

4.1.3 Theorem. *Let $T\colon \mathcal{A} \to \mathcal{B}$ be a completely positive mapping between unital pre–C^*–algebras $\mathcal{A}$ and $\mathcal{B}$. Then there exists a contractive pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module E with a* cyclic *vector $\xi \in E$ (i.e. $E = \mathsf{span}(\mathcal{A}\xi\mathcal{B})$) such that T has the form* (4.1.2). *Conversely, if E' is another pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module with cyclic vector ξ' such that* (4.1.2) *generates T, then $\xi \mapsto \xi'$ extends as a two-sided isomorphism $E \to E'$.*

Proof. $\mathcal{A} \otimes \mathcal{B}$ is an $\mathcal{A}$–$\mathcal{B}$–module in an obvious fashion. We define a sesquilinear mapping by setting

$$\langle a \otimes b, a' \otimes b' \rangle \;=\; b^* T(a^*a') b'.$$

Clearly, this mapping is a semi-inner product which turns $\mathcal{A} \otimes \mathcal{B}$ into a semi-Hilbert $\mathcal{A}$–$\mathcal{B}$–module. Since T is bounded, the whole construction extends to the completion $\overline{\mathcal{A}}$, so that E must be contractive. We set $E = \mathcal{A} \otimes \mathcal{B}/\mathcal{N}_{\mathcal{A}\otimes\mathcal{B}}$ and $\xi = \mathbf{1} \otimes \mathbf{1} + \mathcal{N}_{\mathcal{A}\otimes\mathcal{B}}$. Then the pair (E, ξ) has the desired properties. The statement on uniqueness is obvious. ∎

4.1.4 Observation. The preceding construction works also for unbounded T. However, in this case E is no longer contractive.

4.1.5 Corollary. *We have*

1. *T is hermitian (i.e. $T(a^*) = T(a)^*$).*
2. *$\|T\| = \|T(\mathbf{1})\|$.*
3. *$T(a^*b)T(b^*a) \leq \|T(b^*b)\| \, T(a^*a)$.*
4. *$T(a^*)T(a) \leq \|T\| \, T(a^*a)$.*

4.1.6 Corollary. *For a mapping $T \in \mathcal{B}(\mathcal{A}, \mathcal{B})$ the following conditions are equivalent.*

1. *T is completely positive.*
2. *The mapping $T^{(n)} \colon M_n(\mathcal{A}) \to M_n(\mathcal{B})$, defined by setting*
$$T^{(n)}(A)_{ij} \;=\; T(a_{ij})$$
for $A = \big(a_{ij}\big) \in M_n(\mathcal{A})$, is positive for all $n \in \mathbb{N}$.
3. *$T^{(n)}$ is comletely positive for all $n \in \mathbb{N}$.*

Proof. Clearly, $3 \Rightarrow 2 \Rightarrow 1$, and $1 \Rightarrow 3$ follows directly from Examples 1.7.7 and 4.1.2. ■

4.1.7 Definition. We refer to the pair (E, ξ) as the *GNS-construction* for T with *cyclic vector* ξ and *GNS-module* E .

4.1.8 Observation. ξ is a *unit vector* (i.e. $\langle \xi, \xi \rangle = \mathbf{1}$), if and only if T is unital.

4.1.9 Remark. Suppose $\mathcal{B}$ is a pre–C^*–algebra of operators on some pre-Hilbert space G and consider the Stinespring representations (H, η, ρ) of E and $\mathcal{A}$. Then Theorem 4.1.3, interpreted in terms of G, H and operators between these two spaces, asserts that there exists a pre-Hilbert space $H = E \odot G$, a representation ρ of $\mathcal{A}$ on H and a mapping $L_\xi = \eta(\xi) \in \mathcal{B}^a(G, H)$ such that
$$T(a) = L_\xi^* \rho(a) L_\xi$$
and $H = \mathsf{span}(\rho(\mathcal{A}) L_\xi G)$. If T is unital, then L_ξ is an isometry.

This is the usual *Stinespring construction*. In other words, we have decomposed the Stinespring construction into the GNS-construction, which does not depend on a representation of $\mathcal{B}$, and the Stinespring representation of the GNS-module, which does depend on a representation of $\mathcal{B}$.

4.1.10 Example. What are the completely positive mappings that have as GNS-module one of the modules $\mathcal{B}_\vartheta$ for some unital endomorphism ϑ on $\mathcal{B}$ as considered in Example 1.6.7? For that the cyclic vector ξ must be an element in $\mathcal{B}$ such that $\vartheta(b)\xi b'$ spans all of $\mathcal{B}$. This is the case independently of ϑ, if the right ideal in $\mathcal{B}$ generated by ξ is all of $\mathcal{B}$, for instance, if ξ is invertible. In particular, if $\xi = \mathbf{1}$, then we establish $(\mathcal{B}_\vartheta, \mathbf{1})$ as the GNS-construction for ϑ. In general, the GNS-module of a completely positive mapping $T(b) = \xi^* b\xi$ for some $\xi \in \mathcal{B}$ is contained in $\mathcal{B}$. And if we pass to $T \circ \vartheta$, then the GNS-module is contained in $\mathcal{B}_\vartheta$.

We will see in the remainder of these notes, particularly in the following section, that the GNS-construction is superior concerning algebraic questions due to its behaviour under composition. Whereas, the Stinespring construction has advantages, whenever we need the strong topology, like in Chapter 3 or in the following theorem where we investigate *unitizations* of completely positive mappings.

4.1.11 Theorem. *Let $T\colon \mathcal{A} \to \mathcal{B}$ be a completely positive contraction.*

1. *T is completely positive also as a mapping $\mathcal{A} \to M(\mathcal{B})$ or, more generally, as mapping into any pre–C^*–algebra which contains the $*$–subalgebra of $\mathcal{B}$ generated by $T(\mathcal{A})$ as an ideal.*

2. *If $\mathcal{B}$ is unital, then the unital extension of T to a mapping $\widetilde{\mathcal{A}} \to \mathcal{B}$ is a completely positive contraction, too.*

3. *The unital extension $\widetilde{T}$ of T to a mapping $\widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is a completely positive contraction.*

4. *Suppose that T is strictly continuous and that $\mathcal{B}$ is a unital C^*–algebra. Then there exists $b_{\mathbf{1}} \in \mathcal{B}$ such that $\lim_\lambda T(u_\lambda) = b_{\mathbf{1}}$ for each approximate unit $\bigl(u_\lambda\bigr)_{\lambda\in\Lambda}$ for $\mathcal{A}$, and the extension of T to a mapping $\widetilde{\mathcal{A}} \to \mathcal{B}$ by $\widetilde{\mathbf{1}} \mapsto b_{\mathbf{1}}$ is a completely positive contraction.*

Proof. 1. If $\mathcal{B}'$ is a pre–C^*–algebra containing $\mathsf{alg}^*(T(\mathcal{A}))$ as an ideal, then $\mathcal{B}'$ has a homomorphic image in $M(\mathcal{B})$. Therefore, it is enough to show only the satement for $\mathcal{B}' = M(\mathcal{B})$. Choose an approximate unit $\bigl(v_\mu\bigr)_{\mu\in M}$ for $\mathcal{B}$ and replace in (4.1.1) all $b_i \in M(\mathcal{B})$ by $b_i v_\mu \in \mathcal{B}$ (and, of course b_i^* by $v_\mu b_i^*$). Then (4.1.1) is the (strict) limit over μ of positive elements and, therefore, also positive.

2. Let $\bigl(u_\lambda\bigr)_{\lambda\in\Lambda}$ be an approximate unit for $\mathcal{A}$. Furthermore, assume that $\mathcal{B}$ is represented as an algebra of operators on a pre-Hilbert space G. Consider the net $\xi_\lambda = u_\lambda \otimes \mathbf{1} \in E \subset \mathcal{B}(\overline{G}, \overline{H} = E \bar{\odot} G)$, where $E = \mathcal{A} \otimes \mathcal{B}/\mathcal{N}_{\mathcal{A}\otimes\mathcal{B}}$ is the same module as in the GNS-construction, of course, without a cyclic vector, if $\mathcal{A}$ is non-unital. Choose $g \in G$ and define the bounded positive functional $\varphi(a) = \langle g, T(a)g\rangle$ on $\mathcal{A}$. We find

$$\|(\xi_\lambda - \xi_{\lambda'})g\| \;=\; \varphi\bigl((u_\lambda - u_{\lambda'})^2\bigr),$$

which is close to 0 for λ, λ' sufficiently big, if u_λ approximates the standard approximate unit for $\overline{\mathcal{A}}$ as explained in Appendix A.7. Of course, the net $\big(\xi_\lambda\big)$ is bounded by $\|T\|$. Therefore, ξ_λ converges strongly to an element $\xi \in \mathcal{B}(\overline{G}, \overline{H})$ and $\langle\xi,\xi\rangle \in \overline{\mathcal{B}}^s$. From $\rho(u_\lambda)\xi = \xi_\lambda$ we conclude that $T(u_\lambda a u_\lambda) \to T(a)$ weakly for all $a \in \mathcal{A}$ and $T(u_\lambda u_\lambda) \to \langle\xi,\xi\rangle$, which shows that $a \mapsto T(a)$, $\widetilde{1} \mapsto \langle\xi,\xi\rangle$ is a completely positive mapping $\widetilde{\mathcal{A}} \to \overline{\mathcal{B}}^s$. Now set $\widehat{\xi} = \sqrt{\mathbf{1} - \langle\xi,\xi\rangle} \in \overline{\mathcal{B}}^s$ and let $\widehat{E}$ be the pre-Hilbert $\widetilde{\mathcal{A}}$–$\overline{\mathcal{B}}^s$–module $\widehat{\xi}\,\overline{\mathcal{B}}^s \subset \overline{\mathcal{B}}^s$ with left multiplication defined by $a\widehat{\xi} = 0$ for $a \in \mathcal{A}$ and $\widetilde{\mathbf{1}}\widehat{\xi} = \widehat{\xi}$ as in Example 1.6.5. Then $\widetilde{\xi} = \xi \oplus \widehat{\xi}$ is a unit vector in $\widetilde{E} = E \oplus \widehat{E}$ and the unital extension of T can be written in the form $a \mapsto \langle\widetilde{\xi}, a\widetilde{\xi}\rangle$ $(a \in \widetilde{\mathcal{A}})$ and, therefore, is a completely positive contraction.

3. By 1 $a \mapsto T(a)$ is completely positive also as mapping into $\widetilde{\mathcal{B}}$ so that we may apply 2 to that mapping.

4. Since $\mathcal{B}$ is unital, the strict topology of $\mathcal{B}$ coincides with the norm topology. Therefore, T sends strict Cauchy nets in $\mathcal{A}$ to norm convergent nets in $\mathcal{B}$ (here we need completeness of $\mathcal{B}$). In particular, $b_{\mathbf{1}} \in \mathcal{B}$ exists for each approximate unit and, of course, does not depend on the particular choice. If follows that the mappings T_λ defined by setting $T_\lambda(a) = T(u_\lambda a u_\lambda)$ converge pointwise in norm for any $a \in \widetilde{\mathcal{A}}$. The limit mapping is completely positive as limit of completely positive mappings, its restriction to $\mathcal{A}$ is T, and its value at $\widetilde{\mathbf{1}}$ is $b_{\mathbf{1}}$. ■

4.1.12 Remark. The proof of 2 shows that the GNS-construction works also for non-unital algebras in the sense that we find a two-sided pre-Hilbert module E and a cyclic vector $\xi \in \overline{E}^s$ such that $T(a) = \langle\xi, a\xi\rangle$ and $E = \mathsf{span}(\mathcal{A}\xi\mathcal{B}) \subset \overline{E}^s$. Alternatively, we find pre-Hilbert $\mathcal{B}$–module $\widetilde{E}$ with unit vector $\widetilde{\xi}$ and a representation j of $\mathcal{A}$ (degenerate, if ξ is not a unit vector) such that $T(a) = \langle\widetilde{\xi}, j(a)\widetilde{\xi}\rangle$ and $\widetilde{E} = \mathsf{span}(j(\mathcal{A})\widetilde{\xi}\mathcal{B} \cup \widetilde{\xi}\mathcal{B})$. The situation is improved in 4 where $\xi \in E$ (not in the closure). Also here $\mathsf{span}(\mathcal{A}\xi\mathcal{B})$ may not contain all of $\xi\mathcal{B}$. In any case, in 4 the canonical representation of $\mathcal{A}$ on E is total so that $\overline{E} = \overline{\mathsf{span}}(\mathcal{A}\xi\mathcal{B})$.

It is also important to notice that in 2 we temporarily considered strong closures in order to assure existence of some strong limits. However, after the construction everything reduces again to the pre–C^*–algebras with which we started. This enabled us to formulate the statement in 2 purely algebraic without any reference to a representation space.

Of course, the results can be applied also to a completely positive mapping T which is not necessarily contractive, by considering the contraction $T/\|T\|$ rather than T itself.

4.1.13 Proposition. *Let $T\colon \mathcal{A} \to \mathcal{B}$ be a normal completely positive mapping between von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$. Then the strong closure $\overline{E}^s$ of the GNS-module E is a von Neumann $\mathcal{A}$–$\mathcal{B}$–module.*

PROOF. Let $\mathcal{B}$ act on a Hilbert space G and set $H = E \,\bar{\odot}\, G$. By Lemma 3.3.2 it is sufficient to show that the representation ρ of $\mathcal{A}$ on H is normal. So let $\big(a_\lambda\big)$ be a bounded increasing net in $\mathcal{A}$. This net converges strongly to some $a \in \mathcal{A}$. Then for each $b \in \mathcal{A}$ also the net $\big(b^* a_\lambda b\big)$ is bounded and increasing, and it converges strongly to b^*ab, because multiplication in $\mathcal{A}$ is separately strongly continuous. Since T is normal, we have $\lim_\lambda T(b^* a_\lambda b) = T(b^*ab)$ strongly. Let $g \in G$ be a unit vector and define the normal state $\langle g, \bullet g \rangle$ on $\mathcal{B}$. Then for $f = (b \otimes \mathbf{1} + \mathcal{N}_{\mathcal{A}\otimes\mathcal{B}}) \odot g \in E \odot G$ we have

$$\lim_\lambda \langle f, \rho(a_\lambda) f \rangle = \lim_\lambda \langle g, T(b^* a_\lambda b) g \rangle = \langle g, T(b^* ab) g \rangle) = \langle f, \rho(a) f \rangle \tag{4.1.3}$$

where f ranges over all vectors of the form $x \odot g$ which form a total subset, if b and g range over $\mathcal{A}$ and G, respectively. Therefore, as explained in Appendix A.9, $\lim_\lambda \rho(a_\lambda) = \rho(a)$ in the strong topology. ■

4.1.14 Remark. This proposition asserts as a special application that the Stinespring construction of a normal completely positive mapping results in a normal representation. Of course, the separation of the Stinespring construction into the construction of the GNS-module $\overline{E}^s$ and the Stinespring representation in the sense of Definition 2.3.6 leads to much more general statements in connection with Theorem 3.2.17 and Corollary 3.3.3.

4.1.15 Example. Let T be a normal completely positive mapping on $\mathcal{B}(G)$. By Example 3.3.4 we know that the (von Neumann) GNS-module E has the form $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$ for $\mathfrak{H}$ being the central Hilbert space. Expanding the cyclic vector $\xi = \sum_{\beta \in B} b_\beta \otimes e_\beta$ for some ONB $\big(e_\beta\big)_{\beta \in B}$ of $\mathfrak{H}$, we see that $T(b) = \sum_{\beta \in B} b_\beta^* b b_\beta$ (strong convergence). T is unital, if and only if $\sum_{\beta \in B} b_\beta^* b_\beta = \mathbf{1}$.

4.2 Tensor product

Let $\mathcal{B}$ and $\mathcal{C}$ be pre–C^*–algebras. Furthermore, let E be a pre-Hilbert $\mathcal{B}$–module, and let F be a contractive pre-Hilbert $\mathcal{B}$–$\mathcal{C}$–module. There are, at least, two different ways to see that the sesquilinear mapping, defined by setting

$$\langle x \otimes y, x' \otimes y' \rangle \;=\; \big\langle y, \langle x, x' \rangle y' \big\rangle, \tag{4.2.1}$$

is a semiinner product on the right $\mathcal{C}$–module $E \otimes F$. Of course, the only property in question is positivity.

Firstly, we know from Example 1.7.6 that for $X = (x_1, \ldots, x_n) \in E_n$ the matrix $B = \langle X, X \rangle = \big(\langle x_i, x_j \rangle\big)$ is positive in $M_n(\mathcal{B})$ and, therefore, has a square root $\sqrt{B}$ in $M_n(\overline{\mathcal{B}})$. Thus, for $Y = (y_1, \ldots, y_n) \in F^n \subset \overline{F}^n$ we see that $\sum_{ij} \langle x_i \otimes y_i, x_j \otimes y_j \rangle = \langle \sqrt{B} Y, \sqrt{B} Y \rangle \geq 0$.

Secondly, we take any faithful representation π of $\mathcal{C}$ on a pre-Hilbert space K and construct the Stinespring representation $(G = F \odot K, \eta, \rho)$. In other words, we identify elements $y \in F$ with mappings $L_y \in \mathcal{B}^a(K, G)$ and $\mathcal{B}$ acts on L_y via the representation ρ of $\mathcal{B}$ on G. Then we construct the Stinespring representation of E associated with the representation ρ. We find

$$\pi\big(\langle x \otimes y, x' \otimes y' \rangle\big) \;=\; L_y^* L_x^* L_{x'} L_{y'}, \tag{4.2.2}$$

from which positivity is immediate.

The first argument is comparably elegant, but requires C^*–algebraic methods. In this form it is limited to contractive modules, because it is necessary to extend the action of $\mathcal{B}$ to all elements of $\overline{\mathcal{B}}$. (It is possible to apply the argument also to the non-contractive case, roughly speaking, by (semi-)norming $\mathcal{B}$ with the operator norm by considering it as an algebra of operators on F. Then, of course, F becomes contractive.) The second argument is based on elementary Hilbert space arguments like cyclic decomposition. It applies immediately to the non-contractive case.

4.2.1 Definition. The *tensor product* of E and F is the pre-Hilbert $\mathcal{C}$–module $E \odot F = E \otimes F / \mathcal{N}_{E \otimes F}$. For unital $\mathcal{B}$, we identify always $E \odot \mathcal{B}$ and E (via $x \odot b = xb$), and we identify always $\mathcal{B} \odot F$ and F (via $b \odot y = by$).

Often, this tensor product is refered to as the *interior* tensor product in contrast with the exterior tensor product; see Section 4.3.

Observe that $E \bar{\odot} \mathcal{B} = E$, also if $\mathcal{B}$ is nonunital (and similarly for F). This follows by using an approximate unit.

4.2.2 Observation. Since $\mathcal{B}$ does not appear explicitly in the inner product (4.2.1), the tensor product is independent of the precise "size" of $\mathcal{B}$. More precisely, if $\mathcal{B}'$ is another pre–C^*–algebra containing $\mathcal{B}_E$ as an ideal, and acting on F via a representation such that the action of the elements of $\mathcal{B}_E$ is the same, then $E \odot F$ is still the same pre-Hilbert module.

4.2.3 Example. We already met the tensor product $E \odot G$ of a pre-Hilbert $\mathcal{B}$–module E with a pre-Hilbert space G carrying a (non-degenerate) representation of $\mathcal{B}$. Clearly, G is a pre-Hilbert $\mathcal{B}$–$\mathbb{C}$–module and in this interpretation the pre-Hilbert space (i.e. pre-Hilbert $\mathbb{C}$–module) $E \odot G$, indeed, coincides with the pre-Hilbert module tensor product of E and G.

4.2.4 Example. Let E be a pre-Hilbert $\mathcal{B}$–module, and let $j \colon \mathcal{B} \to \mathcal{C}$ be a (contractive) unital homomorphism. Then $\mathcal{C}$ is a (contractive) pre-Hilbert $\mathcal{B}$–$\mathcal{C}$–module with its natural pre-Hilbert $\mathcal{C}$–module structure (see Example 1.1.5) and the left multiplication $b.c = j(b)c$. (Obviously, $\mathcal{C}$ is nothing but the GNS-module of the completely positive mapping j, and $\mathbf{1}_{\mathcal{C}}$ the cyclic vector.) Then $E \odot \mathcal{C}$ is called the ***extension*** of E by $\mathcal{C}$. Notice that $E \odot j(\mathcal{B})$ is just the quotient of the semi-Hilbert $j(\mathcal{B})$–module E with semiinner product $\langle x, x' \rangle_j = j(\langle x, x' \rangle)$ and right multiplication $x.j(b) = xb$, by its length-zero elements.

4.2.5 Proposition. *Let E_1, E_2 be pre-Hilbert $\mathcal{B}$–modules, let F be a pre-Hilbert $\mathcal{B}$–$\mathcal{C}$–module, and let $a \in \mathcal{B}^a(E_1, E_2)$. Then*

$$a \odot \mathsf{id} \colon x \odot y \longmapsto ax \odot y$$

defines an operator on $E_1 \odot F \to E_2 \odot F$ with adjoint $a^* \odot \mathsf{id}$. If F is contractive, then

$$\|a \odot \mathsf{id}\| \;\; \leq \;\; \|a\|\,.$$

PROOF. One easily checks that $a^* \odot \mathsf{id}$ is, indeed, an adjoint of $a \odot \mathsf{id}$. Therefore, by Corollary 1.4.3 $a \odot \mathsf{id}$ is a well defined element of $\mathcal{L}^a(E_1, E_2)$. If F is contractive, then we may complete. Therefore, the norm of the element $a^*a \odot \mathsf{id}$ cannot be bigger than the norm of a^*a in the C^*–algebra $\mathcal{B}^a(\overline{E_1})$. ■

4.2.6 Corollary. *Let $x \in E \subset \mathcal{B}^a(\mathcal{B}, E)$. Then $x \odot \mathsf{id} \colon y \mapsto x \odot y$ is a mapping $F = \widetilde{\mathcal{B}} \odot F \to E \odot F$, and $x^* \odot \mathsf{id} \colon x' \odot y \mapsto \langle x, x' \rangle y$ is its adjoint. If x is a unit vector, then $x \odot \mathsf{id}$ is an isometry. In particular, $(x^* \odot \mathsf{id})(x \odot \mathsf{id}) = x^*x \odot \mathsf{id} = \mathsf{id}_F$ and $(x \odot \mathsf{id})(x^* \odot \mathsf{id}) = xx^* \odot \mathsf{id}$ is a projection onto the range of $x \odot \mathsf{id}$. If F is contractive, then $\|x \odot \mathsf{id}\| \leq \|x\|$.*

4.2.7 Corollary. *Let E be a contractive pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module, and let F be a contractive pre-Hilbert $\mathcal{B}$–$\mathcal{C}$–module. Then $E \odot F$ is a contractive pre-Hilbert $\mathcal{A}$–$\mathcal{C}$–module.*

PROOF. The mapping $\mathcal{A} \to \mathcal{B}^a(E) \to \mathcal{B}^a(E \odot F)$ is a composition of contractive mappings. ■

We come to the most important application of the tensor product in these notes.

4.2.8 Example. Let $T \colon \mathcal{A} \to \mathcal{B}$ and $S \colon \mathcal{B} \to \mathcal{C}$ be completely positive contractions with GNS-modules E and F and with cyclic vectors ξ and ζ, respectively. (For simplicity, suppose that all algebras are unital.) Then we have $S \circ T(a) = \langle \xi \odot \zeta, a\xi \odot \zeta \rangle$ so that $S \circ T$ is completely positive by Example 4.1.2. Let G be the GNS-module of the composition $S \circ T$ with cyclic vector χ. Then the mapping

$$\chi \longmapsto \xi \odot \zeta$$

extends (uniquely) as a two-sided isometric homomorphism $G \to E \odot F$. Observe that $E \odot F = \mathsf{span}(\mathcal{A}\xi\mathcal{B} \odot \mathcal{B}\zeta\mathcal{C}) = \mathsf{span}(\mathcal{A}\xi \odot \mathcal{B}\zeta\mathcal{C}) = \mathsf{span}(\mathcal{A}\xi\mathcal{B} \odot \zeta\mathcal{C})$. By the above isometry we may identify G as the submodule $\mathsf{span}(\mathcal{A}\xi \odot \zeta\mathcal{C})$ of $E \odot F$. In other words, inserting a unit $\mathbf{1}$ in $\chi = \xi \odot \zeta$ in between ξ and ζ amounts to an isometry. Varying, instead, $b \in \mathcal{B}$ in $\xi b \odot \zeta = \xi \odot b\zeta$, we obtain a set which generates all of $E \odot F$.

This operation is crucial in the construction of tensor product systems. We explain immediately, why the Stinespring construction cannot do the same job. Suppose that $\mathcal{B}$ and $\mathcal{C}$ are algebras of operators on some pre-Hilbert spaces. Then, unlike the GNS-construction, the knowledge of the Stinespring construction for the mapping T does not help in finding the Stinespring construction for $S \circ T$. What we need is the Stinespring construction for T based on the representation of $\mathcal{B}$ arising from the Stinespring construction for S; cf. (4.2.2). The GNS-construction, on the other hand, is *representation free*. It is sufficient to do it once for each completely positive mapping. Yet in other words, we can formulate as follows.

4.2.9 Functoriality. A pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module E is a *functor* sending representations of $\mathcal{B}$ on F to (non-degenerate) representations of $\mathcal{A}$ on $E \odot F$, and the composition of two such functors is the tensor product. The Stinespring construction is a dead end for this functoriality.

4.2.10 Example. Clearly, the tensor product of two-sided pre-Hilbert modules is associative. Applying this to Example 4.2.3, we find $H = E \odot (F \odot K) = (E \odot F) \odot K$. In other words, the Stinespring representation of $E \odot F$ allows us to identify $E \odot F$ as the subspace $\mathsf{span}\{L_x L_y \ (x \in, y \in F)\}$ of $\mathcal{B}^a(K, H)$; see again (4.2.2).

4.2.11 Corollary. *If in Proposition 4.2.5 F is contractive, then the mapping $a \mapsto a \odot \mathsf{id}$ extends to a contraction $\mathcal{B}^r(E_1, E_2) \to \mathcal{B}^r(E_1 \odot F, E_2 \odot F)$.*

Proof. Precisely, as in the proof of Lemma 2.3.7 we show with the help of a cyclic decomposition of K that in order to compute the norm of the operator $x \odot y \odot k \mapsto ax \odot y \odot k$ it is sufficient to take the supremum over elementary tensors. ■

4.2.12 Example. In the first argument, showing positivity of the inner product (4.2.1), we interpreted $\sum_i x_i \odot y_i \in E \odot F$ as the elementary tensor $X \odot Y \in E_n \odot F^n$. Reversing this interpretation, we show that

$$E_n \odot F^n \ \cong \ E \odot F$$

for all $n \in \mathbb{N}$.

On the contrary, one easily checks that the mapping $X \odot Y \mapsto Z$ where $X \in E^n, Y \in F_m$ and $Z = \big(x_i \odot y_j\big)_{ij}$ establishes a two-sided isomorphism of the pre-Hilbert $M_n(\mathcal{A})$–$M_m(\mathcal{C})$–module $E^n \odot F_m$ and $M_{nm}(E \odot F)$ as introduced in Example 1.7.7. A generalization of these computation rules is $M_{n\ell}(E) \odot M_{\ell m}(F) = M_{nm}(E \odot F)$ (via the identification $\big(X \odot Y\big)_{ij} = \sum\limits_k x_{ik} \odot y_{kj}$) which contains the former as the special case $n = m = 1$ and the latter as $\ell = 1$.

Of course, for $X \in M_{n\ell}(E)$ the mapping $A \mapsto \langle X, AX\rangle$ is completely positive, and so is $A \mapsto \langle X \odot Y, AX \odot Y\rangle$ for $Y \in M_{\ell m}$. For $\ell = m = n$ and $X = \big(\delta_{ij} x_i\big)$ ($x_i \in E$), $Y = \big(\delta_{ij} y_i\big)$ ($y_i \in F$) as in Example 1.7.7, we find that $T\colon \big(a_{ij}\big) \mapsto \big(\langle x_i, a_{ij} x_j\rangle\big)$ and $S\colon \big(b_{ij}\big) \mapsto \big(\langle y_i, b_{ij} y_j\rangle\big)$ are completely positive mappings and that their *Schur composition* is given by $S \circ T\colon \big(a_{ij}\big) \mapsto \big(\langle x_i \odot y_i, a_{ij} x_j \odot y_j\rangle\big)$ and, therefore, also completely positive.

Setting $E = F = \mathcal{B}$, we find $\mathcal{B}_n \odot \mathcal{B}^n = \mathcal{B}$, $\mathcal{B}^n \odot \mathcal{B}_m = M_{nm}(\mathcal{B})$, and $M_{n\ell}(\mathcal{B}) \odot M_{\ell m}(\mathcal{B}) = M_{nm}(\mathcal{B})$. Also here the identifications include the correct two-sided pre-Hilbert module structure.

4.2.13 Example. For Hilbert spaces $G, H, \mathfrak{H}$ let $E = \mathcal{B}(G, H)$ be an (arbitrary) von Neumann $\mathcal{B}(G)$–module, and let $F = \mathcal{B}(G, G \mathbin{\bar{\otimes}} \mathfrak{H})$ be an (arbitrary) two-sided von Neumann $\mathcal{B}(G)$–module (cf. Examples 3.1.2 and 3.3.4). Then the identification

$$x \odot y \;=\; (x \otimes \mathsf{id})y$$

establishes an isomorphism $E \mathbin{\bar{\odot}^s} F \to \mathcal{B}(G, H \mathbin{\bar{\otimes}} \mathfrak{H})$. In particular, if $E_i = \mathcal{B}(G, G \mathbin{\bar{\otimes}} \mathfrak{H}_i)$ ($i = 1, 2$) are two-sided $\mathcal{B}(G)$–modules, then

$$\mathcal{B}(G, G \mathbin{\bar{\otimes}} \mathfrak{H}_1) \mathbin{\bar{\odot}^s} \mathcal{B}(G, G \mathbin{\bar{\otimes}} \mathfrak{H}_2) \;=\; \mathcal{B}(G, G \mathbin{\bar{\otimes}} \mathfrak{H}_1 \mathbin{\bar{\otimes}} \mathfrak{H}_2).$$

Taking into account also Proposition 3.3.5, we see not only that the centers $\mathfrak{H}_i$ compose like tensor products of Hilbert spaces, but also that this composition is associative.

In the preceding example we have symmetry of the tensor product in the sense that $E_1 \odot E_2 \cong E_2 \odot E_1$. However, the following example shows that this is by far not always so. In Proposition 4.2.15 we see that centered Hilbert modules have this symmetry. In Section 8.2 we give another example.

4.2.14 Example. Let $\mathcal{B}$ be unital, let E be a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module, and let F be pre-Hilbert $\mathcal{B}$–module. Then E has a natural pre-Hilbert $\widetilde{\mathcal{B}}$–$\widetilde{\mathcal{B}}$–module structure (extending the $\mathcal{B}$–$\mathcal{B}$–module structure), and we may equip F with the pre-Hilbert $\widetilde{\mathcal{B}}$–$\widetilde{\mathcal{B}}$–module structure as in Example 1.6.5. Then

$$x \odot y \;=\; x\mathbf{1} \odot y \;=\; x \odot \mathbf{1}y \;=\; 0$$

so that $E \odot F = \{0\}$. However, if there is a vector $\zeta \in F$ such that $\langle \zeta, \zeta \rangle = \mathbf{1} \in \mathcal{B}$, then $\zeta \odot \mathsf{id}$, defined as in Corollary 4.2.6, is an isometry $E \to F \odot E$. Therefore, if $E \neq \{0\}$, then certainly also $F \odot E \neq \{0\}$. We see that, in general, we may not expect that $E \odot F \cong F \odot E$.

4.2.15 Proposition. *Let E and F be centered pre-Hilbert $\mathcal{B}$–modules. Then $C_{\mathcal{B}}(E) \odot C_{\mathcal{B}}(F) \subset C_{\mathcal{B}}(E \odot F)$ (so that also $E \odot F$ is centered), and the mapping*

$$\mathcal{F}\colon x \odot y \longmapsto y \odot x \qquad (x \in C_{\mathcal{B}}(E), y \in C_{\mathcal{B}}(F)) \tag{4.2.3}$$

extends to a two-sided flip *isomorphism $E \odot F \to F \odot E$.*

Proof. The first statement is obvious. For the second statement it is sufficient to show that (4.2.3) preserves inner products (see Remark A.1.5). So let $x, x' \in C_{\mathcal{B}}(E)$, and $y, y' \in C_{\mathcal{B}}(F)$. By Proposition 3.4.2, $\langle x, x' \rangle$ and $\langle y, y' \rangle$ are in $C_{\mathcal{B}}(\mathcal{B})$. We find

$$\langle x \odot y, x' \odot y' \rangle = \langle y, \langle x, x' \rangle y' \rangle = \langle y, y' \rangle \langle x, x' \rangle = \langle x, x' \rangle \langle y, y' \rangle = \langle y \odot x, y' \odot x' \rangle. \blacksquare$$

In Proposition 4.2.5 we have seen that there is a natural unital (in general, non-injective) 'embedding' of $\mathcal{B}^a(E)$ into $\mathcal{B}^a(E \odot F)$ as $\mathcal{B}^a(E) \odot \mathsf{id}$. For $\mathcal{B}^a(F)$ this is, in general, not possible.

4.2.16 Example. Let $\mathcal{B}$ be a unital pre–C^*–algebra and consider $\mathcal{B} \odot \mathcal{B}$. Then the attempt to define an operator $\mathsf{id} \odot b$ for an element $b \in \mathcal{B}^a(\mathcal{B}) = \mathcal{B}$ fails, if b is not in the center of $\mathcal{B}$. Indeed, let $b' \in \mathcal{B}$ such that $b'b \neq bb'$. Then

$$\langle \mathbf{1} \odot \mathbf{1}, (\mathsf{id} \odot b)(b' \odot \mathbf{1} - \mathbf{1} \odot b') \rangle = b'b - bb' \neq 0,$$

but $b' \odot \mathbf{1} - \mathbf{1} \odot b' = 0$. If, however, b is in the center of $\mathcal{B}$ then $\mathsf{id} \odot b$ is a well-defined operator on $\mathcal{B} \odot \mathcal{B}$.

4.2.17 Example. A positive example is the case, when E and F are centered. In this case, for $a \in \mathcal{B}^a(F)$ we define the operator $\mathsf{id} \odot a = \mathcal{F}(a \odot \mathsf{id})\mathcal{F}$ with the help of the flip from Proposition 4.2.15.

The following theorem shows a general possibility.

4.2.18 Theorem. *Let E be pre-Hilbert module over a unital pre–C^*–algebra $\mathcal{B}$ with a unit vector ξ, and let F be a pre-Hilbert $\mathcal{B}$–$\mathcal{C}$–module. Then for each $a \in \mathcal{B}^{bil,a}(F)$ the mapping $x \odot y \to x \odot ay$ extends as a well-defined mapping $\mathsf{id} \odot a$ in $\mathcal{B}^a(E \odot F)$. Moreover, the mapping $a \mapsto \mathsf{id} \odot a$ is an isometric isomorphism from $a \in \mathcal{B}^{bil,a}(F)$ onto the relative commutant of $\mathcal{B}^a(E) \odot \mathsf{id}$ in $\mathcal{B}^a(E \odot F)$, independently of whether F is contractive or not. In other words,*

$$(\mathcal{B}^a(E) \odot \mathsf{id})' = \mathsf{id} \odot \mathcal{B}^{bil,a}(F) \cong \mathcal{B}^{bil,a}(F).$$

PROOF. Let a be a $\mathcal{B}$–$\mathcal{C}$–linear mapping in $\mathcal{B}^a(F)$. Then

$$\langle x \odot y, x' \odot ay' \rangle = \big\langle y, \langle x, x' \rangle ay' \big\rangle = \big\langle y, a\langle x, x' \rangle y' \big\rangle = \big\langle a^* y, \langle x, x' \rangle y' \big\rangle = \langle x \odot a^* y, x' \odot y' \rangle.$$

By Corollary 1.4.3, $\mathsf{id} \odot a$ is a well-defined mapping with adjoint $\mathsf{id} \odot a^*$. Let $a' \in \mathcal{B}^a(E)$. Replacing in the preceding computation x by $a'x$, we see that $\mathsf{id} \odot a$ and $a' \odot \mathsf{id}$ commute. The embedding $\mathcal{B}^{bil,a}(F) \to \mathsf{id} \odot \mathcal{B}^{bil,a}(F) \subset \mathcal{B}^a(E \odot F)$ does certainly not decrease the norm, because $\langle \xi \odot y, (\mathsf{id} \odot a)(\xi \odot y') \rangle = \langle y, ay' \rangle$. On the other hand, for $z \in E \odot F$ by Example 4.2.12 we may choose $n \in \mathbb{N}, X \in E_n, Y \in F^n$ such that $z = X \odot Y \in E_n \odot F^n = E \odot F$. Set $B = \sqrt{\langle X, X \rangle}$ (where we interprete elements of $M_n(\mathcal{B})$ as elements of $M_n(\mathcal{B}^a(\overline{F}))$, if necessary). Then

$$\|(\mathsf{id} \odot a)z\| \;=\; \|X \odot aY\| \;=\; \|BaY\| \;=\; \|aBY\| \;\leq\; \|a\|\,\|BY\| \;=\; \|a\|\,\|z\|\,. \tag{4.2.4}$$

Therefore, $a \mapsto \mathsf{id} \odot a$ is an isometry into $(\mathcal{B}^a(E) \odot \mathsf{id})'$.

Conversely, let $\mathfrak{a} \in \mathcal{B}^a(E \odot F)$ be in $(\mathcal{B}^a(E) \odot \mathsf{id})'$. Set $a = (\xi^* \odot \mathsf{id})\mathfrak{a}(\xi \odot \mathsf{id}) \in \mathcal{B}^a(F)$. By $j(b) = \xi b \xi^*$ we define a (degenerate, unless $\xi \in \mathcal{B}^a(\mathcal{B}, E)$ is unitary) representation of $\mathcal{B}$ on E. Then $j(b) \odot \mathsf{id}$ is an element in $\mathcal{B}^a(E) \odot \mathsf{id}$ and, therefore, commutes with $\mathfrak{a}$. We find

$$ba \;=\; (\xi^* \odot \mathsf{id})(j(b) \odot \mathsf{id})\mathfrak{a}(\xi \odot \mathsf{id}) \;=\; (\xi^* \odot \mathsf{id})\mathfrak{a}(j(b) \odot \mathsf{id})(\xi \odot \mathsf{id}) \;=\; ab,$$

i.e. a is $\mathcal{B}$–$\mathcal{C}$–linear. In particular, $\mathsf{id} \odot a$ is a well-defined element of $\mathcal{B}^a(E \odot F)$. For arbitrary $x \in E$ and $y \in F$ we find

$$\mathfrak{a}(x \odot y) \;=\; \mathfrak{a}(x\xi^* \odot \mathsf{id})(\xi \odot y) \;=\; (x\xi^* \odot \mathsf{id})\mathfrak{a}(\xi \odot y) \;=\; x \odot ay \;=\; (\mathsf{id} \odot a)(x \odot y),$$

where $x\xi^* \odot \mathsf{id}$ is an element of $\mathcal{B}^a(E) \odot \mathsf{id}$ and, therefore, commutes with $\mathfrak{a}$. In other words, $\mathfrak{a} = \mathsf{id} \odot a$. ■

4.2.19 Remark. Where the assumptions of Theorem 4.2.18 and Example 4.2.17 intersect $\mathsf{id} \odot a$ means the same operator in both cases.

4.2.20 Observation. The estimate in (4.2.4) does depend neither on existence of a unit vector in E nor on adjointability of a. Therefore, $a \mapsto \mathsf{id} \odot a$ always defines a contraction $\mathcal{B}^{bil}(F) \to \mathcal{B}^r(E \odot F)$, whose range commutes with $\mathcal{B}^r(E) \odot \mathsf{id}$.

4.2.21 Observation. Let E, F, F', G be two-sided pre-Hilbert modules and let $\beta \colon F \to F'$ be an isometric two-sided homomorphism of two-sided pre-Hilbert modules. Then also the mapping $\mathsf{id} \odot \beta \odot \mathsf{id} \colon E \odot F \odot G \to E \odot F' \odot G$ is an isometric two-sided homomorphism of two-sided pre-Hilbert modules.

Until now our considerations were at a rather algebraic level. We close this section with some more topological results.

Recall from Appendix C.4 that the tensor product *over* $\mathcal{B}$ of a right $\mathcal{B}$–module E and a left $\mathcal{B}$–module F is $E \underline{\odot} F := E \otimes F / \mathcal{N}_{\mathcal{B}}$ where $\mathcal{N}_{\mathcal{B}} = \mathsf{span}\{xb \otimes y - x \otimes by\}$. Of course, $\mathcal{N}_{\mathcal{B}} \subset \mathcal{N}_{E\otimes F}$. We repeat a result by Lance [Lan95] which asserts that under some completeness conditions the two tensor products coincide.

4.2.22 Proposition. *Let $\mathcal{B}$ be a C^*–algebra, let E be Hilbert $\mathcal{B}$–module, and let F be a pre-Hilbert $\mathcal{B}$–$\mathcal{C}$–module. Then*

$$E \odot F \;=\; E \underline{\odot} F.$$

Proof. Let $z = \sum_{i=1}^{n} x_i \otimes y_j$ be in $\mathcal{N}_{E\otimes F}$. We show $X \otimes Y = X'B \otimes Y - X' \otimes BY$ for suitable $X' \in E_n$ and $B \in M_n(\mathcal{B})$, from which $z \in \mathcal{N}_{\mathcal{B}}$ follows.

Let $B = \sqrt[4]{\langle X, X\rangle}$ and define the function

$$f_m(t) \;=\; \begin{cases} m^{\frac{1}{4}} & \text{for } 0 \le t \le \frac{1}{m} \\ t^{-\frac{1}{4}} & \text{for } \frac{1}{m} < t. \end{cases}$$

Then $Xf_m(B)$ is a Cauchy sequence and, therefore, converges to some $X' \in E_n$. Clearly, $X'B = X$. On the other hand,

$$\|BY\|^2 = \left\|\langle Y, B^2 Y\rangle\right\| \le \|Y\|\,\|B^2Y\| = \|Y\|\sqrt{\|\langle Y, B^4Y\rangle\|} = \|Y\|\,\|z\| = 0.$$

In other words, $X'B \otimes Y - X' \otimes BY = X \otimes Y$. ■

4.2.23 Corollary. *Let E be a two-sided Hilbert module over a C^*–algebra $\mathcal{B}$. Then*

$$E \odot \ldots \odot E \;=\; E \underline{\odot} \ldots \underline{\odot} E.$$

4.2.24 Proposition. *Let E be a von Neumann $\mathcal{A}$–$\mathcal{B}$–module, and let F be a von Neumann $\mathcal{B}$–$\mathcal{C}$–module where $\mathcal{C}$ acts on a Hilbert space K. Then the strong closure $E \mathbin{\bar{\odot}^s} F$ of $E \odot F$ in $\mathcal{B}(K, E \mathbin{\bar{\odot}} F \mathbin{\bar{\odot}} K)$ is a von Neumann $\mathcal{A}$–$\mathcal{C}$–module.*

Proof. Let ρ be the canonical representation of $\mathcal{B}$ on $G = F \mathbin{\bar{\odot}} K$ which is normal by Lemma 3.3.2. Then by Corollary 3.3.3 the canonical representation ρ' on $H = E \mathbin{\bar{\odot}} G = E \mathbin{\bar{\odot}} F \mathbin{\bar{\odot}} K$ associates with ρ is normal, too. Therefore, again by Lemma 3.3.2, $E \mathbin{\bar{\odot}^s} F$ is a two-sided von Neumann module. ■

4.2.25 Remark. There is a simple application of Theorem 4.1.11, which allows to generalize the Stinespring construction from $\mathcal{B} \subset \mathcal{B}(G)$ for some Hilbert space G to $\mathcal{B} \subset \mathcal{B}^a(F)$ for some Hilbert $\mathcal{C}$–module F. By Point 4 of the theorem (together with Remark 4.1.12) for a strict completely positive mapping T from a pre–C^*–algebra into $\mathcal{B}^a(F)$ there exists a Hilbert module E with a cyclic vector ξ such that $T(a) = \langle \xi, a\xi \rangle$ and $E = \overline{\mathsf{span}}(\mathcal{A}\xi\mathcal{B}^a(F))$. Then the pre-Hilbert $\mathcal{C}$–module $E \odot F$ carries the total representation $a \mapsto a \odot \mathsf{id}$ of $\mathcal{A}$, which is strict (on bounded subsets), and $\xi \odot \mathsf{id}$ is a mapping in $\mathcal{B}^a(F, E \odot F)$ such that $T(a) = (\xi \odot \mathsf{id})^*(a \odot \mathsf{id})(\xi \odot \mathsf{id})$. This result is known as KSGNS-construction; see [Lan95].

4.3 Exterior tensor product

Besides the tensor product discussed in Section 4.2, there is a second one. This *exterior* tensor product is based on the observation that the (vector space) tensor product $E_1 \otimes E_2$ of a pre-Hilbert $\mathcal{B}_i$–modules E_i $(i = 1, 2)$ is a $\mathcal{B}_1 \otimes \mathcal{B}_2$–module in an obvious way. Also here there are several ways to show that the sesquilinear mapping on $E_1 \otimes E_2$, defined by setting

$$\langle x \otimes y, x' \otimes y' \rangle \;=\; \langle x, x' \rangle \otimes \langle y, y' \rangle, \tag{4.3.1}$$

is a semi-inner product.

By Example 1.7.6 the matrices $\big(\langle x_i, x_j \rangle\big)$ and $\big(\langle y_i, y_j \rangle\big)$ are positive and, therefore, can be written in the form B^*B with $B \in M_n(\overline{\mathcal{B}_1})$ and C^*C with $C \in M_n(\overline{\mathcal{B}_2})$, respectively. We find that also the matrix

$$\big(\langle x_i \otimes y_i, x_j \otimes y_j \rangle\big) \;=\; \sum_{k,\ell} (b_{ki} \otimes c_{\ell i})^*(b_{kj} \otimes c_{\ell j}) \;\in\; M_n(\overline{\mathcal{B}_1} \otimes \overline{\mathcal{B}_2})$$

is positive.

Another possibility is to reduce the exterior tensor product to the tensor product. Consider the extension $E_2 \odot (\widetilde{\mathcal{B}}_1 \otimes \widetilde{\mathcal{B}}_2)$ of E_2 by $\widetilde{\mathcal{B}}_1 \otimes \widetilde{\mathcal{B}}_2$. Obviously, $E_2 \odot (\widetilde{\mathcal{B}}_1 \otimes \widetilde{\mathcal{B}}_2) = \widetilde{\mathcal{B}}_1 \otimes E_2$ equipped with the (semi-)inner product defined by (4.3.1). (Actually, we should say modulo length-zero elements. However, we will see immediately that there are no length-zero elements in $\widetilde{\mathcal{B}}_1 \otimes E_2$ different from 0.) On $\widetilde{\mathcal{B}}_1 \otimes E_2$ we have a natural (non-degenerate) left action of $\widetilde{\mathcal{B}}_1$. Then also $E_1 \odot (\widetilde{\mathcal{B}}_1 \otimes E_2)$ and $E_1 \otimes E_2$ have the same (semi-)inner product (again up to potential length-zero elements). Positivity of the former implies positivity of the latter.

4.3.1 Proposition. *The semi-inner product on $E_1 \otimes E_2$ is inner.*

Proof. Let $z = \sum\limits_i x_i \otimes y_i$ be an arbitrary element of $E_1 \otimes E_2$. We may assume that the x_i form a ($\mathbb{C}$–)linearly independent set. If $\langle z, z \rangle = 0$, then by (1.2.1) we have $\langle u \otimes v, z \rangle =$

$\sum_i \langle u, x_i\rangle \otimes \langle v, y_i\rangle = 0$ for all $u \in E_1, v \in E_2$. For an arbitrary state φ on $\mathcal{B}_2$ we define $\Phi_v \colon z \mapsto \sum_i x_i \varphi(\langle v, y_i\rangle)$. We have $\langle u, \Phi_v(z)\rangle = (\mathsf{id} \otimes \varphi)(\langle u \otimes v, z\rangle) = 0$ for all $u \in E_1$, hence $\Phi_v(z) = 0$. From linear independence of the x_i we conclude that $\varphi(\langle v, y_i\rangle) = 0$ for all i. Since φ and v are arbitrary, we find $y_i = 0$ for all i, i.e. $z = 0$. ■

4.3.2 Definition. $E_1 \otimes E_2$ with inner product defined by (4.3.1) is called the *exterior* tensor product of E_1 and E_2.

Even if the $\mathcal{B}_i$ are C^*–algebras, then there are, in general, several ways to define a C^*–norm on $\mathcal{B}_1 \otimes \mathcal{B}_2$. Of course, also the norm on $E_1 \otimes E_2$ depends on this choice. If one of the algebras is nuclear (for instance, commutative or finite-dimensional), then the norm is unique.

If the E_i are pre-Hilbert $\mathcal{A}_i$–$\mathcal{B}_i$–modules, then $E_1 \otimes E_2$ is an $\mathcal{A}_1 \otimes \mathcal{A}_2$–$\mathcal{B}_1 \otimes \mathcal{B}_2$–module in an obvious way. To see that it is a pre-Hilbert $\mathcal{A}_1 \otimes \mathcal{A}_2$–$\mathcal{B}_1 \otimes \mathcal{B}_2$–module we must check whether the elements of $\mathcal{A}_1 \otimes \mathcal{A}_2$ act as bounded operators. But this follows from the observation that we may complete the E_i and that by Proposition 4.3.1 $E_1 \otimes E_2$ is contained in $\overline{E}_1 \otimes \overline{E}_2$ as a submodule. Now the C^*–algebras $\mathcal{B}^a(\overline{E}_i)$ embed isometrically as $\mathcal{B}^a(\overline{E}_1) \otimes \mathbf{1}$ and $\mathbf{1} \otimes \mathcal{B}^a(\overline{E}_2)$, respectively, into $\mathcal{B}^a(\overline{E}_1 \otimes \overline{E}_2)$. (The embeddings are contractions like any homomorphism form a C^*–algebra into operators on an inner product space. By simple computations on elementary tensors in $E_1 \otimes E_2$ we see that the embedding are not norm decreasing.) As $a_i \in \mathcal{A}_i$ defines an element in $\mathcal{B}^a(E_i)$, $a_1 \otimes a_2 = (a_1 \otimes \mathbf{1})(\mathbf{1} \otimes a_2) = (\mathbf{1} \otimes a_2)(a_1 \otimes \mathbf{1})$ defines an operator in $\mathcal{B}^a(E_1 \otimes E_2)$.

Even if the E_i are contractive, we can guarantee contractivity of $E_1 \otimes E_2$ only, if we equip $\mathcal{A}_1 \otimes \mathcal{A}_2$ with the *projective C^*–norm*, i.e. the greatest C^*–norm, or (as follows by Proposition 4.3.3) if we equip $\mathcal{B}_1 \otimes \mathcal{B}_2$ with the *spatial* norm, i.e. the least C^*–norm. (In fact, setting $E_i = \mathcal{B}_i = \mathcal{A}_i$, we find immediately a counter example, by equipping $\mathcal{B}_1 \otimes \mathcal{B}_2$ with a norm bigger than that of $\mathcal{A}_1 \otimes \mathcal{A}_2$.) Recall that we obtain the spatial norm of $b \in \mathcal{B}_1 \otimes \mathcal{B}_2$ as the operator norm of $(\pi_1 \otimes \pi_2)(b)$ where π_i are arbitrary isometric representations of $\mathcal{B}_i$ on pre-Hilbert spaces G_i.

4.3.3 Proposition. *If $\mathcal{B}_1 \otimes \mathcal{B}_2$ is equipped with the spatial C^*–norm, then the canonical representation $\mathcal{B}^a(E_1) \otimes \mathcal{B}^a(E_2) \to \mathcal{B}^a(E_1 \otimes E_2)$ is an isometry for the spatial norm.*

PROOF. Let (G_i, π_i) $(i = 1, 2)$ be isometric representations of $\mathcal{B}_i$. Then the Stinespring representations $(H_i = E_i \odot G_i, \rho_i)$ are isometric representations of $\mathcal{B}^a(E_i)$ so that the representation $(H_1 \otimes H_2, \rho_1 \otimes \rho_2)$ of $\mathcal{B}^a(E_1) \otimes \mathcal{B}^a(E_2)$ is isometric for the spatial norm on $\mathcal{B}^a(E_1) \otimes \mathcal{B}^a(E_2)$. By definition, the representation $(G_1 \otimes G_2, \pi_1 \otimes \pi_2)$ is an isometric representation of $\mathcal{B}_1 \otimes \mathcal{B}_2$. Therefore, the Stinespring representation $(H = (E_1 \otimes E_2) \odot (G_1 \otimes G_2), \rho)$ is an isometric representation of $\mathcal{B}^a(E_1 \otimes E_2)$. Obviously, $(x_1 \otimes x_2) \odot (g_1 \otimes g_2) \mapsto (x_1 \odot g_1) \otimes (x_2 \odot g_2)$ defines a unitary

$$H = (E_1 \otimes E_2) \odot (G_1 \otimes G_2) \longrightarrow (E_1 \odot G_1) \otimes (E_2 \odot G_2) = H_1 \otimes H_2 \tag{4.3.2}$$

so that the restriction of (H, ρ) to $\mathcal{B}^a(E_1) \otimes \mathcal{B}^a(E_2)$ is unitarily equivalent to $\rho_1 \otimes \rho_2$. It follows that the operator norm of ρ equals that of $\rho_1 \otimes \rho_2$ and, therefore, the spatial one. ■

4.3.4 Remark. The unitary equivalence of $\rho_1 \otimes \rho_2$ and ρ expressed in (4.3.2) allows to identify $E_1 \otimes E_2$ as a subset of $\mathcal{B}^a(G_1, H_1) \otimes \mathcal{B}^a(G_2, H_2) \subset \mathcal{B}^a(G_1 \otimes G_2, H_1 \otimes H_2)$ including the correct norm. If E_i are von Neumann modules, then by the von Neumann module $E_1 \bar{\otimes}^s E_2$ over the von Neumann algebra $\mathcal{B}_1 \bar{\otimes}^s \mathcal{B}_2$ on $G_1 \bar{\otimes} G_2$ we understand the strong closure of $E_1 \otimes E_2$ in $\mathcal{B}(G_1 \bar{\otimes} G_2, H_1 \bar{\otimes} H_2)$. The von Neumann algebras $\mathcal{B}^a(E_1) \bar{\otimes}^s \mathcal{B}^a(E_2)$ and $\mathcal{B}^a(E_1 \bar{\otimes}^s E_2)$ are isomorphic. Moreover, if the E_i are von Neumann $\mathcal{A}_i$–$\mathcal{B}_i$–modules, then the representation $\rho_1 \otimes \rho_2$ extends to a normal representation of the von Neumann algebra $\mathcal{A}_1 \bar{\otimes}^s \mathcal{A}_2$. In other words, $E_1 \bar{\otimes}^s E_2$ is a von Neumann $\mathcal{A}_1 \bar{\otimes}^s \mathcal{A}_2$–$\mathcal{B}_1 \bar{\otimes}^s \mathcal{B}_2$–module.

4.3.5 Corollary. *If $\big(a_\lambda\big)_{\lambda \in \Lambda}$ is an increasing bounded net in $\mathcal{A}_1$ (thus, converging to some $a \in \mathcal{A}_1$), then $\big(a_\lambda \otimes \mathbf{1}\big)_{\lambda \in \Lambda}$ defines an increasing bounded net in $\mathcal{B}^a(E_1 \bar{\otimes}^s E_2)$ such that*

$$\lim_\lambda a_\lambda \otimes \mathbf{1} = a \otimes \mathbf{1}.$$

4.3.6 Remark. If the E_i are pre-Hilbert $\mathbb{C}$–$\mathbb{C}$–modules (i.e. pre-Hilbert spaces), then the (interior) tensor product and the exterior tensor product both coincide with the usual tensor product of pre-Hilbert spaces.

4.3.7 Observation. Proposition 4.3.3 has an obvious analogue for the canonical mapping $\mathcal{B}^a(E_1, F_1) \otimes \mathcal{B}^a(E_2, F_2) \to \mathcal{B}^a(E_1 \otimes E_2, F_1 \otimes F_2)$, where F_i is another pair of pre-Hilbert $\mathcal{B}_i$–modules. (The spatial norm on the left hand side comes by restriction from the spatial norm on $\mathcal{B}^a(E_1 \oplus F_1) \otimes \mathcal{B}^a(E_2 \oplus F_2)$.)

4.3.8 Example. (4.3.2) defines a two-sided unitary also, when the G_i are replaced by arbitrary pre-Hilbert $\mathcal{B}_i$–$\mathcal{C}_i$–modules. We see immediately that the matrix pre-Hilbert modules in Example 4.2.12 have a close relation with the exterior tensor product. Indeed, we easily see that $M_{nm}(E) = E \otimes M_{nm}$. So if we put in (4.3.2) $E_1 = E$, $G_1 = F$, $E_2 = M_{n\ell}$, and $G_2 = M_{\ell m}$, then the identification $M_{n\ell}(E) \odot M_{\ell m}(F) = M_{nm}(E \odot F)$ can be traced back to $M_{n\ell} \odot M_{\ell m} = M_{nm}$.

4.3.9 Example. For us, the most important exterior tensor product is that of a pre-Hilbert $\mathcal{B}$–module E and a pre-Hilbert space $\mathfrak{H}$. By Proposition 4.3.3, $\mathfrak{H}_E := E \otimes \mathfrak{H}$ carries a (non-degenerate) representation of $\mathcal{B}^a(E) \otimes \mathcal{B}^a(\mathfrak{H})$ which is isometric (and, therefore, faithful) for the spatial norm and contractive for any other norm. In particular, if E is a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module and $\mathfrak{H}$ a pre-Hilbert $\mathcal{C}$–$\mathbb{C}$–module, then $\mathfrak{H}_E$ is a pre-Hilbert $\mathcal{A} \otimes \mathcal{C}$–$\mathcal{B}$–module. Clearly, the prototype of a centered module $\mathfrak{H}_\mathcal{B} = \mathcal{B} \otimes \mathfrak{H}$ is a special case of the exterior tensor product.

Let us choose in Observation 4.3.7 the modules $E_1 = F_1 = E$, and $F_2 = \mathcal{B}_2 = \mathbb{C}$, $E_2 = \mathfrak{H}$. We find that for all $f \in \mathfrak{H}$

$$\mathsf{id} \otimes f^* \colon x \otimes g \;\longmapsto\; x\langle f, g\rangle$$

is a mapping in $\mathcal{B}^a(\mathfrak{H}_E, E)$ with with norm $\|f\|$ and with adjoint $(\mathsf{id} \otimes f) \colon x \mapsto x \otimes f$. Clearly, these mappings extend to the completion and, in the case of a von Neumann algebra $\mathcal{B}$ on a Hilbert space G to the strong closures of E in $\mathcal{B}(G, H)$ (with $H = E \,\bar{\odot}\, G$) and of $\mathfrak{H}_E$ in $\mathcal{B}(G, H \,\bar{\otimes}\, \mathfrak{H})$. If $v \in \mathcal{B}^a(\mathfrak{H})$ is an isometry, then

$$(\mathsf{id} \otimes (vf)^*)(x \otimes vg) \;=\; (\mathsf{id} \otimes f^*)(x \otimes g). \tag{4.3.3}$$

Let E, E' be two-sided pre-Hilbert modules and let $\mathfrak{H}, \mathfrak{H}'$ be pre-Hilbert spaces. Then the unitary in (4.3.2) (extended to modules as explained in Example 4.3.8) tells us that $\mathfrak{H}_E \odot \mathfrak{H}'_{E'} = (\mathfrak{H} \otimes \mathfrak{H}')_{E \odot E'}$. In particular, we have a natural left multiplication by elements of $\mathcal{B}^a(\mathfrak{H} \otimes \mathfrak{H}')$.

We close our discussion of the exterior tensor product by looking at Example 4.3.9, when $\mathfrak{H} = L^2(M, \Sigma, \mu) = L^2(M)$ for some measure space (M, Σ, μ) (usually, $\mathbb{R}$ with the Lebesgue measure). We use the notions of Appendix B about function spaces.

4.3.10 Definition. By $L^2(M, E)$ we denote the (contractive) Hilbert $\mathcal{B}^a(\overline{E}) \,\bar{\otimes}\, \mathcal{B}(L^2(M))$–$\overline{\mathcal{B}}$–module (where $\mathcal{B}^a(\overline{E}) \,\bar{\otimes}\, \mathcal{B}(L^2(M))$ denotes the completed spatial tensor product of pre–C^*–algebras.) obtained from $E \otimes L^2(M)$ by norm completion.

By Proposition 4.3.3 the canonical representation of $\mathcal{B}^a(\overline{E}) \,\bar{\otimes}\, \mathcal{B}(L^2(M))$ on $L^2(M, E)$ is faithful. If E is a contractive pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module, then $L^2(M, E)$ is a (contractive, of course) Hilbert $\overline{\mathcal{A}}$–$\overline{\mathcal{B}}$–module, via the canonical homomorphisms $\overline{\mathcal{A}} \to \mathcal{B}^a(\overline{E}) \to \mathcal{B}^a(\overline{E}) \,\bar{\otimes}\, \mathcal{B}(L^2(M))$. In the sequel, we assume that E and $\mathcal{B}$ are complete.

4.3.11 Observation. By Example 4.3.9 we have $L^2(M_1, E_1) \,\bar{\odot}\, L^2(M_2, E_2) = L^2(M_1 \times M_2, E_1 \odot E_2)$. So operators in $\mathcal{B}(L^2(M_1 \times M_2))$, in particular, multiplication operators like indicator functions to measurable subsets of $M_1 \times M_2$ act naturally on $L^2(M_1, E_1) \bar{\odot} L^2(M_2, E_2)$.

Clearly, the integrable simple functions $\mathfrak{E}_0(M,E)$ may be identified with the subspace $E \otimes \mathfrak{E}_0(M)$ of $E \otimes L^2(M)$. Since $\mathfrak{E}_0(M)$ is dense in $L^2(M)$, $\mathfrak{E}_0(M,E)$ is dense in $L^2(M,E)$. By the estimate

$$\|x\|^2 \;=\; \Big\| \int \langle x(t), x(t) \rangle \mu(dt) \Big\| \;\leq\; \int \|x(t)\|^2 \mu(dt) \tag{4.3.4}$$

for $x \in \mathfrak{E}_0$ we see that the Bochner square integrable functions $L^2_B(M,E)$ are contained in $L^2(M,E)$ and that $\langle x, y \rangle = \int \langle x(t), y(t) \rangle$ for $x, y \in L^2_B(M,E)$ in the sense of Example B.1.9. We use this notation for all $x, y \in L^2(M,E)$, although the following example shows that, in general, $L^2(E,M)$ is bigger than $L^2_B(M,E)$.

If μ is finite, then by Paragraph B.1.14 and (4.3.4) we find the estimate

$$\|x\| \;\leq\; \|x\|_2 \;\leq\; \|x\|_{ess} \sqrt{\mu(M)}. \tag{4.3.5}$$

4.3.12 Example. By Example B.1.10, $L^2_B(M,E)$ is a Banach $L^\infty(M, \mathcal{B}^a(E))$–$L^\infty(M,\mathcal{B})$–module. By Corollary 2.3.10, $L^2(M,E)$ is at least a left Banach $L^\infty(M, \mathcal{B}^a(E))$–module and, similarly, by a simple application of Lemma B.1.6, the canonical representation of $L^\infty(M, \mathcal{B}^a(E))$ on $L^2(M,E)$ is isometric. In other words, $L^\infty(M, \mathcal{B}^a(E)) = \mathcal{B}^a(E) \bar{\otimes} L^\infty(M)$ is a C^*–subalgebra of $\mathcal{B}^a(L^2(M,E))$. (We may identify $L^\infty(M, \mathcal{B}^a(E))$ and $\mathcal{B}^a(E) \bar{\otimes} L^\infty(M)$, because the latter is a dense subspace of the former, and because the commutative C^*–algebra $L^\infty(M)$ is nuclear, i.e. there is one and only one C^*–norm on the tensor product.)

Nothing like this is true for the right action of $L^\infty(M,\mathcal{B})$. We consider $\ell^2(\mathcal{B}(G))$ (i.e. $L^2(\mathbb{N}, \mathcal{B}(G))$ equipped with the counting measure on $\mathbb{N}$) where G is a Hilbert space with an ONB $\big(e_n\big)_{n\in\mathbb{N}}$. Let $\big(c_n\big)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ converging to 0, but not square summable, and let $x \in \ell^2(\mathcal{B}(G))$ be the function defined by $x(n) = c_n p_n$ where p_n is the projection onto $\mathbb{C}e_n$. Observe that $\|x\|^2 = \big\| \sum_{n\in\mathbb{N}} |c_n|^2 p_n \big\| = \max |c_n|^2 < \infty$ and that $\big\| \sum_{n\geq N} |c_n|^2 p_n \big\| = \max_{n\geq N} |c_n|^2 \to 0$ for $N \to \infty$ so that x is a norm limit of step functions. Now let $b \in \ell^\infty(\mathcal{B}(G)) = L^\infty(\mathbb{N}, \mathcal{B}(G))$ be the function defined by setting $b(n) = e_n e_1^*$. Then $\|xb\|^2 = \|p_1\| \sum_{n\in\mathbb{N}} |c_n|^2 = \infty$. Therefore, xb is not in $\ell^2(\mathcal{B}(G))$ and *a fortiori* not in $\ell^2_B(\mathcal{B}(G))$. Consequently, x cannot be in $\ell^2_B(\mathcal{B}(G))$ either, because otherwise xb would be in $\ell^2_B(\mathcal{B}(G))$.

Even if not all $x \in L^2(M,E)$ are Bochner square integrable, we still could hope that all x are represented (at least μ–a.e.) by a function (measurable in a suitable sense) with values in E. However, also this is not true, in general. We borrow the following Example from Hellmich [Hel01].

4.3.13 Example. Let $I = [0,1]$ and consider the Hilbert $\mathcal{B}(L^2(I))$–module $L^2\big(I, \mathcal{B}(L^2(I))\big)$ as a subset of $\mathcal{B}(L^2(I), L^2(I \times I))$. The function $K(s,t) = |s-t|^{-\frac{1}{4}}$ defines an operator

K sending the function $f\colon I \to \mathbb{C}$ to the function $Kf\colon I \times I \to \mathbb{C}$ defined by setting $(Kf)(s,t) = K(s,t)f(s)$. For $f \in L^2(I)$ an application of *Fubini's theroem* yields

$$\begin{aligned} \|Kf\|^2 &= \int_0^1 ds \int_0^1 dt\, |s-t|^{-\frac{1}{2}} |f(s)|^2 \\ &= \int_0^1 ds\, |f(s)|^2 \Big[\int_0^s dt\,(s-t)^{-\frac{1}{2}} + \int_s^1 dt\,(t-s)^{-\frac{1}{2}}\Big] \\ &= 2\int_0^1 ds\, |f(s)|^2 (\sqrt{s} + \sqrt{1-s}) \;\le\; 4\|f\|^2. \end{aligned}$$

In other words, K is an element of $\mathcal{B}(L^2(I), L^2(I \times I))$. We show that $K \in L^2\big(I, \mathcal{B}(L^2(I))\big)$. To that goal let $S_\varepsilon = \{(s,t) \in I \times I\colon |s-t| \ge \varepsilon\}$ $(1 > \varepsilon > 0)$ and set $K_\varepsilon = I\!I_{S_\varepsilon} K$. Obviously, $(K - K_\varepsilon)^2 = K^2 - K_\varepsilon^2$ so that $\|(K-K_\varepsilon)f\|^2 = \|Kf\|^2 - \|K_\varepsilon f\|^2$. We find

$$\begin{aligned} \|K_\varepsilon f\|^2 &= \int_\varepsilon^1 ds\, |f(s)|^2 \int_0^{s-\varepsilon} dt\,(s-t)^{-\frac{1}{2}} + \int_0^{1-\varepsilon} ds\, |f(s)|^2 \int_{s+\varepsilon}^1 dt\,(t-s)^{-\frac{1}{2}} \\ &= 2\int_\varepsilon^1 ds\, |f(s)|^2 (\sqrt{s} - \sqrt{\varepsilon}) + 2\int_0^{1-\varepsilon} ds\, |f(s)|^2 (\sqrt{1-s} - \sqrt{\varepsilon}) \end{aligned}$$

and, therefore,

$$\|(K-K_\varepsilon)f\|^2 \;=\; 2\int_0^1 ds\, |f(s)|^2 (\min(\sqrt{s}, \sqrt{\varepsilon}) + \min(\sqrt{1-s}, \sqrt{\varepsilon})) \;\le\; 4\sqrt{\varepsilon}\,\|f\|^2.$$

Clearly, $K_\varepsilon \in L^2\big(I, \mathcal{B}(L^2(I))\big)$ so that also $K = \lim\limits_{\varepsilon \to 0} K_\varepsilon \in L^2\big(I, \mathcal{B}(L^2(I))\big)$. However, none of the operators K_t mapping $f(s)$ to $(K_t f)(s) = (Kf)(s,t)$ is bounded on $L^2(I)$, because the function $s \mapsto K(s,t)$ is not essentially bounded.

Any *measure preserving transformation* $\mathcal{T}$ on M (i.e. a bijection $\mathcal{T}\colon M \to M$ such that $\mu \circ \mathcal{T}(S) = \mu(S)$ for all $S \in \Sigma$) gives rise to a unitary $u_\mathcal{T}\colon f \mapsto f \circ \mathcal{T}$ on $\mathfrak{E}_0(M)$ which extends to a unitary (also denoted by $u_\mathcal{T}$) on $L^2(M)$. By Example 4.3.9, $\mathsf{id} \otimes u_\mathcal{T}$ is a unitary on $E \otimes L^2(M)$ which extends to a unitary on $L^2(M, E)$ (also denoted by $u_\mathcal{T}$). Obviously, $u_\mathcal{T}$ commutes with all operators of the form $a \otimes \mathbf{1}$ where $a \in \mathcal{B}^a(E)$. In other words, if E is a Hilbert $\mathcal{A}$–$\mathcal{B}$–module, then $u_\mathcal{T}$ is $\mathcal{A}$–$\mathcal{B}$–linear. For the mappings $\mathsf{id} \otimes f^*$ $(f \in L^2(M))$ we find

$$(\mathsf{id} \otimes (u_\mathcal{T} f)^*) u_\mathcal{T} \;=\; \mathsf{id} \otimes f^*. \tag{4.3.6}$$

We use the same symbol for the restriction of $u_\mathcal{T}$ to $L^2(K, E)$ where $K \in \Sigma$. In this case, we consider $u_\mathcal{T}$ as isometry $L^2(K, E) \to L^2(K', E)$ where $K' \in \Sigma$ can be any set such that $\mathcal{T}^{-1}(K) \subset K'$. (This follows from $u_\mathcal{T} I\!I_K = I\!I_{\mathcal{T}^{-1}(K)} u_\mathcal{T}$, where the multiplication operator $I\!I_K$ is the $(L^\infty(M, \mathcal{B}^a(E)))$–$\mathcal{B}$–linear) projection from $L^2(M, E)$ onto the submodule $L^2(K, E)$.) In particular, $u_\mathcal{T}$ is a unitary $L^2(K, E) \to L^2(\mathcal{T}^{-1}(K), E)$. If $\mathcal{T}^{-1}(K) \subset K$, then $u_\mathcal{T}$ defines an isometric endomorphism of $L^2(K, E)$. Equation (4.3.6) remains true for the subset K. If K' does not contain all of $\mathcal{T}^{-1}(K)$, then we still may consider the partial isometry $I\!I_{K'} u_\mathcal{T}$.

4.3.14 Example. Let $\mathcal{G}$ be a *locally compact group* equipped with the *right Haar measure* h on $\mathfrak{B}(\mathcal{G})$. Then by definition of the Haar measure, for all $g \in \mathcal{G}$ the right shift $\mathcal{T}_g \colon h \mapsto hg^{-1}$ is a measure preserving transformation on $\mathcal{G}$. Moreover, $\mathcal{T}_g \circ \mathcal{T}_h = \mathcal{T}_{gh}$. In other words, $g \mapsto u_{\mathcal{T}_g}$ is a unitary representation of $\mathcal{G}$ on $L^2(\mathcal{G}, E)$.

For the real line with the Lebesgue measure we use a special notation. By the ***time shift*** $\mathfrak{s}_t$ on $L^2(\mathbb{R}, E)$ we mean (the extension of) the mapping $x \to \mathfrak{s}_t x$ $(t \in \mathbb{R})$ where $[\mathfrak{s}_t x](s) = x(s-t)$ (from simple functions to all of $L^2(\mathbb{R}, E)$). Obviously, $\mathfrak{s}_t$ is the unitary $u_{\mathcal{T}_t}$ comming from the right shift $\mathcal{T}_t \colon s \mapsto s - t$ on $\mathbb{R}$. As $\mathcal{T}_t^{-1}(\mathbb{R}_+) \subset \mathbb{R}_+$ for all $t \in \mathbb{R}_+$, we find that the unitary representation $\mathfrak{s} = \bigl(\mathfrak{s}_t\bigr)_{t \in \mathbb{R}}$ of $\mathbb{R}$ on $L^2(\mathbb{R}, E)$ restricts to a representation of $\mathbb{R}_+$ by isometries on $L^2(\mathbb{R}_+, E)$. Following our convention, we do not distinguish between the unitary $\mathfrak{s}_t$ on $L^2(\mathbb{R}, E)$, its restriction to $L^2(\mathbb{R}_+, E)$ and the unitaries $\mathfrak{s}_t \in \mathcal{B}^a(L^2(K, E), L^2(K+t, E))$ for some Borel set K where $K+t = \mathcal{T}_t^{-1}(K) = \{s+t \colon s \in K\}$.

4.3.15 Proposition. *$\mathfrak{s}$ is strongly continuous on $L^2(\mathbb{R}, E)$ (and, consequently, on any submodule $L^2(K, E)$).*

Proof. $\mathfrak{s}$ is bounded, so it is sufficient to check the statement for elements $\mathbb{I}_{[r,s]} \in \mathfrak{S}(\mathbb{R})$, because by Appendix B.2 $\mathfrak{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ and $E \otimes L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R}, E)$. For $|t| < (s-r)$ we have $\bigl|\mathbb{I}_{[r,s]+t} - \mathbb{I}_{[r,s]}\bigr|^2 = \mathbb{I}_I$ for a suitable union I of two intervals where $\lambda(I) < 2t$. From this it follows that $\mathfrak{s}_t \mathbb{I}_{[r,s]} \to \mathbb{I}_{[r,s]}$ for $t \to 0$. ■

Among other (dense) submodules of E–valued functions contained in $L^2(\mathbb{R}, E)$ the most important for us is the space $\mathcal{C}_c(\mathbb{R}, E)$ of continuous function with compact support. Clearly, $\mathcal{C}_c(\mathbb{R}, E) \subset L^2_B(\mathbb{R}, E)$ so that the inner product $\langle x, y \rangle = \int \langle x(t), y(t) \rangle \, dt$ makes sense as a Bochner integral. However, we may understand it also as Riemann integral as discussed in Appendix B.1. Clearly, $\mathcal{C}_c(\mathbb{R}, B)$ is invariant under left *and* right multiplication by elements in $\mathcal{C}(\mathbb{R}, \mathcal{B}^a(E)) \subset L^\infty_{loc}(\mathbb{R}, \mathcal{B}^a(E))$, although by Example 4.3.12 right multiplication may act unboundedly. Notice that $\mathfrak{s}_t$ leaves invariant $\mathcal{C}_c(\mathbb{R}, E)$ and all other dense submodules of $L^2(\mathbb{R}, E)$ considered so far.

4.3.16 Definition. Let $E \subset \mathcal{B}(G, H)$ be a von Neumann $\mathcal{B}$–module where $\mathcal{B}$ is a von Neumann algebra on a Hilbert space G and $H = E \bar{\odot} G$. By $L^{2,s}(M, E)$ we denote the von Neumann $\mathcal{B}^a(E) \bar{\otimes}^s \mathcal{B}(L^2(M))$–$\mathcal{B}$–module $E \bar{\otimes}^s L^2(M) \subset \mathcal{B}(G, H \bar{\otimes} L^2(M)) = \mathcal{B}(G, L^2(M, H))$ (where $\mathcal{B}^a(E) \bar{\otimes}^s \mathcal{B}(L^2(M)) = \mathcal{B}^a(L^{2,s}(M, E))$ is the tensor product of von Neumann algebras, i.e. the strong closure of the spatial tensor product in $\mathcal{B}(H \bar{\otimes} L^2(M))$.)

All operators on $L^2(M, E)$ described so far extend strongly continuously to $L^{2,s}(M, E)$. Proposition 4.3.15 remains true in the strong topology of $\mathcal{B}(G, L^2(\mathbb{R}, E))$ by the same reasoning. Additionally, we have the strongly dense submodule $\mathcal{C}^s_c(\mathbb{R}, E)$ of strongly continuous functions with compact support which is invariant under the action of functions in $\mathcal{C}^s(\mathbb{R}, \mathcal{B}^a(E))$. Now the inner product may be understood as Bochner or Riemann integral in the weak sense, i.e. $\langle g, \langle x, x' \rangle g' \rangle = \int \langle g, \langle x(t), x'(t) \rangle g' \rangle \, dt$ for all $g, g' \in G$.

4.4 Conditional expectations

Unital completely positive mappings may be considered as a generalization of the notion of a *state* on a unital $*$–algebra $\mathcal{A}$, that is a linear functional $\varphi \colon \mathcal{A} \to \mathbb{C}$ which is *unital* (i.e. $\varphi(\mathbf{1}) = 1$) and positivity preserving (i.e. $a \geq 0$ implies $\varphi(a) \geq 0$), to mappings on $\mathcal{A}$ with values in another $*$–algebra $\mathcal{B}$. But this is not the only thinkable possibility for a generalization.

On the one hand, we can consider unital mappings $T \colon \mathcal{A} \to \mathcal{B}$ which are just positive, not necessarily completely positive. Under such weak conditions we are not able to guarantee that the $\mathcal{B}$–valued inner product induced by T on $\mathcal{A}$ extends to the right $\mathcal{B}$–module $\mathcal{A} \otimes \mathcal{B}$. Also constructions like the tensor product are no longer possible. Remarkably enough, in the framework of von Neumann algebras there exist quite a lot of deep results on semigroups of unital positive mappings on a von Neumann with an invariant (faithful, normal) weight; see e.g. [GL99] and related works. The basis for our applications are GNS-construction and tensor product. Consequently, we do not tackle the difficult problems arround positive mappings on C^*–algebras.

On the other hand, we can require additional conditions for T. For instance, if φ is a state on $\mathcal{A}$, then we may define the mapping $T \colon \mathcal{A} \to \mathcal{B} = \mathbb{C}\mathbf{1} \subset \mathcal{A}$ by setting $T(a) = \varphi(a)\mathbf{1}$. This mapping is a projection, it has norm 1, it is $\mathcal{B}$–$\mathcal{B}$–linear (because it is $\mathbb{C}$–linear) and, therefore it is completely positive.

4.4.1 Definition. Let $\mathcal{A}$ be a pre–C^*–algebra and let $\mathcal{B}$ be a pre–C^*–subalgebra of $\mathcal{A}$. A *conditional expectation* is a surjective positive $\mathcal{B}$–$\mathcal{B}$–linear mapping $\varphi \colon \mathcal{A} \to \mathcal{B}$ of norm 1. We say a conditional expectation is *faithful*, if $\varphi(a^*a) = 0$ implies $a = 0$, and we say it is *essential*, if its GNS-representation (φ is completely positive by Remark 4.4.2!) is isometric.

Following Voiculescu [Voi95] and Speicher [Spe98], by a *$*$–$\mathcal{B}$–algebra* we understand a unital $*$–algebra $\mathcal{A}$ with a $*$–subalgebra $\mathcal{B}$ containing the unit of $\mathcal{A}$. We use similar notions for (pre–)C^*–algebras or von Neumann algebras (if $\mathcal{B}$ is a von Neumann algebra). In this case we require that the restriction of further structures from $\mathcal{A}$ to $\mathcal{B}$ gives back the correct structures on $\mathcal{B}$. A *$\mathcal{B}$–quantum probability space* is a pair $(\mathcal{A}, \varphi)$ of a pre–C^*–$\mathcal{B}$–algebra $\mathcal{A}$ and a conditional expectation onto $\mathcal{B}$.

4.4.2 Remark. From positivity and $\mathcal{B}$–$\mathcal{B}$–linearity we conclude that a conditional expectation $\varphi\colon \mathcal{A} \to \mathcal{B}$ is completely positive. By Theorem 4.1.11(3) also the unital extension $\widetilde{\varphi}\colon \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is completely positive and has norm 1. If we identify the new units of $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$, then we still have that $\widetilde{\mathcal{B}} \subset \widetilde{\mathcal{A}}$ and it is easy to check that $\widetilde{\varphi}$ is a conditional expectation onto $\widetilde{\mathcal{B}}$.

4.4.3 Remark. A conditional expectation is a projection onto $\mathcal{B}$. (For unital $\mathcal{B}$ this follows from surjectivitiy and the observation that $\varphi(\mathbf{1})$ is a central projection such that $\varphi(\mathcal{B}) = \varphi(\mathbf{1})\mathcal{B}$. For non-unital $\mathcal{B}$ we apply this to $\widetilde{\varphi}$ as in Remark 4.4.2.) Conversely, one can show that any norm-one projection is a conditional expectation; see e.g. [Tak79].

4.4.4 Remark. Let $\mathcal{B}$ be unital. Then for any $a \in \mathcal{A}$ for which $a \geq \mathbf{1}_\mathcal{B}$ and $\|a\| \leq 1$ we have $\varphi(a) = \mathbf{1}_\mathcal{B}$. (Indeed, $\mathbf{1}_\mathcal{B} = \varphi(\mathbf{1}_\mathcal{B}) \leq \varphi(a) = \varphi(\mathbf{1}_\mathcal{B} a \mathbf{1}_\mathcal{B}) \leq \|a\| \mathbf{1}_\mathcal{B} \leq \mathbf{1}_\mathcal{B}$, whence $0 \leq \varphi(a) - \mathbf{1}_\mathcal{B} \leq 0$.) In particular, if $\mathcal{A}$ is unital, then $\varphi(\mathbf{1}_\mathcal{A}) = \mathbf{1}_\mathcal{B}$.

4.4.5 Observation. Let $(\widetilde{E}, \widetilde{\xi})$ denote the GNS-construction of $\widetilde{\varphi}$. From $\left\|b\widetilde{\xi} - \widetilde{\xi}b\right\|^2 = \widetilde{\varphi}(b^*b) - \widetilde{\varphi}(b^*)b - b^*\widetilde{\varphi}(b) + b^*b = 0$ for all $b \in \widetilde{\mathcal{B}}$, we conclude that $b\widetilde{\xi} = \widetilde{\xi}b$. In particular, if $\mathcal{B}$ is unital, then we have $a\widetilde{\xi} = a\widetilde{\xi}\mathbf{1}_\mathcal{B}$ for all $a \in \mathcal{A}$ (because $\langle x, a\widetilde{\xi}\rangle = \langle x, a\widetilde{\xi}\rangle \mathbf{1}_\mathcal{B} = \langle x, a\widetilde{\xi}\mathbf{1}_\mathcal{B}\rangle$ for all $x \in \widetilde{E}$), so that $E = \mathcal{A}\xi$ is a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module with a cyclic unit vector $\xi = \widetilde{\xi}\mathbf{1}_\mathcal{B}$ such that $\varphi(a) = \langle \xi, a\xi\rangle$. Moreover, $b\xi = \xi b$ for all $b \in \mathcal{B}$. For unital $\mathcal{B}$ we refer, as usual, to the pair (E, ξ) as the GNS-construction of φ.

Notice that in the GNS-construction for a conditional expectation, we may start with $\mathcal{A}$ (instead of $\mathcal{A} \otimes \mathcal{B}$) and turn it into a semi-Hilbert $\mathcal{B}$–module with semiinner product $\langle a, a'\rangle = \varphi(a^*a')$. This is more similar to the GNS-construction for states. We see that also in this sense the conditional expectations are that generalization of states which behaves most similar to states.

4.4.6 Example. Let $\mathcal{A}$ be a pre–C^*–algebra with a projection p. Then the *compression* $\varphi_p\colon a \mapsto pap$ is a conditional expectation onto the unital pre–C^*–subalgebra $p\mathcal{A}p$ of $\mathcal{A}$. Obviously, the GNS-module is just $\mathcal{A}p$ (i.e. the left ideal in $\mathcal{A}$ generated by p), and the cyclic vector is p. The ideal $\mathsf{span}(\mathcal{A}p\mathcal{A})$ in $\mathcal{A}$ generated by p consists precisely of the finite-rank operators on $\mathcal{A}p$.

4.4.7 Proposition. *φ_p is faithful, if and only if $p = \mathbf{1}$, whence $\varphi_p = \mathsf{id}_\mathcal{A}$.*

Proof. If $p = \mathbf{1}$, then there is nothing to show. Thus, suppose that p is not a unit for $\mathcal{A}$. If $\mathcal{A}$ is unital, then for $p_\perp = \mathbf{1} - p \neq 0$ we find $\varphi_p(p_\perp^* p_\perp) = 0$ so that φ_p is not faithful.

If $\mathcal{A}$ is non-unital, we consider $p_\perp = \widetilde{\mathbf{1}} - p \in \widetilde{\mathcal{A}}$ and find again that $\varphi_p(p_\perp^* p_\perp) = 0$. By Cauchy-Schwartz inequalitiy we conclude that $\varphi_p((a-ap)^*(a-ap)) = \varphi_p((a^*ap_\perp)^*p_\perp) = 0$. If φ_p was faithful, then this implies $a = ap$ for all $a \in \mathcal{A}$ contradicting the assumption that p is not a unit for $\mathcal{A}$. ■

4.4.8 Proposition. *φ_p is essential, if and only if the ideal $\operatorname{\mathsf{span}}(\mathcal{A}p\mathcal{A})$ generated by p is essential.*

Proof. φ_p is essential, if and only if for all $a \neq 0$ in $\overline{\mathcal{A}}$ there exists $a'p$ in the GNS-module $\mathcal{A}p$ of φ_p such that $aa'p \neq 0$ (in $\overline{\mathcal{A}}$). If this condition is fulfilled, then by Proposition A.7.3(1) $aa'pa'^* \neq 0$, so that $\operatorname{\mathsf{span}}(\mathcal{A}p\mathcal{A})$ is essential. On the other hand, if there exists $a \in \overline{\mathcal{A}}$ different from 0 such that $aa'p = 0$ for all $a' \in \mathcal{A}$, then $ac = 0$ for all $c \in \operatorname{\mathsf{span}}(\mathcal{A}p\mathcal{A})$ so that $\operatorname{\mathsf{span}}(\mathcal{A}p\mathcal{A})$ is not essential. ■

4.4.9 Example. We show that an algebraic version of 'essential' is not sufficient. Consider the $*$–algebra $\mathbb{C}\langle x\rangle$ of polynomials in one self-adjoint indeterminate. By $p \mapsto p(x)$ we define a homomorphism from $\mathbb{C}\langle x\rangle$ into the C^*–algebra of continuous functions on the subset $\{0\} \cup [1,2]$ of $\mathbb{R}$. Denote by $\mathcal{A}$ the image of $\mathbb{C}\langle x\rangle$ under this homomorphism. Furthermore, choose the ideal I in $\mathcal{A}$ consisting of all functions which vanish at 0. Clearly, I separates the points of $\mathcal{A}$. But, the completion of $\mathcal{A}$ contains just all continuous functions. These are no longer separated by I as $I\!I_{\{0\}} \in \mathcal{C}(\{0\} \cup [1,2])$ and $cI\!I_{\{0\}} = 0$ for all $c \in I$. Another way to say this is that the conditional expectation $\varphi\colon \mathcal{A} \to I$ defined by $[\varphi(f)](x) = f(x) - f(0)$ (being a homomorphism extending to $\overline{\mathcal{A}}$, the norm of φ is 1) is faithful, but not essential.

4.4.10 Example. Let E be a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module where $\mathcal{A}$ is a pre–C^*–algebra with a unital C^*–subalgebra $\mathcal{B}$, and let ξ be a unit vector in E. Then $\varphi(a) = \langle \xi, a\xi\rangle$ defines a conditional expectation, if and only if ξ commutes with all $b \in \mathcal{B}$.

On the other hand, if E is a pre-Hilbert $\mathcal{B}$–module, then for any unit vector $\xi \in E$ we define a faithful representation $j(b) = \xi b \xi^*$ of $\mathcal{B}$ on E and ξ *intertwines* $j(b)$ and b, i.e. $j(b)\xi = \xi b$. Observe that E is a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module via j, if and only if E is isomorphic to $\mathcal{B}$. Indeed, if j acts non-degenerately, then for all $x \in E$ we have $x = j(\mathbf{1})x = \xi\langle \xi, x\rangle$. Therefore, $\xi \mapsto \mathbf{1}$ extends as a two-sided isomorphism $E \to \mathcal{B}$. In general, $p := j(\mathbf{1}) = \xi\xi^*$ is only a projection.

By Example 4.4.6, $\varphi_p(a) = pap = j(\langle \xi, a\xi\rangle)$ defines a conditional expectation $\mathcal{B}^a(E) \to j(\mathcal{B})$. We could tend to identify $\mathcal{B}$ with $j(\mathcal{B})$. (Then φ_p maybe considered as a conditional expectation $\mathcal{B}^a(E) \to \mathcal{B}$.) However, if E is the pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module considered before, then we have to distinguish clearly between the action of $\mathcal{B} \subset \mathcal{A}$ (which, for instance, is unital, if $\mathbf{1}_\mathcal{B} = \mathbf{1}_\mathcal{A}$) and $j(\mathcal{B})$.

4.4.11 Example. We obtain an extension of Example 4.4.10, if we replace $\mathcal{B}$ by a pre-Hilbert $\mathcal{B}$–$\mathcal{C}$–module F, in the following sense. Let again ξ be a unit vector in E. Then $a \mapsto (\xi^* \odot \mathsf{id})a(\xi \odot \mathsf{id})$ defines a conditional expectation $\mathcal{B}^a(E \odot F) \to \mathcal{B}^a(F)$ (cf. the proof of Theorem 4.2.18). We denote this conditional expectation by $\varphi_p \odot \mathsf{id}$, because its restriction to $\mathcal{B}^a(E) \odot \mathcal{B}^{a,bil}(F)$, indeed, maps $a \odot a'$ to $\varphi_p(a)a'$. Also here we may embed $\mathcal{B}^a(F)$ (in general, not unit preservingly) into $\mathcal{B}^a(E \odot F)$ via $k\colon a \mapsto (\xi \odot \mathsf{id})a(\xi^* \odot \mathsf{id})$. We find $(\varphi_p \odot \mathsf{id}) \circ k = \mathsf{id}$ and $k \circ (\varphi_p \odot \mathsf{id}) = \varphi_{p \odot \mathsf{id}}$.

In general, unlike for operators on the tensor product of Hilbert spaces we do not have the possibility to embed $\mathcal{B}^a(F)$ into $\mathcal{B}^a(E \odot F)$ unitally. For this it is (more or less) necessary that E is already a left $\mathcal{B}^a(F)$–module. In order that $\varphi_p \odot \mathsf{id}$ defines a conditional expectation onto this subalgebra, it is necessary and sufficient that $\xi \odot \mathsf{id}$ commutes with $\mathcal{B}^a(F)$.

On the contrary, if F is a two-sided pre-Hilbert $\mathcal{B}$–module, and if ζ is a unit vector in the center of F (whence, $\zeta \in \mathcal{B}^a(\mathcal{B}, F)$ and ζ^* are $\mathcal{B}$–$\mathcal{B}$–linear mappings), then $\mathsf{id} \odot \langle \zeta, \bullet \zeta \rangle\colon a \mapsto (\mathsf{id} \odot \zeta^*)a(\mathsf{id} \odot \zeta)$ defines a conditionally expectation $\mathcal{B}^a(E \odot F) \to \mathcal{B}^a(E)$. Here $a \mapsto \big[(\mathsf{id} \odot \langle \zeta, \bullet \zeta \rangle)(a)\big] \odot \mathsf{id}$ is, indeed, a conditional expectation $\mathcal{B}^a(E \odot F) \to \mathcal{B}^a(E) \odot \mathsf{id} \subset \mathcal{B}^a(E \odot F)$, whereas, $\varphi_{\mathsf{id} \odot \zeta\zeta^*}$ is a conditional expectation $\mathcal{B}^a(E \odot F) \to \mathcal{B}^a(E) \odot \zeta\zeta^*$.

4.4.12 Example. We consider the centered pre-Hilbert $\mathcal{B}$–module $\mathfrak{H}_{\mathcal{B}} = \mathcal{B} \otimes \mathfrak{H}$ for a unital pre–C^*–algebra $\mathcal{B}$ and for some pre-Hilbert space $\mathfrak{H}$ and choose a vector $\Omega \in \mathfrak{H}$. Then $\omega = \mathbf{1} \otimes \Omega$ is in $C_{\mathcal{B}}(\mathfrak{H}_{\mathcal{B}})$ such that by setting $\varphi(a) = \langle \omega, a\omega \rangle$ we define a conditional expectation $\varphi\colon \mathcal{B}^a(\mathfrak{H}_{\mathcal{B}}) \to \mathcal{B}$. Clearly, $\mathfrak{H}_{\mathcal{B}}$ is the GNS-module of φ and ω its cyclic vector. (Observe that $\mathcal{B}^a(\mathfrak{H}_{\mathcal{B}})$ contains $\mathcal{B} \otimes \mathcal{B}^a(\mathfrak{H})$ as a strongly dense subalgebra.)

Notice that here $\mathcal{B} \subset \mathcal{B}^a(\mathfrak{H}_{\mathcal{B}})$ contains the unit of $\mathcal{B}^a(\mathfrak{H}_{\mathcal{B}})$. Now set $p = \omega\omega^* \in \mathcal{B}^a(\mathfrak{H}_{\mathcal{B}})$. Then $\varphi_p = pap = \omega\varphi(a)\omega^* = j \circ \varphi(a)$ where $j(b) = \omega^* b\omega$ is another faithful but (by Proposition 4.4.7) usually non-unital representation of $\mathcal{B}$ on E. So again the operators b and $j(b)$ on E are, in general, very much different. However, notice that $bj(\mathbf{1}) = j(\mathbf{1})b = j(b)$.

Let assume that $\mathcal{B} \subset \mathcal{B}(G)$, where G is some pre-Hilbert space, and do the Stinespring representation so that any element $a \in \mathcal{B}^a(\mathfrak{H}_{\mathcal{B}})$ may be identified with an operator on $G \otimes \mathfrak{H}$. In this identification we have $p = j(\mathbf{1}_{\mathcal{B}}) = \mathsf{id} \otimes \Omega\Omega^*$, whereas $\mathbf{1}_{\mathcal{B}} = \mathsf{id} \otimes \mathsf{id}$. Notice that $p = \omega\omega^*$ shows clearly that p leaves invariant E, whereas this statement appears already less clear, if we write only $p = \mathsf{id} \otimes \Omega\Omega^*$. Finally, if we start with a submodule $E \subset \mathfrak{H}_{\mathcal{B}} \subset \mathcal{B}^a(G, G \otimes \mathfrak{H})$, then invarince of E under $p = \mathsf{id} \otimes \Omega\Omega^*$ appears to be unclear, whereas the statement is clear for $p = \omega\omega^*$, as soon as $\omega \in E$. (This criterion is sufficient, but not necessary. This follows immediately by considering $\mathbb{C}_{\mathcal{B}} = \mathcal{B}$ with a submodule $E = q\mathcal{B}$ where q is a central projection. Then $\omega = \mathbf{1}$ is not in E, but $p = \mathbf{1}$, clearly, leaves invariant E.)

If $\mathcal{B} = \mathcal{B}(G)$ where G and $\mathfrak{H}$ are Hilbert space G, then (by strong closure) we obtain the usual conditional expectation $\varphi\colon \mathcal{B}^a(\overline{\mathfrak{H}_{\mathcal{B}}}^s) = \mathcal{B}(G \bar{\otimes} \mathfrak{H}) \to \mathcal{B} = \mathcal{B}(G)$. By Example 3.3.4, Proposition 4.1.13, and Observation 4.4.5 it follows that all normal conditional expectations from $\mathcal{B}(H)$ onto a von Neumann subalgebra isomorphic to $\mathcal{B}(G)$ containing the unit of $\mathcal{B}(H)$ must be of the stated form.

4.4.13 Example. Let G be a Hilbert space with ONB $\left(e_i\right)_{i\in\mathcal{I}}$ and consider the two-sided von Neumann $\mathcal{B}(G)$–module $E = \overline{G_{\mathcal{B}(G)}}^s = \mathcal{B}(G, G \bar{\otimes} G)$. By $\xi = \sum\limits_{i\in\mathcal{I}} (e_i \otimes e_i)e_i^*$ we define a unit vector in E. The range of the mapping $\varphi\colon \mathcal{B}(G) \to \mathcal{B}(G)$ defined by setting $\varphi(a) = \langle \xi, a\xi \rangle = \sum\limits_{i\in\mathcal{I}} e_i \langle e_i, ae_i \rangle e_i^*$ is the commutative subalgebra $\mathcal{C}_e$ of $\mathcal{B}(G)$ consisting of all elements of the form $c = \sum\limits_{i\in\mathcal{I}} c_i e_i e_i^*$ $(\left(c_i\right) \in \ell^\infty)$. Observe that $\mathcal{C}_e$ contains the unit of $\mathcal{B}(G)$. Embedding $\mathcal{C}_e \subset \mathcal{B}(G)$, we see that φ is a conditional expectation. One can show that, if G is infinite-dimensional and separable, then all commutative von Neumann subalgebras (containing the unit of $\mathcal{B}(G)$) onto which there exists a (normal) conditional expectation are of the form $\mathcal{C}_e$ for a suitable ONB of G; see Stormer [Sto72].

As two-sided von Neumann $\mathcal{B}(G)$–module, E is generated by ξ. (Indeed, $e_j e_i^* \xi e_i e_k^* = (e_j \otimes e_i)e_k^*$ so that the strongly closed submodule of E generated by ξ contains a dense subset of the finite-rank operators and, hence, all of E.) However, E is not (the strong closure of) the GNS-module of φ, because the $\mathcal{B}(G)$–$\mathcal{C}_e$–submodule of E generated by ξ is the strongly closed span of operators of the form $(e_j \otimes e_i)e_i^*$. Therefore, (the strong closure of) the GNS-module is the exterior tensor product $G \bar{\otimes}^s \mathcal{C}_e$ with the natural action $\mathcal{B}(G) \otimes \mathsf{id}$ of $\mathcal{B}(G)$ and, of course, $\xi = \sum\limits_{i\in\mathcal{I}} e_i \otimes (e_i e_i^*)$.

4.4.14 Example. After having investigated conditional expectations from $\mathcal{B}(H)$, in some sense the most noncommutative case, to $\mathcal{B}(G)$ and to a commutative algebra $\mathcal{C}_e$, we come to the purely commutative case, describing the procedure of *conditioning* in classical probability.

Let (M, Σ, μ) and (M', Σ', μ') be *probability spaces* such that $M' = M$, $\Sigma' \subset \Sigma$, and $\mu' = \mu \restriction \Sigma'$. Since the simple functions $\mathfrak{E}(M')$ on M' are contained in the simple function $\mathfrak{E}(M)$ on M, we find that $L^\infty(M')$ is a (von Neumann) subalgebra of the (von Neumann) algebra $L^\infty(M)$. Obviously, $\mathbf{1} = 𝟙_M = 𝟙_{M'} \in L^\infty(M') \subset L^\infty(M)$. Since μ is a *probability measure*, we have $\mathfrak{E}_0(M) = \mathfrak{E}(M)$ and similarly for M'. It follows that also $L^2(M') \subset L^2(M)$. By Example B.1.10 the canonical representations of $L^\infty(M^{(\prime)})$ on $L^2(M^{(\prime)})$ are isometric.

Denote by $p \in \mathcal{B}(L^2(M))$ the projection onto $L^2(M')$. Besides having norm 1 and being (completely) positive, the mapping $\varphi_p\colon \mathcal{B}(L^2(M)) \to \mathcal{B}(L^2(M'))$ leaves invariant $L^\infty(M')$ (indeed, for $\psi \in L^\infty(M')$ and $f \in L^2(M')$ we have $p\psi p f = p\psi f = \psi f$) so that the range of $\varphi_p \restriction L^\infty(M)$ contains $L^\infty(M')$, and (by a similar computation) φ_p is

$L^\infty(M')$–$L^\infty(M')$–linear. If we show now that φ_p maps into $L^\infty(M')$, then we establish φ_p as a conditional expectation and, additionally, we see that the GNS-module of φ_p is $\mathcal{B}(L^2(M'), L^2(M))$ with the canonical mapping $i\colon L^2(M') \to L^2(M)$ as cyclic vector. However, to see this property, we need to work slightly more.

Let $\psi \in L^\infty(M)$. Then $\mu'_\psi\colon S \mapsto \int_S \psi(x)\,\mu(dx)$ is a ($\mathbb{C}$–valued) measure on (M', Σ'), absolutely continuous with respect to μ'. Therefore, by the *Radon-Nikodym theorem* there exists a unique element $\varphi(\psi) \in L^\infty(M')$ such that $\mu'_\psi(S) = \int_S [\varphi(\psi)](x)\,\mu'(dx)$. Obviously, φ is a conditional expectation. We show that $\varphi = \varphi_p$. An operator a on $L^2(M')$ is determined by the matrix elements $\langle f, ag\rangle$ where $f, g \in L^2(M')$. It is even sufficient to compute $\langle f, ag\rangle$ only for $f = \mathbb{I}_S, g = \mathbb{I}_{S'}$ where either $S = S' \in \Sigma'$ or $S \cap S' = 0$. We find $\langle f, \varphi_p(\psi)g\rangle = \langle f, \psi g\rangle = \delta_{SS'} \int_S \psi(x)\,\mu(dx) = \delta_{SS'} \int_S [\varphi(\psi)](x)\,\mu'(dx) = \langle f, \varphi(\psi)g\rangle$.

The preceding construction depends on the measure μ. It enters the *Radon-Nikodym derivative*, but also the projection p changes with μ, because μ determins which pairs of elements in $L^2(M)$ are orthogonal. However, $L^\infty(M)$ is determined by μ only up to *equivalence of measures* (two measures μ and ν are equivalent, if each is absolutely continuous with respect to the other). We see that there are many conditional expectations $L^\infty(M) \to L^\infty(M')$. However, by definition of φ we see that φ leaves *invariant* the state $\varphi_\mu\colon \psi \mapsto \int \psi(x)\,\mu(dx)$, i.e. $\varphi_\mu(\psi) = \varphi_\mu \circ \varphi(\psi)$. Equality of φ and φ_p also shows that φ is determined uniquely by this invariance condition.

Chapter 5

Kernels

Until now we presented more or less well-known results on Hilbert modules, maybe, in a new presentation, particularly suitable for our applications. Now we start presenting new results. The results in this chapter appeared probably first in Barreto, Bhat, Liebscher and Skeide [BBLS00].

Positive definite kernels on some set S with values in $\mathbb{C}$ (i.e. functions $k\colon S \times S \to \mathbb{C}$ such that $\sum\limits_{i,j} \overline{c}_i k^{\sigma_i,\sigma_j} c_j \geq 0$ for all choices of finitely many $c_i \in \mathbb{C}, \sigma_i \in S$) are well-established objects. There are basically two important results on such kernels.

One is the *Kolmogorov decomposition* which provides us with a Hilbert space H and an embedding $i\colon S \to H$ (unique, if the set $i(S)$ is total) such that $k^{\sigma,\sigma'} = \langle i(s), i(s') \rangle$. Literally every dilation theorem in older literature, be it the *Stinespring construction* for a single completely positive mapping, be it the reconstruction theorem for quantum stochastic processes from Accardi, Frigerio and Lewis [AFL82], be it the *Daniell-Kolmogorov construction* for classical Markov processes, or be it the construction of the weak dilation of a CP-semigroup in Bhat and Parthasarathy [BP94, BP95], start by writing down *ad hoc* a positive definite kernel, and then show that the Hilbert space obtained by Kolmogorv decomposition carries the desired structures (for instance, the Stinespring representation in the Stinespring construction; cf. Remark 4.1.9). It is not always easy to establish that the kernel in question is positive definite. For instance, the construction of a *weak Markov flow* by Belavkin [Bel85] (as for the rest, being very similar to that of [BP94]) starts with the assumption that the kernel be positive definite.

The other main result is that the *Schur product* of two positive definite kernels (i.e. the pointwise product on $S \times S$) is again positive definite. Semigroups of such kernels were studied, for instance, in Guichardet [Gui72] or Parthasarathy and Schmidt [PS72]. The kernel ℓ obtained by (pointwise) derivative at $t = 0$ of such a semigroup is *conditionally positive definite* ($\sum\limits_{i,j} \overline{c}_i \ell^{\sigma_i,\sigma_j} c_j \geq 0$ for all choices of finitely many $c_i \in \mathbb{C}, \sigma_i \in S$ such that $\sum\limits_i c_i = 0$), and any such kernel defines a positive definite semigroup via (pointwise) exponential.

The goal of this chapter is to find suitable generalizations of the preceding notions to the $\mathcal{B}$–valued case. Suitable means, of course, that we will have plenty of occasions to see these notions at work. Positive definite $\mathcal{B}$–valued kernels together with the Kolmogorov decomposition generalize easily (Section 5.1). They are, however, not sufficient, mainly, because for noncommutative $\mathcal{B}$ the pointwise product of two kernels does not preserve positive definiteness. For this reason we have to pass to *completely positive definite kernels* (Section 5.2). These kernels take values in the bounded mappings on the C^*–algebra $\mathcal{B}$, fulfilling a condition closely related to complete positivity. Instead of the pointwise product of elements in $\mathcal{B}$ we consider the composition (pointwise on $S \times S$) of mappings on $\mathcal{B}$. Also here we have a Kolmogorov decomposition for a completely positive definite kernel, we may consider *Schur semigroups* of such (CPD-semigroups) and their generators (Section 5.4).

Both completely positive mappings and completely positive definite kernels have realizations as *matrix elements* with vectors of a suitably constructed two-sided Hilbert module. In both cases we can understand the composition of two such objects in terms of the tensor product of the underlying Hilbert modules (GNS-modules or Kolmogorov modules). In fact, we find the results for completely positive definite kernels by reducing the problems to completely positive mappings (between $n \times n$–matrix algebras) with the help of Lemmata 5.2.1 and 5.4.6, and then applying the crucial Examples 1.7.7 and 4.2.12. In both cases the tensor product plays a distinguished role. An attempt to realize a whole semigroup, be it of mappings or of kernels, on the same Hilbert module, leads us directly to the notion of tensor product systems of Hilbert modules. We follow this idea in Part III.

It is a feature of CPD-semigroups on S that they restrict to (and are straightforward generalizations of) CP-semigroups, when $S = \{s\}$ consists of a single element. Sometimes, the proofs of analogue statements are straightforward analogues. However, often they are not. In this chapter we put emphasis on the first type of statements which, therefore, will help us in Part III to analyze product systems. To prove the other type of statements we have to wait for Part III.

Although slightly different, our notion of completely positive definite kernels is inspired very much by the corresponding notion in Accardi and Kozyrev [AK99]. The idea to consider CP-semigroups on $M_n(\mathcal{B})$ (of which the CPD-semigroups "generated" by certain *exponential units* as explained in Section 7.3 are a direct generalization) is entirely due to [AK99].

5.1 Positive definite kernels

5.1.1 Definition. Let S be a set and let $\mathcal{B}$ be a pre–C^*–algebra. A *$\mathcal{B}$–valued kernel* or short *kernel* on S is a mapping $\mathfrak{k}\colon S \times S \to \mathcal{B}$. We say a kernel $\mathfrak{k}$ is *positive definite*, if

$$\sum_{\sigma,\sigma' \in S} b_\sigma^* \mathfrak{k}^{\sigma,\sigma'} b_{\sigma'} \;\geq\; 0 \tag{5.1.1}$$

for all choices of $b_\sigma \in \mathcal{B}$ $(\sigma \in S)$ where only finitely many b_σ are different from 0.

5.1.2 Observation. Condition (5.1.1) is equivalent to

$$\sum_{i,j} b_i^* \mathfrak{k}^{\sigma_i,\sigma_j} b_j \;\geq\; 0 \tag{5.1.2}$$

for all choices of finitely many $\sigma_i \in S, b_i \in \mathcal{B}$. To see this, define b_σ $(\sigma \in S)$ to be the sum over all b_i for which $\sigma_i = \sigma$. Then (5.1.2) transforms into (5.1.1). The converse direction is trivial.

5.1.3 Proposition. *Let $\mathcal{B}$ be a unital pre–C^*–algebra and let $\mathfrak{k}$ be a positive definite $\mathcal{B}$–valued kernel on S. Then there exists a pre-Hilbert $\mathcal{B}$–module E and a mapping $i\colon S \to E$ such that*

$$\mathfrak{k}^{\sigma,\sigma'} \;=\; \langle i(\sigma), i(\sigma') \rangle$$

and $E = \operatorname{span}\bigl(i(S)\mathcal{B}\bigr)$. Moreover, if (E', i') is another pair with these properties, then $i(\sigma) \mapsto i'(\sigma)$ establishes an isomorphism $E \to E'$.

PROOF. Let $S_\mathcal{B}$ denote the *free right $\mathcal{B}$–module generated by* S (i.e. $\bigoplus_{\sigma \in S} \mathcal{B} = \bigl\{ \bigl(b_\sigma\bigr)_{\sigma \in S} \colon b_\sigma \in \mathcal{B}, \#\{\sigma \in S \colon b_\sigma \neq 0\} < \infty \bigr\}$ or, in other words, $S_\mathbb{C} \otimes \mathcal{B}$ where $S_\mathbb{C}$ is a vector space with basis S). Then by (5.1.1)

$$\bigl\langle \bigl(b_\sigma\bigr), \bigl(b'_\sigma\bigr) \bigr\rangle \;=\; \sum_{\sigma,\sigma' \in S} b_\sigma^* \mathfrak{k}^{ss'} b'_{\sigma'}$$

defines a semiinner product on $S_\mathcal{B}$. We set $E = S_\mathcal{B}/\mathcal{N}_{S_\mathcal{B}}$ and $i(\sigma) = \bigl(\delta_{\sigma\sigma'}\mathbf{1}\bigr)_{\sigma' \in S} + \mathcal{N}_{S_\mathcal{B}}$. Then the pair (E, i) has all desired properties. Uniqueness is clear. ∎

5.1.4 Remark. If $\mathcal{B}$ is non-unital, then we still may construct E as before as a quotient of $S_\mathbb{C} \otimes \mathcal{B}$, but we do not have the mapping i. We have, however, a mapping $\widehat{i}\colon S \times \mathcal{B} \to E$, sending (σ, b) to $\bigl(\delta_{\sigma\sigma'} b\bigr)_{\sigma' \in S} + \mathcal{N}_{S_\mathcal{B}}$, such that $b^* \mathfrak{k}^{\sigma,\sigma'} b' = \langle \widehat{i}(\sigma, b), \widehat{i}(\sigma', b') \rangle$ with similar cyclicity and uniqueness properties.

5.1.5 Definition. We refer to the pair (E, i) as the *Kolmogorov decomposition* for $\mathfrak{k}$ and to E as its *Kolmogorov module*.

5.1.6 Example. For $\mathbb{C}$–valued positive definite kernels we recover the usual Kolmogorov decomposition. For instance, usual proofs of the Stinespring construction for a completely positive mapping $T\colon \mathcal{A} \rightarrow \mathcal{B}^a(G)$ start with a Kolmogorov decomposition for the kernel $\big((a,g),(a',g')\big) \mapsto \langle g, T(a^*a')g'\rangle$ on $\mathcal{A} \times G$ and obtain in this way the pre-Hilbert space $H = E \odot G$ where E is the GNS-module of T; cf. Remark 4.1.9.

For $\mathcal{B} = \mathcal{B}^a(F)$ for some pre-Hilbert $\mathcal{C}$–module F we recover the Kolmogorov decomposition in the sense of Murphy [Mur97]. He recovers the module $E \odot F$ of the KSGNS-construction for a completely positive mapping $T\colon \mathcal{A} \rightarrow \mathcal{B}^a(F)$ (cf. Remark 4.2.25) as Kolmogorov decomposition for the kernel $\big((a,y),(a',y')\big) \mapsto \langle y, T(a^*a')y'\rangle$ on $\mathcal{A} \times F$.

5.2 Completely positive definite kernels

For $\mathbb{C}$–valued kernels there is a positivity preserving product, namely, the *Schur product* which consists in multiplying two kernels pointwise. For non-commutative $\mathcal{B}$ this operation is also possible, but will, in general, not preserve positive definiteness. It turns out that we have to consider kernels which take as values mappings between algebras rather than kernels with values in algebras. Then the pointwise multiplication in the Schur product is replaced by pointwise composition of mappings. Of course, this includes the usual Schur product of $\mathbb{C}$–valued kernels, if we interpret $z \in \mathbb{C}$ as mapping $w \mapsto zw$ on $\mathbb{C}$.

5.2.1 Lemma. *Let S be a set and let $\mathfrak{K}\colon S \times S \rightarrow \mathcal{B}(\mathcal{A}, \mathcal{B})$ be a kernel with values in the bounded mappings between pre–C^*–algebras $\mathcal{A}$ and $\mathcal{B}$. Then the following conditions are equivalent.*

1. *We have*
$$\sum_{i,j} b_i^* \mathfrak{K}^{\sigma_i,\sigma_j}(a_i^* a_j) b_j \;\geq\; 0$$
for all choices of finitely many $\sigma_i \in S$, $a_i \in \mathcal{A}$, $b_i \in \mathcal{B}$.

2. *The kernel $\mathfrak{k}\colon (\mathcal{A} \times S) \times (\mathcal{A} \times S) \rightarrow \mathcal{B}$ with $\mathfrak{k}^{(a,\sigma),(a',\sigma')} = \mathfrak{K}^{\sigma,\sigma'}(a^*a')$ is positive definite.*

3. *The mapping*
$$a \;\longmapsto\; \sum_{i,j} b_i^* \mathfrak{K}^{\sigma_i,\sigma_j}(a_i^* a a_j) b_j$$
is completely positive for all choices of finitely many $\sigma_i \in S$, $a_i \in \mathcal{A}$, $b_i \in \mathcal{B}$.

4. *For all choices* $\sigma_1, \ldots, \sigma_n \in S$ $(n \in \mathbb{N})$ *the mapping*

$$\mathfrak{K}^{(n)} \colon \bigl(a_{ij}\bigr) \longmapsto \bigl(\mathfrak{K}^{\sigma_i,\sigma_j}(a_{ij})\bigr)$$

from $M_n(\mathcal{A})$ *to* $M_n(\mathcal{B})$ *is completely positive.*

5. *For all choices* $\sigma_1, \ldots, \sigma_n \in S$ $(n \in \mathbb{N})$ *the mapping* $\mathfrak{K}^{(n)}$ *is positive.*

Moreover, each of these conditions implies the following conditions.

6. *The mapping*

$$a \longmapsto \sum_{\sigma,\sigma' \in S} b_\sigma^* \mathfrak{K}^{\sigma,\sigma'}(a) b_{\sigma'}$$

is completely positive for all choices of $b_\sigma \in \mathcal{B}$ $(\sigma \in S)$ *where only finitely many* b_σ *are different from* 0.

7. *The mapping*

$$a \longmapsto \sum_{\sigma,\sigma' \in S} \mathfrak{K}^{\sigma,\sigma'}(a_\sigma^* a a_{\sigma'})$$

is completely positive for all choices of $a_\sigma \in \mathcal{A}$ $(\sigma \in S)$ *where only finitely many* a_σ *are different from* 0.

Proof. 1 and 2 are equivalent by Observation 5.1.2.

3 means

$$\sum_{k,\ell \in K} \sum_{i,j \in I} \beta_k^* b_i^* \mathfrak{K}^{\sigma_i,\sigma_j}(a_i^* \alpha_k^* \alpha_\ell a_j) b_j \beta_\ell \;\geq\; 0 \tag{5.2.1}$$

for all finite sets I, K and $a_i, \alpha_k \in \mathcal{A}$ and $b_i, \beta_k \in \mathcal{B}$. To see $3 \Rightarrow 1$ we choose K consisting of only one element and we replace α_k and β_k by an approximate unit for $\mathcal{A}$ and an approximate unit for $\mathcal{B}$, respectively. By a similar procedure we see $3 \Rightarrow 6$ and $3 \Rightarrow 7$.

To see $1 \Rightarrow 3$, we choose $P = I \times K$, $\sigma_{(i,k)} = \sigma_i$, $a_{(i,k)} = \alpha_k a_i$, and $b_{(i,k)} = b_i \beta_k$. Then (5.2.1) transforms into

$$\sum_{p,q \in P} b_p^* \mathfrak{K}^{\sigma_p,\sigma_q}(a_p^* a_q) b_q \;\geq\; 0,$$

which is true by 1.

To see $2 \Rightarrow 4$, we do the Kolmogorov decomposition $(E, \widehat{\imath})$ for the kernel $\mathfrak{k}$ in the sense of Remark 5.1.4. If $\mathcal{A}$ and $\mathcal{B}$ are unital, then we set $x_j = \widehat{\imath}(\mathbf{1}, \sigma_j, \mathbf{1}) \in E$ $(j = 1, \ldots, n)$. Then the mapping in 4 is completely positive by Example 1.7.7. If $\mathcal{A}$ and $\mathcal{B}$ are not necessarily unital, then we set $x_j = \widehat{i}(u_\lambda, \sigma_j, v_\mu)$ for some approximate units $\bigl(u_\lambda\bigr)$ and $\bigl(v_\mu\bigr)$ for $\mathcal{A}$ and $\mathcal{B}$, respectively, and we obtain the mapping in 4 as limit (pointwise in norm of $M_n(\mathcal{B})$) of completely positive mappings.

4 and 5 are equivalent by an application of Corollary 4.1.6 to $\mathfrak{K}^{(n)}$.

To see 5 $\Rightarrow$ 1 we apply 5 to the positive element $A = \bigl(a_i^* a_j\bigr) \in M_n(\mathcal{A})$ which means that $\bigl\langle B, \mathfrak{K}^{(n)}(A)B\bigr\rangle$ is positive for all $B = (b_1, \ldots, b_n) \in \mathcal{B}^n$ and, therefore, implies 1. ■

5.2.2 Definition. We call a kernel $\mathfrak{K}\colon S \times S \to \mathcal{B}(\mathcal{A}, \mathcal{B})$ *completely positive definite*, if it fulfills one of the Conditions 1 – 5 in Lemma 5.2.1. By $\mathcal{K}_S(\mathcal{A}, \mathcal{B})$ we denote the set of completely positive definite kernel on S from $\mathcal{A}$ to $\mathcal{B}$. If $\mathcal{A} = \mathcal{B}$, then we write $\mathcal{K}_S(\mathcal{B})$. A kernel fulfilling Condition 6 and Condition 7 in Lemma 5.2.1 is called *completely positive definite for $\mathcal{B}$* and *completely positive definite for $\mathcal{A}$*, respectively.

5.2.3 Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital, and let $\mathfrak{K}$ be in $\mathcal{K}_S(\mathcal{A}, \mathcal{B})$. Then there exists a contractive pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module E and a mapping $i\colon S \to E$ such that*

$$\mathfrak{K}^{\sigma,\sigma'}(a) \;=\; \langle i(\sigma), ai(\sigma')\rangle,$$

and $E = \mathsf{span}\bigl(\mathcal{A}i(S)\mathcal{B}\bigr)$. Moreover, if (E', i') is another pair with these properties, then $i(\sigma) \mapsto i'(\sigma)$ establishes an isomorphism $E \to E'$.

Conversely, if E is a contractive pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module and S a collection of elements of E, then $\mathfrak{K}$ defined by setting $\mathfrak{K}^{\sigma,\sigma'}(a) = \langle \sigma, a\sigma'\rangle$ is completely positive definite.

5.2.4 Corollary. *A kernel $\mathfrak{K} \in \mathcal{K}_S(\mathcal{A}, \mathcal{B})$ is* hermitian, *i.e. $\mathfrak{K}^{\sigma,\sigma'}(a^*) = \mathfrak{K}^{\sigma',\sigma}(a)^*$. (This remains true, also if $\mathcal{A}$ and $\mathcal{B}$ are not necessarily unital.)*

Proof of Theorem 5.2.3. By Proposition 5.1.3 we may do the Kolmogorov decomposition for the kernel $\mathfrak{k}$ and obtain a pre-Hilbert $\mathcal{B}$–module E with an embedding $i_{\mathfrak{k}}$. We have

$$\langle i_{\mathfrak{k}}(a', \sigma'), i_{\mathfrak{k}}(aa'', \sigma'')\rangle \;=\; \langle i_{\mathfrak{k}}(a^*a', \sigma'), i_{\mathfrak{k}}(a'', \sigma'')\rangle.$$

Therefore, by Corollary 1.4.3 setting $ai_{\mathfrak{k}}(a', \sigma') = i_{\mathfrak{k}}(aa', \sigma')$ we define a left action of $\mathcal{A}$ on E. This action is non-degenerate, because $\mathcal{A}$ is unital, and the unit acts as unit on E. It is contractive, because all mappings $\mathfrak{K}^{\sigma,\sigma'}$ are bounded, so that in the whole construction we may assume that $\mathcal{A}$ is complete. Setting $i(\sigma) = i_{\mathfrak{k}}(\mathbf{1}, \sigma)$, the pair (E, i) has the desired properties.

The converse direction follows from Example 1.7.7. ■

5.2.5 Definition. We refer to the pair (E, i) as the *Kolmogorov decomposition* for $\mathfrak{K}$ and to E as its *Kolmogorov module*.

5.2.6 Observation. If $\mathcal{B}$ is a von Neumann algebra, then we may pass to the strong closure $\overline{E}^s$. It is not necessary that also $\mathcal{A}$ is a von Neumann algebra, and also if $\mathcal{A}$ is a von Neumann algebra, then $\overline{E}^s$ need not be a two-sided von Neumann module. We see, however, like in Proposition 4.1.13 that for *normal* kernels (i.e. all mappings $\mathfrak{K}^{\sigma,\sigma'}$ are σ–weak) $\overline{E}^s$ is a von Neumann $\mathcal{A}$–$\mathcal{B}$–module.

Our notion of completely positive definite kernels differs from that given by Accardi and Kozyrev [AK99]. Their completely positive definite kernels fulfill only our requirement for kernels completely positive definite for $\mathcal{B}$. The weaker requirement in [AK99] is compensated by an additional property of their concrete kernel which is mirrored in the assumptions of the following result.

5.2.7 Lemma. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital, and let $\mathfrak{K}\colon S \times S \to \mathcal{B}(\mathcal{A}, \mathcal{B})$ be completely positive definite for $\mathcal{B}$. Let (E, i) denote the Kolmogorov decomposition for the positive definite kernel $\mathfrak{k}^{\sigma,\sigma'} = \mathfrak{K}^{\sigma,\sigma'}(\mathbf{1})$. Assume that for each $a \in \mathcal{A}$, $\sigma \in S$ there exist (possibly uncountably many) $\sigma' \in S$, $b_{\sigma'} \in \mathcal{B}$ such that $x_{a,\sigma} = \sum\limits_{\sigma' \in S} i(\sigma')b_{\sigma'}$ exists in $\overline{E}^s$ in the strong topology (comming from some faithful representation of $\mathcal{B}$) and fulfills*

$$\langle i(\sigma''), x_{a,\sigma} \rangle \;=\; \mathfrak{K}^{\sigma'',\sigma}(a)$$

for all $\sigma'' \in S$. Then $\mathfrak{K}$ is completely positive definite and $\overline{E}^s$ contains the Kolmogorov decomposition for $\mathfrak{K}$ as a strongly dense subset.

Proof. The assumptions guarantee that (like in the proof of Theorem 5.2.3) $\overline{E}^s$ carries a contractive strongly total representation of $\mathcal{A}$ which fulfills and is determined by $ai(\sigma) = x_{a,\sigma}$. Therefore, by the second part of Theorem 5.2.3 the kernel $\mathfrak{K}$ is completely positive definite. Now the remaining statements are clear. ■

5.2.8 Remark. The assumptions are fulfilled, for instance, whenever $\overline{E}^s$ is a centered module. This is the case in [AK99] where $\mathcal{A} = \mathcal{B} = \mathcal{B}(G)$ and the kernel is that of *central exponential units* on the *symmetric Fock module* $\Gamma^s(L^2(\mathbb{R}, \mathcal{B}(G)) = \mathcal{B}\big(G, G \bar{\otimes} \Gamma(L^2(\mathbb{R}))\big)$; see Remark 8.1.7.

Starting from a kernel $\mathfrak{K} \in \mathcal{K}_S(\mathcal{A}, \mathcal{B})$ we may always achieve the assumption to be fulfilled by passing to a kernel on the set $\mathcal{A} \times S$ or any subset of $\mathcal{A} \times S$ generating the Kolmogorov decomposition of $\mathfrak{K}$ as right module.

5.3 Partial order of kernels

We say, a completely positive mapping T *dominates* another S, if the difference $T-S$ is also completely positive. In this case, we write $T \geq S$. Obviously, $\geq$ defines a partial order. As shown by Arveson [Arv69] in the case of $\mathcal{B}(G)$ and extended by Paschke [Pas73] to arbitrary von Neumann algebras, there is an order isomorphism from the set of all completely positive mappings dominated by a fixed completely positive mapping T and certain mappings on the GNS-module of T (or the representation space of the Stinespring representation in the case of $\mathcal{B}(G)$).

In this section we extend these notions and the result to kernels and their Kolmogorov decomposition. Theorem 5.3.3 is the basis for Theorem 13.4.3 which provides us with a powerful tool to establish whether a dilation of a completely positive semigroup is its *GNS-dilation*. In Lemma 5.3.2 we need self-duality. So we stay with von Neumann modules.

5.3.1 Definition. We say, a kernel $\mathfrak{K}$ on S from $\mathcal{A}$ to $\mathcal{B}$ *dominates* another kernel $\mathfrak{L}$, if the difference $\mathfrak{K}-\mathfrak{L}$ is in $\mathcal{K}_S(\mathcal{A},\mathcal{B})$. For $\mathfrak{K} \in \mathcal{K}_S(\mathcal{A},\mathcal{B})$ we denote by $\mathcal{D}_{\mathfrak{K}} = \{\mathfrak{L} \in \mathcal{K}_S(\mathcal{A},\mathcal{B})\colon \mathfrak{K} \geq \mathfrak{L}\}$ the set of all completely positive definite kernels *dominated* by $\mathfrak{K}$.

5.3.2 Lemma. *Let $\mathcal{A}$ be a unital C^*–algebra, let $\mathcal{B}$ be a von Neumann algebra on a Hilbert space G, and let $\mathfrak{K} \geq \mathfrak{L}$ be kernels in $\mathcal{K}_S(\mathcal{A},\mathcal{B})$. Let (E,i) denote the Kolmogorov decomposition for $\mathfrak{K}$. Then there exists a unique positive contraction $w \in \mathcal{B}^{a,bil}(\overline{E}^s)$ such that $\mathfrak{L}^{\sigma,\sigma'}(a) = \langle i(\sigma), wai(\sigma')\rangle$.*

Proof. Let (F,j) denote the GNS-construction for $\mathfrak{L}$. As $\mathfrak{K}-\mathfrak{L}$ is completely positive, the mapping $v\colon i(\sigma) \mapsto j(\sigma)$ extends to an $\mathcal{A}$–$\mathcal{B}$–linear contraction $E \to F$. Indeed, for $x = \sum_k a_k i(\sigma_k) b_k$ we find

$$\langle x,x\rangle - \langle vx,vy\rangle \;=\; \sum_{k,\ell} b_k^*(\mathfrak{K}^{\sigma_k,\sigma_\ell} - \mathfrak{L}^{\sigma_k,\sigma_\ell})(a_k^* a_\ell) b_\ell \;\geq\; 0,$$

such that $\|x\| \geq \|vx\|$. By Proposition 3.1.5 v extends further to a contraction $\overline{E}^s \to \overline{F}^s$. Since von Neumann modules are self-dual, v has an adjoint $v^* \in \mathcal{B}^a(\overline{F}^s, \overline{E}^s)$. Since adjoints of bilinear mappings and compositions among them are bilinear, too, it follows that also $w = v^*v$ is bilinear. Of course, $\langle i(\sigma), wai(\sigma')\rangle = \langle i(\sigma), v^*vai(\sigma')\rangle = \langle j(\sigma), aj(\sigma')\rangle = \mathfrak{L}^{\sigma,\sigma'}(a)$. ■

5.3.3 Theorem. *Let S be a set, let $\mathcal{A}$ be a unital C^*–algebra, let $\mathcal{B}$ be a von Neumann algebra on a Hilbert space G, and let $\mathfrak{K}$ be a kernel in $\mathcal{K}_S(\mathcal{A},\mathcal{B})$. Denote by (E,i) the Kolmogorov decomposition of $\mathfrak{K}$. Then the mapping $\mathfrak{O}\colon w \mapsto \mathfrak{L}_w$ with*

$$\mathfrak{L}_w^{\sigma,\sigma'}(a) \;=\; \langle i(\sigma), wai(\sigma')\rangle$$

establishes an order isomorphism from the positive part of the unit ball in $\mathcal{B}^{a,bil}(\overline{E}^s)$ *onto* $\mathcal{D}_{\mathfrak{K}}$.

Moreover, if (F, j) *is another pair such that* $\mathfrak{K}^{\sigma,\sigma'}(a) = \langle j(\sigma), aj(\sigma')\rangle$, *then* $\mathfrak{O}$ *is still a surjective order homomorphism. It is injective, if and only if* (F, j) *is (unitarily equivalent to) the Kolmogorov decomposition of* $\mathfrak{K}$.

Proof. Let us start with the more general (F, j). Clearly, $\mathfrak{O}$ is order preserving. As $E \subset F$ and $\mathcal{B}^a(\overline{E}^s) = p\mathcal{B}^a(\overline{F}^s)p \subset \mathcal{B}^a(\overline{F}^s)$ where p is the projection onto $\overline{E}^s$, Lemma 5.3.2 tells us that $\mathfrak{O}$ is surjective. If p is non-trivial, then $\mathfrak{O}$ is certainly not injective, because $\mathfrak{L}_p = \mathfrak{L}_{\mathbf{1}}$. Otherwise, it is injective, because the elements $j(\sigma)$ are strongly total, hence, separate the elements of $\mathcal{B}^a(\overline{F}^s)$. It remains to show that in the latter case also the inverse $\mathfrak{O}^{-1}$ is order preserving. But this follows from Lemma 1.5.2. ■

5.3.4 Remark. By restriction to completely positive mappings (i.e. $\#S = 1$) we obtain Paschke's result [Pas73]. Passing to $\mathcal{B} = \mathcal{B}(G)$ and doing the Stinespring construction, we find Arveson's result [Arv69].

We close with a reconstruction result which is a direct generalization from [AK99]. The proof is very much the same, but notation and generality differ considerably. Also here it is a very hidden domination property which keeps things working. But Observation 5.3.6 shows that it is different from the situation discribed before.

If $T\colon \mathcal{A} \to \mathcal{A}$ is a completely positive mapping, then with $\mathfrak{K} \in \mathcal{K}_S(\mathcal{A}, \mathcal{B})$ also the kernel $\mathfrak{K} \circ T$ is completely positive definite. This is a special case of Theorem 5.4.2, but can also easily be seen directly. More generally, let (E, i) be the Kolmogorov decomposition of $\mathfrak{K}$ and define the kernel $\mathfrak{K}_E \in \mathcal{K}_S(\mathcal{B}^a(\overline{E}^s), \mathcal{B})$ by setting $\mathfrak{K}_E^{\sigma,\sigma'}(a) = \langle i(\sigma), ai(\sigma')\rangle$. Then for any completely positive mapping $T\colon \mathcal{A} \to \mathcal{B}^a(\overline{E}^s)$ also the kernel $\mathfrak{K}_E \circ T$ is in $\mathcal{K}_S(\mathcal{A}, \mathcal{B})$. If T is unital, then $\mathfrak{K}^{\sigma,\sigma'}(\mathbf{1}) = \mathfrak{K}_E^{\sigma,\sigma'} \circ T(\mathbf{1})$. Under the assumptions of Lemma 5.2.7 we have also the converse result.

5.3.5 Lemma. *Let* $\mathfrak{K}, \mathfrak{L}$ *be kernels in* $\mathcal{K}_S(\mathcal{A}, \mathcal{B})$ *such that* $\mathfrak{K}$ *fulfills the assumptions of Lemma 5.2.7 and* $\mathfrak{K}^{\sigma,\sigma'}(\mathbf{1}) = \mathfrak{L}^{\sigma,\sigma'}(\mathbf{1})$. *Then there exists a unique unital completely positive mapping* $T\colon \mathcal{A} \to \mathcal{B}^a(\overline{E}^s)$, *such that* $\mathfrak{L} = \mathfrak{K}_E \circ T$.

Proof. Recall that $i(S)$ generates E as a right module. Let $x = \sum_i i(\sigma_i)b_i$ be an arbitray element in E. Then (because $\mathfrak{L}^{(n)}$ is completely positive)

$$\Big\|\sum_{i,j} b_i^*\mathfrak{L}^{\sigma_i,\sigma_j}(a)b_j\Big\| \;\leq\; \|a\|\,\Big\|\sum_{i,j} b_i^*\mathfrak{L}^{\sigma_i,\sigma_j}(\mathbf{1})b_j\Big\| \;=\; \|a\|\,\Big\|\sum_{i,j} b_i^*\mathfrak{K}^{\sigma_i,\sigma_j}(\mathbf{1})b_j\Big\| \;=\; \|a\|\,\|x\|^2$$

for all $a \in \mathcal{A}$. In other words, the $\mathcal{B}$–sesquilinear form A_a on E defined, by setting $A_a(i(\sigma), i(\sigma')) = \mathfrak{L}^{\sigma,\sigma'}(a)$, fulfills $\|A_a(x, x)\| \leq \|a\|\,\|x\|^2$. If $a \geq 0$, then so is $A_a(x, x)$.

By (1.2.1) we find $\|A_a(x,y)\| \leq \|a\|\,\|x\|\,\|y\|$. Since all $a \in \mathcal{A}$ can be written as a linear combination of not more than four positive elements, it follows that A_a is bounded for all a. By Corollary 1.4.8 there is a unique operator $T(a) \in \mathcal{B}^a(\overline{E}^s)$ such that $\mathfrak{L}^{\sigma,\sigma'}(a) = A_a(i(\sigma), i(\sigma')) = \langle i(\sigma), T(a)i(\sigma')\rangle = \mathfrak{K}_E^{\sigma,\sigma'} \circ T(a)$. Complete positivity of T follows directly from completely positive definiteness of $\mathfrak{L}$. ■

5.3.6 Observation. If $\mathcal{A}$ acts faithfully on E, then certainly $\mathfrak{K} \not\geq \mathfrak{L}$, because $\mathsf{id} - T$ compleletely positive implies $\|\mathsf{id} - T\| = \|\mathbf{1} - T(\mathbf{1})\| = 0$.

5.4 Schur product and semigroups of kernels

Now we come to products, or better, compositions of kernels. The following definition generalizes the Schur product of a matrix of mappings and a matrix as discussed in Example 1.7.7.

5.4.1 Definition. Let $\mathfrak{K} \in \mathcal{K}_S(\mathcal{A}, \mathcal{B})$ and let $\mathfrak{L} \in \mathcal{K}_S(\mathcal{B}, \mathcal{C})$. Then the *Schur product* of $\mathfrak{L}$ and $\mathfrak{K}$ is the kernel $\mathfrak{L} \circ \mathfrak{K} \in \mathcal{K}_S(\mathcal{A}, \mathcal{C})$, defined by setting $(\mathfrak{L} \circ \mathfrak{K})^{\sigma,\sigma'}(a) = \mathfrak{L}^{\sigma,\sigma'} \circ \mathfrak{K}^{\sigma,\sigma'}(a)$.

5.4.2 Theorem. *If $\mathfrak{K}$ and $\mathfrak{L}$ are completely positive definite, then so is $\mathfrak{L} \circ \mathfrak{K}$.*

Proof. If all algebras are unital, then this follows directly from Theorem 5.2.3 and Example 4.2.12. Indeed, by the forward direction of Theorem 5.2.3 we have the Kolmogorov decompositions (E, i) and (F, j) for $\mathfrak{K}$ and $\mathfrak{L}$, respectively. Like in Example 4.2.12 we find $\mathfrak{L}^{\sigma,\sigma'} \circ \mathfrak{K}^{\sigma,\sigma'}(a) = \langle i(\sigma) \odot j(\sigma), ai(\sigma') \odot j(\sigma')\rangle$ from which $(\mathfrak{L} \circ \mathfrak{K})^{\sigma,\sigma'}$ is completely positive definite by the backward direction of Theorem 5.2.3. If the algebras are not necessarily unital, then (as in the proof of 2 $\Rightarrow$ 4 in Lemma 5.2.1) we may apply the same argument, replacing $i(\sigma)$ by $\widehat{i}(u_\lambda, \sigma, v_\mu)$ (and similarly for j) and approximating in this way $\mathfrak{L} \circ \mathfrak{K}$ by completely positive definite kernels. ■

5.4.3 Observation. The proof shows that, like the GNS-construction of completely positive mappings, the Kolmogorov decomposition of the composition $\mathfrak{L} \circ \mathfrak{K}$ can be obtained from those for $\mathfrak{K}$ and $\mathfrak{L}$. More precisely, we obtain it as the two-sided submodule of $E \odot F$ generated by $\{i(\sigma) \odot j(\sigma) \colon \sigma \in S\}$ and the embedding $i \odot j \colon \sigma \mapsto i(\sigma) \odot j(\sigma)$.

5.4.4 Definition. A family $\big(\mathfrak{T}_t\big)_{t \in \mathbb{R}_+}$ of kernels on S from $\mathcal{B}$ to $\mathcal{B}$ is called a *(uniformly continuous) Schur semigroup* of kernels, if for all $\sigma, \sigma' \in S$ the mappings $\mathfrak{T}_t^{\sigma,\sigma'}$ form a (uniformly continuous) semigroup on $\mathcal{B}$; see Definition A.5.1. A *(uniformly continuous) CPD-semigroup* of kernels, is a (uniformly continuous) Schur semigroup of completely positive definite kernels. In a similar manner we define *Schur C_0–semigroups* and *CPD–C_0–semigroups*.

Like for CP-semigroups, the generators of (uniformly continuous) CPD-semigroups can be characterized by a *conditional* positivity condition.

5.4.5 Definition. A kernel $\mathfrak{L}$ on S from $\mathcal{B}$ to $\mathcal{B}$ is called *conditionally completely positive definite*, if

$$\sum_{i,j} b_i^* \mathfrak{L}^{\sigma_i,\sigma_j}(a_i^* a_j) b_j \;\geq\; 0 \tag{5.4.1}$$

for all choices of finitely many $\sigma_i \in S$, $a_i, b_i \in \mathcal{B}$ such that $\sum\limits_i a_i b_i = 0$.

5.4.6 Lemma. *For a kernel $\mathfrak{L}$ on S from $\mathcal{B}$ to $\mathcal{B}$ the following conditions are equivalent.*

1. *$\mathfrak{L}$ is conditionally completely positive definite.*
2. *For all choices $\sigma_1, \ldots, \sigma_n \in S$ $(n \in \mathbb{N})$ the mapping*

$$\mathfrak{L}^{(n)} \colon \big(a_{ij}\big) \;\longmapsto\; \big(\mathfrak{L}^{\sigma_i,\sigma_j}(a_{ij})\big)$$

on $M_n(\mathcal{B})$ is conditionally completely positive*, i.e. for all $A^k, B^k \in M_n(\mathcal{B})$ such that $\sum\limits_k A^k B^k = 0$ we have $\sum\limits_{k,\ell} B^{k*} \mathfrak{L}^{(n)}(A^{k*} A^\ell) B^\ell \geq 0$.*

Proof. By Lemma 1.5.2 an element $\big(b_{ij}\big) \in M_n(\mathcal{B})$ is positive, if and only if $\sum\limits_{i,j} b_i^* b_{ij} b_j \geq 0$ for all $b_1, \ldots, b_n \in \mathcal{B}$. Therefore, Condition 2 is equivalent to

$$\sum_{i,j,p,q,k,\ell,r} b_i^* b_{pi}^{k*} \mathfrak{L}^{\sigma_p,\sigma_q}(a_{rp}^{k*} a_{rq}^\ell) b_{qj}^\ell b_j \;\geq\; 0$$

for all $\sigma_1, \ldots, \sigma_n \in S$, $b_1, \ldots, b_n \in \mathcal{B}$ $(n \in \mathbb{N})$, and finitely many $\big(a_{ij}^k\big) \in M_n(\mathcal{A})$, $\big(b_{ij}^k\big) \in M_n(\mathcal{B})$ such that $\sum\limits_{p,k} a_{ip}^k b_{pj}^k = 0$ for all i, j. Assume that 1 is true, choose $b_i \in \mathcal{B}$, and choose $a_{rp}^k, b_{pi}^k \in \mathcal{B}$ such that $\sum\limits_{p,k} a_{rp}^k b_{pi}^k = 0$ for all r, i. Then $\sum\limits_{p,k} a_{rp}^k \big(\sum\limits_i b_{pi}^k b_i\big) = 0$ for all r and 1 implies that $\sum\limits_{i,j,p,q,k,\ell} b_i^* b_{pi}^{k*} \mathfrak{L}^{\sigma_p,\sigma_q}(a_{rp}^{k*} a_{rq}^\ell) b_{qj}^\ell b_j \geq 0$ for each r separately. (Formally, we pass to indices (p, k) and set $\sigma_{(p,k)} = \sigma_p$ as in the proof of Lemma 5.2.1.) Summing over r we find 2.

Conversely, assume that 2 is true and choose $a_i, b_i \in \mathcal{B}$ such that $\sum\limits_i a_i b_i = 0$. Set $a_{rp} = \delta_{1r} a_p$ and $b_{pi} = b_p$. Then $\sum\limits_p a_{rp} b_{pi} = \delta_{1r} \sum\limits_p a_p b_p = 0$ for all r, i and 2 implies that the matrix $\big(\sum\limits_{p,q,r} b_{pi}^* \mathfrak{L}^{\sigma_p,\sigma_q}(a_{rp}^* a_{rq}) b_{qj}\big)_{i,j} = \big(\sum\limits_{p,q} b_p^* \mathfrak{L}^{\sigma_p,\sigma_q}(a_p^* a_q) b_q\big)_{i,j}$ is positive. As any of the (equal) diagonal entries $\sum\limits_{p,q} b_p^* \mathfrak{L}^{\sigma_p,\sigma_q}(a_p^* a_q) b_q$ must be positive in $\mathcal{B}$, we find 1. ■

5.4.7 Theorem. *Let $\mathcal{B}$ be a unital C^*–algebra and let S be a set. Then the formula*

$$\mathfrak{T}_t \;=\; e^{t\mathfrak{L}} \tag{5.4.2}$$

(where the exponential is that for the Schur product of kernels) establishes a one-to-one correspondence between uniformly continuous CPD-semigroups $\left(\mathfrak{T}_t\right)_{t\in\mathbb{R}_+}$ of positive definite kernels on S from $\mathcal{B}$ to $\mathcal{B}$ and hermitian (see Corollary 5.2.4) conditionally completely positive definite kernels on S from $\mathcal{B}$ to $\mathcal{B}$. We say $\mathfrak{L}$ is the generator *of $\mathfrak{T}$.*

Proof. First of all, let us remark that (5.4.2) establishes a on-to-one correspondence between uniformly continuous Schur semigroups and kernels $\mathfrak{L}\colon S\times S\to\mathcal{B}(\mathcal{B})$. This follows simply by the same statement for the uniformly continuous semigroups $\mathfrak{T}_t^{\sigma,\sigma'}$ and their generators $\mathfrak{L}^{\sigma,\sigma'}$. So the only problem we have to deal with is positivity.

Let $\mathfrak{T}$ by a CPD-semigroup. By Lemma 5.2.1 (4) this is equivalent to complete positivity of the semigroup $\mathfrak{T}_t^{(n)}$ on $M_n(\mathcal{B})$ for each choice of $\sigma_1,\ldots,\sigma_n\in S$ $(n\in\mathbb{N})$. So let us choose $A^k, B^k\in M_n(\mathcal{B})$ such that $\sum\limits_k A^kB^k=0$. Then

$$\sum_{k,\ell} B^{k*}\mathfrak{L}^{(n)}(A^{k*}A^\ell)B^\ell \;=\; \lim_{t\to 0}\frac{1}{t}\sum_{k,\ell} B^{k*}\mathfrak{T}_t^{(n)}(A^{k*}A^\ell)B^\ell \;\geq\; 0.$$

In other words, $\mathfrak{L}^{(n)}$ is conditionally completely positive and by Lemma 5.4.6 (2) $\mathfrak{L}$ is conditionally completely positive definite. As limit of hermitian kernels, also $\mathfrak{L}$ must be hermitian.

Conversely, let $\mathfrak{L}$ be hermitian and conditionally completely positive definite, so that $\mathfrak{L}^{(n)}$ is hermitian and conditionally completely positive for each choice of $\sigma_1,\ldots,\sigma_n\in S$ $(n\in\mathbb{N})$. We follow Evans and Lewis [EL77, Theorem 14.2 (3 $\Rightarrow$ 1)] to show that $\mathfrak{T}_t^{(n)}$ is positive, which by Lemma 5.2.1 (5) implies that $\mathfrak{T}_t$ is completely positive definite.

Let $A\geq 0$ and B in $M_n(\mathcal{B})$ such that $AB=0$. Then by Proposition A.7.3 also $\sqrt{A}B=0$, whence $B^*\mathfrak{L}^{(n)}(A)B\geq 0$, because $\mathfrak{L}^{(n)}$ is conditionally completely positive. Let $0\leq\varepsilon<\left\|\mathfrak{L}^{(n)}\right\|$, whence $\mathsf{id}-\varepsilon\mathfrak{L}^{(n)}$ is invertible. Now let $A=A^*$ be an arbitrary self-adjoint element in $M_n(\mathcal{B})$. We show that $A\geq 0$ whenever $(\mathsf{id}-\varepsilon\mathfrak{L}^{(n)})(A)\geq 0$, which establishes the hermitian mapping $(\mathsf{id}-\varepsilon\mathfrak{L}^{(n)})^{-1}$ as positive. We write $A=A_+-A_-$ where A_+,A_- are unique positive elements fulfilling $A_+A_-=0$. Therefore, $A_-\mathfrak{L}^{(n)}(A_+)A_-\geq 0$. Hence,

$$\begin{aligned}0 \;\leq\; A_-(\mathsf{id}-\varepsilon\mathfrak{L}^{(n)})(A)A_- \;&=\; A_-(\mathsf{id}-\varepsilon\mathfrak{L}^{(n)})(A_+)A_- - A_-(\mathsf{id}-\varepsilon\mathfrak{L}^{(n)})(A_-)A_-\\ &=\; -\,\varepsilon A_-\mathfrak{L}^{(n)}(A_+)A_- - A_-^3 + \varepsilon A_-\mathfrak{L}^{(n)}(A_-)A_-,\end{aligned}$$

whence

$$A_-^3 \;\leq\; A_-^3+\varepsilon A_-\mathfrak{L}^{(n)}(A_+)A_- \;\leq\; \varepsilon A_-\mathfrak{L}^{(n)}(A_-)A_-.$$

If $A_- \neq 0$, then $\|A_-\|^3 = \|A_-^3\| \leq \|\varepsilon A_- \mathfrak{L}^{(n)}(A_-)A_-\| \leq \varepsilon \|\mathfrak{L}^{(n)}\| \|A_-\|^3 < \|A_-\|^3$, a contradiction, whence $A_- = 0$. By Proposition A.5.4 we have $\mathfrak{T}_t^{(n)} = \lim_{m\to\infty} \bigl(\mathbf{1} - \frac{t}{m}\mathfrak{L}^{(n)}\bigr)^{-m}$ which is positive as limit of compositions of positive mappings. ■

By appropriate applications of Lemmata 5.2.1 and 5.4.6 to a kernel on a one-element set S, we find the following well-known result.

5.4.8 Corollary. *The formula $T_t = e^{t\mathcal{L}}$ establishes a one-to-one correspondence between uniformly continuous* CP-semigroups *on $\mathcal{B}$ (i.e. semigroups of completely positive mappings on $\mathcal{B}$) and hermitian conditionally completely positive mappings $\mathcal{L} \in \mathcal{B}(\mathcal{B})$.*

5.4.9 Observation. A CP-semigroup on a von Neumann algebra is normal, if and only if its generator is σ–weak. (This follows from the observation that norm limits of σ–weak mappings are σ–weak.)

We find a simple consequence, by applying this argument to the CP-semigroups $\mathfrak{T}_t^{(n)}$.

5.4.10 Corollary. *A CPD-semigroup $\mathfrak{T}$ on a von Neumann algebra is* normal *(i.e. each mapping $\mathfrak{T}_t^{\sigma,\sigma'}$ is σ–weak), if and only if its generator $\mathfrak{L}$ is σ–weak.*

5.4.11 Remark. It is easily possible to show first Corollary 5.4.8 as in [EL77], and then apply it to $\mathfrak{T}_t^{(n)} = e^{t\mathfrak{L}^{(n)}}$ to show the statement for CPD-semigroups. Notice, however, that also in [EL77] in order to show Corollary 5.4.8, it is necessary to know at least parts of Lemma 5.2.1 in a special case.

We say a CPD-semigroup $\mathfrak{T}$ *dominates* another $\mathfrak{T}'$ (denoted by $\mathfrak{T} \geq \mathfrak{T}'$), if $\mathfrak{T}_t \geq \mathfrak{T}'_t$ for all $t \in \mathbb{T}$. The following lemma reduces the analysis of the order structure of uniformly continuous CPD-semigroups to that of the order structure of their generators.

5.4.12 Lemma. *Let $\mathfrak{T}$ and $\mathfrak{T}'$ be uniformly continuous CPD-semigroups on S in $\mathcal{K}_S(\mathcal{B})$ with generators $\mathfrak{L}$ and $\mathfrak{L}'$, respectively. Then $\mathfrak{T} \geq \mathfrak{T}'$, if and only if $\mathfrak{L} \geq \mathfrak{L}'$.*

PROOF. Since $\mathfrak{T}_0 = \mathfrak{T}'_0$, we have $\frac{\mathfrak{T}_t - \mathfrak{T}'_t}{t} = \frac{\mathfrak{T}_t - \mathfrak{T}_0}{t} - \frac{\mathfrak{T}'_t - \mathfrak{T}'_0}{t} \to \mathfrak{L} - \mathfrak{L}'$ for $t \to 0$ so that $\mathfrak{T} \geq \mathfrak{T}'$ certainly implies $\mathfrak{L} \geq \mathfrak{L}'$. Conversely, assume that $\mathfrak{L} \geq \mathfrak{L}'$. Choose $n \in \mathbb{N}$ and $\sigma_i \in S$ $(i = 1, \ldots, n)$. From the proof of Theorem 5.4.7 we know that $(\mathbf{1} - \varepsilon\mathfrak{L}^{(n)})^{-1} \geq 0$ and $(\mathbf{1} - \varepsilon\mathfrak{L}'^{(n)})^{-1} \geq 0$ for all sufficiently small $\varepsilon > 0$. Moreover, by Theorem 5.4.2

$$(\mathbf{1} - \varepsilon\mathfrak{L}^{(n)})^{-1} - (\mathbf{1} - \varepsilon\mathfrak{L}'^{(n)})^{-1} \;=\; \varepsilon(\mathbf{1} - \varepsilon\mathfrak{L}^{(n)})^{-1}(\mathfrak{L}^{(n)} - \mathfrak{L}'^{(n)})(\mathbf{1} - \varepsilon\mathfrak{L}'^{(n)})^{-1} \;\geq\; 0,$$

because all three factors are ≥ 0. This implies $(\mathbf{1} - \frac{t}{m}\mathfrak{L}^{(n)})^{-m} - (\mathbf{1} - \frac{t}{m}\mathfrak{L}'^{(n)})^{-m} \geq 0$ for m sufficiently big. Letting $m \to \infty$, we find $\mathfrak{T}_t^{(n)} \geq \mathfrak{T}_t'^{(n)}$ and further $\mathfrak{T} \geq \mathfrak{T}'$ by Lemma 5.2.1(4). ■

We close this section with a conjecture (Theorem 5.4.14) for the form of the generators of a uniformly continuous CPD-semigroup, based on the corresponding results for CP-semigroups by Christensen and Evans [CE79] which we report in Appendix A.6. One of the main goals of Part III is to prove Theorem 5.4.14.

Let $\mathcal{B}$ be a unital C^*–algebra, let ζ be an element in a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module F, and let $\beta \in \mathcal{B}$. Then

$$\mathcal{L}(b) = \langle \zeta, b\zeta \rangle + b\beta + \beta^* b \tag{5.4.3}$$

is obviously conditionally completely positive and hermitian so that $T_t = e^{t\mathcal{L}}$ is a uniformly continuous CP-semigroup. We say the generator of T has *Christensen-Evans form* (or is a *CE-generator*). By Theorem A.6.3 generators $\mathcal{L}$ of normal CP-semigroups T on a von Neumann algebra $\mathcal{B}$ have the form (5.4.3) where F is some von Neumann $\mathcal{B}$–$\mathcal{B}$–module. For us it will be extremely important that F can be chosen in a minimal way, as it follows from Lemma A.6.1 (and its Corollary A.6.2 which asserts that bounded derivations with values in von Neumann modules are inner). Therefore, we consider Lemma A.6.1 rather than Theorem A.6.3 (which is a corollary of lemma A.6.1) as the main result of [CE79].

The results in [CE79] are stated for (even non-unital) C^*–algebras $\mathcal{B}$. However, the proof runs (more or less) by embedding $\mathcal{B}$ into the bidual von Neumann algebra $\mathcal{B}^{**}$. Hence, the inner product on F takes values in $\mathcal{B}^{**}$ and also $\beta \in \mathcal{B}^{**}$. Only the combinations in (5.4.3) remain in $\mathcal{B}$. As this causes unpleasant complications in formulations of statements, usually, we restrict to the case of von Neumann algebras.

What can be the analogue for CPD-semigroups on some set S? Let $\mathcal{B}$ be a unital C^*–algebra, let ζ_σ ($\sigma \in S$) be elements in a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module F, and let $\beta_\sigma \in \mathcal{B}$ ($\sigma \in S$). Then the kernel $\mathfrak{L}$ on S defined, by setting

$$\mathfrak{L}^{\sigma,\sigma'}(b) \;=\; \langle \zeta_\sigma, b\zeta_{\sigma'} \rangle + b\beta_{\sigma'} + \beta_\sigma^* b \tag{5.4.4}$$

is conditionally completely positive definite and hermitian. (The first summand is completely positive definite. Each of the remaining summands is conditionally completely positive definite, and the rest follows, because $\mathfrak{L}$ should be hermitian.)

5.4.13 Definition. A generator $\mathfrak{L}$ of a uniformly continuous CPD-semigroup has *Christensen-Evans form* (or is a *CE-generator*), if it can be written in the form (5.4.4).

5.4.14 Theorem. *Let $\mathfrak{T}$ be a normal uniformly continuous CPD-semigroup on S on a von Neumann algebra $\mathcal{B}$ with generator $\mathfrak{L}$. Then there exist a von Neumann $\mathcal{B}$–$\mathcal{B}$–module F with elements $\zeta_\sigma \in F$ ($\sigma \in S$), and elements $\beta_\sigma \in \mathcal{B}$ ($\sigma \in S$) such that $\mathfrak{L}$ has the Christensen-Evans form in* (5.4.4). *Moreover, the strongly closed submodule of F generated* *by the derivations $d_\sigma(b) = b\zeta_\sigma - \zeta_\sigma b$ (see Appendix A.6) is determined by $\mathfrak{L}$ up to (two-sided) isomorphism.*

We prove this Theorem (and semigroup versions of other theorems like Theorem 5.3.3) in Chapter 13 (after Theorem 13.3.1) with the help of product systems. A direct generalization of the methods of [CE79] as explained in Appendix A.6 fails, however. This is mainly due to the following fact.

5.4.15 Observation. Although the von Neumann module F is determined uniquely by the cyclicity condition in Theorem 5.4.14, the concrete choice neither of ζ_σ nor of β_σ is unique. This makes it impossible to extend to $\mathfrak{T}$ what Lemma A.6.1 asserts for $\mathfrak{T}^{(n)}$ with a fixed choice of $\sigma_1, \ldots, \sigma_n \in S$ by an inductive limit over finite subsets of S.

Part II

Fock modules

Fock spaces appear typically as representation spaces of *central limit distributions* for convolutions of probability laws. For instance, the symmetric Fock space carries a representation of the classical brownian motion (being ditributed according to the classical gaußian law), the full Fock space carries a representation of the free brownian motion (being distributed according to the *Wigner law*, i.e. the central limit distribution of Voiculescu's *free probability* [Voi87]). The appearence of such Fock spaces in physics is typical for set-ups where we try to understand the evolution of a small system as an irreversible evolution driven by some *reservoire* or *heat bath* (i.e. a *white noise*). Nowadays, a huge quantity of new convolutions appears in continuation, each comming along with a central limit distribution which may be represented by *creators* and *annihilators* on some Fock like space in the *vacuum state*. This lead Accardi, Lu and Volovich [ALV97] to the notion of *interacting Fock space*, an abstraction with emphasis on the $\mathbb{N}_0$–graduation of the space (the creators being homogeneous of degree 1; see Appendix A.3) and on existence of a cyclic *vacuum vector*.

Voiculescu [Voi95] generalized free probability to *operator-valued free probability* and determined the central limit distribution. Speicher [Spe98] showed that the central limit distribution may be understood as moments of creators and annihilators on a *full Fock module* (introduced independently and for completely different reasons by Pimsner [Pim97]). For *operator valued tensor independence* (an independence paralleling classical independence which works, however, only for centered $\mathcal{B}$–algebras and is, therefore, very restricted) we showed in Skeide [Ske99a] a central limit theorem and that the central limit distribution may be realized in the same way on a *symmetric Fock module*.

In Accardi and Skeide [AS98] we showed that the two aspects, on the one hand, convolutions of probability laws, and on the other hand, operator-valued free probability, are two sides of the same game. Creators and annihilators on an interacting Fock space may represented by creators and annihilators on a canonically associated full Fock module. (In some sense also the converse is true.) We discuss this in Chapter 9. The construction, being very algebraic, makes it necessary to use the new notion of P^*–algebras from Appendix C.

Of course, it would be interesting to speak about all the probabilistic aspects mentioned so far, and to show all the central limit theorems. Meanwhile, this would fill, however, a further book. Therefore, (perhaps, except Section 17.2) we decided to neglect this aspect completely. We discuss the several Fock modules on their own right.

We understand all Fock modules as subspaces of the full Fock module, so that Chapter 6 is indispensable for almost everything in the remainder of these notes. The reader who is only interested in the calculus in Part IV, may skip the remaining chapters of Part II. The time ordered Fock module is the basic example for a product system. Therefore, the reader who is interested mainly in Part III should read also Chapter 7. The remaining chapters of Part II are independent of the rest of these notes. Chapter 8 about the symmetric Fock module (roughly speaking, a subclass of the time ordered Fock modules, but with more interesting operators) is interesting to see better the connection with existing work on the case $\mathcal{B} = \mathcal{B}(G)$. Starting from Section 8.2 we discuss our realization of the *square of white noise* relations (postulated in Accardi, Lu and Volovich [ALV99]) from Accardi and Skeide [AS00a, AS00b]. In Chapter 9 we present our results on the relation between interacting Fock spaces and full Fock modules from Accardi and Skeide [AS98].

Contrary to our habits in Parts I and III we, usually, assume that Fock modules $\mathcal{F}$, Γ, $\mathrm{I\!\Gamma}$, etc. are completed. We do so, because in our main applications completion (for the calculus in Part IV) or, at least, partial completion (for having product systems in Part III or in Appendix D) is necessary. By $\underline{\mathcal{F}}$, $\underline{\Gamma}$, $\underline{\mathrm{I\!\Gamma}}$, etc. we denote purely algebraic Fock modules (i.e. tensor product and direct sum are algebraic). For intermediate objects we use different notations to be introduced where they occur. Throughout Part II, $\mathcal{B}$ is a unital C^*–algebra. As a consequence we may or may not assume that a pre-Hilbert $\mathcal{B}$–module is complete, and all pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules are contractive automatically.

The only exceptions of the preceding conventions are Section 8.2 and Chapter 9, where $\mathcal{B}$ is a P^*–algebra (see Appendix C). Here we cannot speak about completion.

Chapter 6

The full Fock module

6.1 Basic definitions

6.1.1 Definition. Let $\mathcal{B}$ be a unital C^*–algebra and let E be a (pre-)Hilbert $\mathcal{B}$–$\mathcal{B}$–module. Then the *full Fock module* $\mathcal{F}(E)$ *over* E is the completion of the pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module

$$\underline{\mathcal{F}}(E) = \bigoplus_{n=0}^{\infty} E^{\odot n}$$

where $E^{\odot 0} = \mathcal{B}$ and $\omega = \mathbf{1} \in E^{\odot 0}$ is the *vacuum*. If $\mathcal{B}$ is a von Neumann algebra, then by $\mathcal{F}^s(E)$ we denote the von Neumann $\mathcal{B}$–$\mathcal{B}$–module obtained by strong closure of $\mathcal{F}(E)$ in the identification preceding Definition 3.1.1.

6.1.2 Definition. Let $x \in E$. The *creation operator* (or *creator*) $\ell^*(x)$ on $\mathcal{F}(E)$ is defined by setting

$$\ell^*(x)x_n \odot \cdots \odot x_1 = x \odot x_n \odot \cdots \odot x_1$$

for $n \geq 1$ and $\ell^*(x)\omega = x$. The *annihilation operator* (or *annihilator*) is the adjoint operator, i.e.

$$\ell(x)x_n \odot \cdots \odot x_1 = \langle x, x_n \rangle x_{n-1} \odot \cdots \odot x_1$$

for $n \geq 1$ and 0 otherwise. Let $T \in \mathbb{B}^a(E)$. The *conservation operator* (or *conservator*) $p(T)$ on $\mathcal{F}(E)$ is defined by setting

$$p(T)x_n \odot \cdots \odot x_1 = (Tx_n) \odot x_{n-1} \odot \cdots \odot x_1$$

for $n \geq 1$ and 0 otherwise.

6.1.3 Proposition. *The mappings $x \mapsto \ell^*(x)$ and $T \mapsto p(T)$ depend $\mathcal{B}$–$\mathcal{B}$–linearly on their arguments. The mapping $x \mapsto \ell(x)$ depends $\mathcal{B}$–$\mathcal{B}$–anti-linearly on its argument. We have $\|\ell^*(x)\| = \|\ell(x)\| = \|x\|$ and $\|p(T)\| = \|T\|$.*

We have

$$p(TT') = p(T)p(T') \quad \textit{and} \quad p(T^*) = p(T)^*$$

so that $T \mapsto p(T)$ defines an injective homomorphism of C^*–algebras. Finally, we have the relations

$$p(T)\ell^*(x) = \ell^*(Tx) \qquad\qquad \ell(x)p(T) = \ell(T^*x)$$

$$\ell(x)\ell^*(x') = \langle x, x' \rangle. \tag{6.1.1}$$

Proof. The other statements being obvious, we only show $\|p(T)\| = \|T\|$ and postpone $\|\ell^*(x)\| = \|x\|$ to the more general statement in Proposition 6.2.5. We have $p(T) = 0 \oplus T \odot \mathsf{id}$ on $\mathcal{F}(E) = \mathcal{B}\omega \oplus E \bar{\odot} \mathcal{F}(E)$. Therefore, $\|p(T)\| \leq \|T\|$. On the other hand, $p(T) \upharpoonright E^{\odot 1} = T$ so that $\|p(T)\|$ certainly is not smaller than $\|T\|$. ∎

The first formal definitions of full Fock module are due to Pimsner [Pim97] and Speicher [Spe98]. Pimsner used the full Fock module to define the module analogue $\mathcal{O}_E$ of the Cuntz algebras [Cun77] as the C^*–algebra generated by the creators $\ell^*(x)$ $(x \in E)$. In a certain sense, this C^*–algebra is determined by the *generalized Cuntz relations* (6.1.1). Speicher, who introduced also the conservation operators $p(T)$, considered the full Fock module as a space for a potential quantum stochastic calculus (an idea which we realize in Part IV). In [Ske98a] we pointed out that one of the first full Fock modules appeared already in Accardi, Lu [AL96] in the context of the stochastic limit of quantum field theory. We discuss this in Appendix D; see also Example 6.1.7.

6.1.4 Definition. For any mapping $T \in \mathcal{B}^{a,bil}(E)$ we define its *second quantization*

$$\mathcal{F}(T) = \bigoplus_{n \in \mathbb{N}_0} T^{\odot n} \in \mathcal{B}^a(\mathcal{F}(E)) \qquad\qquad (T^{\odot 0} = \mathsf{id}).$$

6.1.5 Example. One of the most important full Fock modules is $\mathcal{F}(L^2(\mathbb{R}, F))$. In Example 4.3.14 we defined the time shift $\mathfrak{s}_t$ in $\mathcal{B}^{a,bil}(L^2(\mathbb{R}, F))$ for some Hilbert $\mathcal{B}$–$\mathcal{B}$–module F. The corresponding second quantized time shift $\mathcal{F}(\mathfrak{s}_t)$ gives rise to the *time shift automorphism group* $\mathcal{S}$ on $\mathcal{B}^a\big(\mathcal{F}(L^2(\mathbb{R}, E))\big)$, defined by setting

$$\mathcal{S}_t(a) \;=\; \mathcal{F}(\mathfrak{s}_t)a\mathcal{F}(\mathfrak{s}_t)^*.$$

$\mathcal{F}(\mathfrak{s}_t)$ is $\mathcal{B}$–$\mathcal{B}$–linear so that $\mathcal{S}$ leaves invariant $\mathcal{B}$. By Proposition 4.3.15, $\mathcal{S}$ is strongly continuous.

By Example 4.4.10, setting $\mathbb{E}_0(a) = \langle \omega, a\omega \rangle$, we define a conditional expectation

$$\mathbb{E}_0 \colon \mathcal{B}^a(\mathcal{F}(E)) \longrightarrow \mathcal{B}\ (\subset \mathcal{B}^a(\mathcal{F}(E))),$$

the *vacuum conditional expectation*. Clearly, $(\mathcal{F}(E), \omega)$ is the GNS-construction for $\mathbb{E}_0$.

Obviously, $\mathbb{E}_0$ is continuous in the strong topology of $\mathcal{B}^a(\mathcal{F}(E))$ and the norm topology of $\mathcal{B}$. However, by Proposition 3.1.5 $\mathbb{E}_0$ is also continuous in the strong topologies of $\mathcal{B}^a(\mathcal{F}(E)) \subset \mathcal{B}(\mathcal{F}(E) \bar{\odot} G)$ and $\mathcal{B} \subset \mathcal{B}(G)$, when $\mathcal{B}$ is represented on a Hilbert space G. In particular, if E is a von Neumann module, then $\mathbb{E}_0$ extends to a normal conditional expectation $\mathcal{B}^a(\mathcal{F}^s(E)) \to \mathcal{B}$.

6.1.6 Example. Let $E = \overline{\mathcal{B}(G) \otimes \mathfrak{H}}^s = \mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$ be an (arbitrary) von Neumann $\mathcal{B}(G)$–$\mathcal{B}(G)$–module (with Hilbert spaces $G, \mathfrak{H}$). By Example 4.2.13 we find $\mathcal{F}^s(E) = \overline{\mathcal{B}(G) \otimes \mathcal{F}(\mathfrak{H})}^s = \mathcal{B}(G, G \bar{\otimes} \mathcal{F}(\mathfrak{H}))$ where $\mathcal{F}(\mathfrak{H})$ is (in accordance with Definition 6.1.1) the usual full Fock space over $\mathfrak{H}$. Moreover, $\mathcal{F}^s(E) \bar{\odot}^s G = G \bar{\otimes} \mathcal{F}(\mathfrak{H})$. By Example 3.1.2 we have $\mathcal{B}^a(\mathcal{F}^s(E)) = \mathcal{B}(G \bar{\otimes} \mathcal{F}(\mathfrak{H}))$. In other words, considering operators on the full Fock module $\mathcal{F}^s(E)$ ammounts to the same as considering operators on the *initial space* G tensor the full Fock space over the center $\mathfrak{H}$ of E. This fact remains true for other Fock modules over $\mathcal{B}(G)$–modules. It fails, in general, for modules over other C^*– (or von Neumann) algebras.

Example 4.4.12 tells us that the vacuum conditional expectation on $\mathcal{B}^a(\mathcal{F}^s(E))$ is precisely the identity on $\mathcal{B}(G)$ tensor the vacuum expectation $\langle \Omega, \bullet\Omega \rangle$ on $\mathcal{F}(\mathfrak{H})$. Other operations appearing simple in the module description, appear, however, considerably more complicated when translated back into the Hilbert space picture. For instance, let $x \in E$ and consider the creator $\ell^*(x)$ on $\mathcal{F}^s(E)$. Expanding $x = \sum_{\beta \in B} b_\beta \otimes e_\beta$ for some ONB $\left(e_\beta\right)_{\beta \in B}$ of $\mathfrak{H}$ as in Example 4.1.15, we see that $\ell^*(x)$ corresponds to the operator $\sum_{\beta \in B} b_\beta \otimes \ell^*(e_\beta)$ on $G \bar{\otimes} \mathcal{F}(\mathfrak{H})$. In general, there is no possibility to write $\ell^*(x)$ as a single tensor $b \otimes \ell^*(h)$. Similarly, expanding an operator $T \in \mathcal{B}^a(E)$ as $T = \sum_{\beta, \beta' \in B} (\mathbf{1} \otimes e_\beta) b_{\beta,\beta'} (\mathbf{1} \otimes e_{\beta'})^*$ we see that $p(T)$ corresponds to the operator $\sum_{\beta,\beta' \in B} b_{\beta,\beta'} \otimes e_\beta e_{\beta'}^*$ on $G \bar{\otimes} \mathcal{F}(\mathfrak{H})$.

6.1.7 Example. Let E denote the completion of the $\mathcal{P}$–$\mathcal{P}$–module E_0 from Example 1.6.11. Then as discussed in Appendix D the limits of the *moments* of certain *field operators* converge in the vacuum conditional expextation to the moments of suitable creators and annihilators on the full Fock module $\mathcal{F}(L^2(\mathbb{R}, E))$.

We mention Corollary 8.4.2, where we realize the relations of the *free square of white noise* (introduced by Sniady [Sni00] as a modification of the *square of white noise* introduced by Accardi, Lu and Volovich [ALV99]) by creators and annihilators on a full Fock module over a P^*–algebra (see Appendix C).

6.2 The full Fock module as graded Banach space

In this section we apply the notions from Appendices A.2 and A.3 to a graded subspace $\mathcal{F}_g(E)$ of $\mathcal{F}(E)$. Following [Ske98b, Ske00d] we introduce the *generalized creators*, which are related to this graduation. They appear naturally, if we want to explain why Arveson's spectral algebra [Arv90a] is the continuous time analogue of the Cuntz algebra [Cun77] in Section 12.5 and they allow to describe most conveniently the algebraic consequences of adaptedness in Section 6.3.

6.2.1 Definition. We define the *homogeneous* subspaces of $\mathcal{F}(E)$ by $E^{(n)} = \overline{E^{\odot n}}$ $(n \in \mathbb{N}_0)$ and $E^{(n)} = \{0\}$ $(n < 0)$. We denote by $\mathcal{F}_g(E)$ and $\mathcal{F}_1(E)$ the algebraic direct sum and the ℓ^1–completed direct sum, respectively, over all $E^{(n)}$. In other words, $\mathcal{F}_1(E)$ consists of all families $\left(x^{(n)}\right)_{n\in\mathbb{Z}}$ $(x^{(n)} \in E^{(n)})$ for which $\|x\|_1 = \sum_{n\in\mathbb{Z}} \|x^{(n)}\| < \infty$. Since $\|x\| \le \|x\|_1$, we have $\underline{\mathcal{F}}(E) \subset \mathcal{F}_g(E) \subset \mathcal{F}_1(E) \subset \mathcal{F}(E)$.

6.2.2 Definition. For $n \in \mathbb{Z}$ we denote by $\mathcal{B}^{(n)} \subset \mathcal{B}^a(\mathcal{F}(E))$ the Banach space consisting of all operators with *offset n in the number of particles*, i.e. $a^{(n)} \in \mathcal{B}^{(n)}$, if $a^{(n)}(E^{\odot m}) \subset E^{\odot(m+n)}$. Also $\mathcal{B}^a(\mathcal{F}(E))$ has a natural graded vector subspace $\mathcal{B}_g$ with $\mathcal{B}^{(n)}$ $(n \in \mathbb{Z})$ being the *homogeneous* subspaces. Any $a \in \mathcal{B}^a(\mathcal{F}(E))$ allows a $*$–strong decomposition into $a = \sum_{n\in\mathbb{Z}} a^{(n)}$ with $a^{(n)} \in \mathcal{B}^{(n)}$. We define the Banach space $\mathcal{B}_1$ as the ℓ^1–completed direct sum. It consists of all $a \in \mathcal{B}^a(\mathcal{F}(E))$ for which $\|a\|_1 = \sum_{n\in\mathbb{Z}} \|a^{(n)}\| < \infty$. Again, we have $\|a\| \le \|a\|_1$, so that $\mathcal{B}_g \subset \mathcal{B}_1 \subset \mathcal{B}^a(\mathcal{F}(E))$.

We have $\ell^*(x) \in \mathcal{B}^{(1)}$, $p(T) \in \mathcal{B}^{(0)}$ and $\ell(x) \in \mathcal{B}^{(-1)}$. Obviously, $\mathcal{B}^{(n)}\mathcal{B}^{(m)} \subset \mathcal{B}^{(n+m)}$ so that the multiplication on $\mathcal{B}_g$ is an even mapping. Notice also that $\mathcal{B}^{(n)^*} \subset \mathcal{B}^{(-n)}$. By Lemma A.3.1 $\mathcal{B}_1$ is a Banach $*$–algebra. $\mathcal{B}^a(\mathcal{F}(E))$ is $*$–strongly complete and, therefore, so is the closed subspace $\mathcal{B}^{(n)}$.

6.2.3 Definition. Let $X \in \mathcal{F}(E)$. By the *generalized creator* $\widehat{\ell}^*(X)$ we mean the operator on $\mathcal{F}(E)$ defined by setting

$$\widehat{\ell}^*(X)Y = X \odot Y$$

for $Y \in E^{\odot n}$, where we identify $\mathcal{F}(E) \odot E^{\odot n}$ as a subset of $\mathcal{F}(E)$ in an obvious way (cf. the proof of Proposition 6.3.1). If $\widehat{\ell}^*(X)$ has an adjoint $\widehat{\ell}(X)$ on $\underline{\mathcal{F}}(E)$, then we call $\widehat{\ell}(X)$ *generalized annihilator* .

6.2.4 Remark. For $Y \in E^{\odot n}$ we easily find $\|\widehat{\ell}^*(X)Y\| \leq \|X\| \, \|Y\|$. However, $\widehat{\ell}^*(X)$ is not necessarily a bounded operator on $\mathcal{F}(E)$. To see this let $X = \sum\limits_{n=0}^{\infty} \frac{e_n}{(n+1)^\alpha} \in \mathcal{F}(\mathbb{C})$ $(\alpha = \frac{2}{3})$ where e_n is the unit vector $1^{\otimes n} \in \mathbb{C}^{\otimes n}$. Taking into account that $\frac{1}{(k+1)(n-k+1)} \geq \frac{2}{n^2}$ $(n \geq k)$ and $2(2\alpha - 1) = \alpha$, we find

$$\|X \otimes X\|^2 = \Big\|\sum_{n=0}^{\infty} e_n \sum_{k=0}^{n} \frac{1}{(k+1)^\alpha (n-k+1)^\alpha}\Big\|^2 \geq \Big\|\sum_{n=0}^{\infty} e_n n \frac{2^\alpha}{n^{2\alpha}}\Big\|^2 = \sum_{n=0}^{\infty} \frac{4^\alpha}{n^\alpha} = \infty.$$

Under certain circumstances $\widehat{\ell}^*(X)$ is bounded. For $X \in E^{(n)}$ we find

$$\widehat{\ell}(X) x_{n+m} \odot \ldots \odot x_1 = \langle X, x_{n+m} \odot \ldots \odot x_{m+1} \rangle x_m \odot \ldots \odot x_1$$

and $\widehat{\ell}(X) E^{\odot m} = \{0\}$, if $m < n$.

6.2.5 Proposition. *Let $X \in E^{(n)}$. Then $\widehat{\ell}^*(X) \in \mathcal{B}^{(n)}$ and $\widehat{\ell}(X) \in \mathcal{B}^{(-n)}$. We have $\|\widehat{\ell}^*(X)\| = \|\widehat{\ell}(X)\| = \|X\|$. For $T \in \mathcal{B}^a(E)$ we have*

$$p(T)\widehat{\ell}^*(X) = \widehat{\ell}^*(p(T)X)$$

where we consider X also as an element of $\mathcal{F}(E)$. Moreover, for $Y \in E^{(m)}$ we have

$$\widehat{\ell}(X)\widehat{\ell}^*(Y) = \widehat{\ell}^*(\widehat{\ell}(X)Y) \quad \text{or} \quad \widehat{\ell}(X)\widehat{\ell}^*(Y) = \widehat{\ell}(\widehat{\ell}(Y)X)$$

depending on whether $n < m$ or $n > m$. For $n = m$ we have

$$\widehat{\ell}(X)\widehat{\ell}^*(Y) = \langle X, Y \rangle. \tag{6.2.1}$$

Proof. We only show $\|\widehat{\ell}^*(X)\| = \|X\|$. This follows easily from (6.2.1), because for $Y \in \mathcal{F}(E)$ we have $\|\widehat{\ell}^*(X)Y\|^2 = \|\langle Y, \langle X, X \rangle Y \rangle\| \leq \|\langle X, X \rangle\| \, \|\langle Y, Y \rangle\| = \|X\|^2 \, \|Y\|^2$. ■

6.2.6 Corollary. *For $X \in \mathcal{F}_1(E)$ we have $\|\widehat{\ell}^*(X)\|_1 = \|\widehat{\ell}(X)\|_1 = \|X\|_1$. In particular, we find for $a \in \mathcal{B}_1$ that $\|\widehat{\ell}^*(a\omega)\| \leq \|\widehat{\ell}^*(a\omega)\|_1 = \|a\omega\|_1 \leq \|a\|_1$ so that $\widehat{\ell}^*(a\omega)$ is a well-defined element of $\mathcal{B}_1 \subset \mathcal{B}^a(\mathcal{F}(E))$.*

6.2.7 Corollary. *Let $a_t \in \mathcal{B}_1$ such that $t \mapsto a_t$ is strongly continuous in $\mathcal{B}^a(\mathcal{F}(E))$. Then both mappings $t \mapsto \widehat{\ell}^*(a_t\omega)$ and $t \mapsto \widehat{\ell}(a_t\omega)$ are $\|\bullet\|_1$–continuous.*

Proof. By an argument very similar to the proof of Lemma A.3.2, we see that $t \mapsto a_t$ is strongly continuous also in $\mathcal{B}_1$. Now the statement follows easily from Corollary 6.2.6. ■

6.2.8 Remark. $\mathcal{F}_1(E)$, equipped with the multiplication obtained from the multiplication of the $\mathcal{B}$–tensor algebra $\mathcal{F}_g(E)$ (see [Ske98a]) and continuous extension in $\|\bullet\|_1$, is a Banach algebra. Corollary 6.2.6 tells us that $\widehat{\ell}^*$ and $\widehat{\ell}$ are an isometric homomorphism and an isometric (anti-linear) anti-homomorphism, respectively, into $\mathcal{B}_1$.

6.3 Adaptedness

6.3.1 Proposition. *Let E, F be Hilbert $\mathcal{B}$–$\mathcal{B}$–modules. Then*

$$\underline{\mathcal{F}}(E \oplus F) \;\cong\; \underline{\mathcal{F}}(E) \odot \Big(\mathcal{B}\omega \;\oplus\; F \odot \underline{\mathcal{F}}(E \oplus F)\Big) \tag{6.3.1}$$

in a canonical way.

Proof. Let $n, m \geq 0$, $x_i \in E$ $(i = 1, \cdots, n)$, $y \in F$, $z_j \in E \oplus F$ $(j = 1, \cdots, m)$. We easily check that the mapping, sending $(x_n \odot \ldots \odot x_1) \odot (y \odot z_m \odot \ldots \odot z_1)$ on the right-hand side to $x_n \odot \ldots \odot x_1 \odot y \odot z_m \odot \ldots \odot z_1$ on the left-hand side (and sending $(x_n \odot \ldots \odot x_1) \odot \omega$ to $x_n \odot \ldots \odot x_1$, and $\omega \odot (y \odot z_m \odot \ldots \odot z_1)$ to $y \odot z_m \odot \ldots \odot z_1$), and, of course, sending $\omega \odot \omega$ to ω) extends as an isometry onto $\underline{\mathcal{F}}(E \oplus F)$. ■

This factorization was found first for Fock spaces by Fowler [Fow95]. We used it independently in [Ske98b] in the context of quantum stochastic calculus, in order to describe adapted operators.

6.3.2 Definition. An operator a in $\mathcal{B}^a(\mathcal{F}(E \oplus F))$ is called *adapted to* E, if there is an operator $a_E \in \mathcal{B}^a(\mathcal{F}(E))$ such that $a = (a_E \odot \mathsf{id})$ in the decomposition according to (6.3.1). Applying $a_E \odot \mathsf{id}$ to vectors of the form $x \odot \omega$, we see that a_E is unique and that $\|a_E\| = \|a\|$.

6.3.3 Observation. By definition, the set of all operators adapted to E is precisely

$$\mathcal{B}^a(\mathcal{F}(E)) \odot \mathsf{id} \cong \mathcal{B}^a(\mathcal{F}(E)).$$

(This identification is an isomorphism of C^*–algebras. The $*$–strong topology is, in general, not preserved.) The identification is canonical in the sense that it identifies creators to the same element $x \in E$. Indeed, the creator $\ell^*(x) \in \mathcal{B}^a(\mathcal{F}(E))$ $(x \in E)$ embedded via $(\ell^*(x) \odot \mathsf{id})$ into $\mathcal{B}^a(\mathcal{F}(E \oplus F))$ coincides with the creator $\ell^*(x) \in \mathcal{B}^a(\mathcal{F}(E \oplus F))$ where now x is considered as an element of $E \oplus F$. The $*$–algebra generated by all creators to elements $x \in E$ is $*$–strongly dense in $\mathcal{B}^a(\mathcal{F}(E))$. (To see this, it is sufficient to show that the $*$–strong closure contains $\omega\omega^*$, because then it contains $\mathfrak{F}(\mathcal{F}(E))$ and by Corollary 2.1.11 all of $\mathcal{B}^a(\mathcal{F}(E))$. So, let us choose an approximate unit $u^\lambda = \sum_{k=1}^{n_\lambda} v_k^\lambda w_k^{\lambda *}$ for $\mathfrak{F}(E)$ with $v_k^\lambda, w_k^\lambda \in E$. Then

$$\omega\omega^* \;=\; \lim_\lambda \Big(\mathbf{1} - \sum_{k=1}^{n_\lambda} \ell^*(v_k^\lambda)\ell(w_k^\lambda)\Big)$$

in the $*$–strong topology. Actually, this shows density of the ball in the ball.) Therefore, we may identify the $*$–subalgebra of $\mathcal{B}^a(\mathcal{F}(E \oplus F))$ consisting of all operators adapted to

E with the $*$–strong closure in $\mathcal{B}^a(\mathcal{F}(E \oplus F))$ of the $*$–algebra generated by all creators on $\mathcal{F}(E \oplus F)$ to elements in $E \subset E \oplus F$.

Under the above isomorphism also the Banach $*$–algebra $\mathcal{B}_1 \subset \mathcal{B}^a(\mathcal{F}(E))$ coincides (isometrically in $\|\bullet\|_1$) with the Banach $*$–algebra of all elements in $\mathcal{B}_1 \subset \mathcal{B}^a(\mathcal{F}(E \oplus F))$ which are adapted to E.

Finally, we remark that by Example 4.4.11 the central vector ω in $\mathcal{B}\omega \oplus F \bar{\odot} \mathcal{F}(E \oplus F)$ defines a conditional expectation $\varphi\colon a \mapsto (\mathsf{id} \odot \omega^*)a(\mathsf{id} \odot \omega)$ onto $\mathcal{B}^a(\mathcal{F}(E)) \subset \mathcal{B}^a(\mathcal{F}(E \oplus F))$ and that $\mathbb{E}_0(a) = \mathbb{E}_0 \circ \varphi(a)$.

6.3.4 Corollary. *Let $x \in E$, $T \in \mathcal{B}^a(E)$ and $X \in \mathcal{F}_1(E)$. Then $\ell^*(x)$, $\ell(x)$, $p(T)$, $\widehat{\ell}^*(X)$ and $\widehat{\ell}(X)$ are adapted to E. Also the identity is adapted. Moreover, $\widehat{\ell}^*(X) \in \mathcal{B}_1$ is adapted to E, if and only if $X \in \mathcal{F}_1(E)$.*

6.3.5 Lemma. *Let $a \in \mathcal{B}_1$ be adapted to E and T in $\mathcal{B}^a(F)$. Then*

$$a\,p(T) = \widehat{\ell}^*(a\omega)p(T) \tag{6.3.2a}$$

and

$$p(T)a = p(T)\widehat{\ell}(a^*\omega). \tag{6.3.2b}$$

Proof. As (6.3.2b) is more or less the adjoint of (6.3.2a), it is sufficient only to prove (6.3.2a).

(6.3.2a) follows from the observation that the range of $p(T)$ is contained in $\big(F \odot \mathcal{F}(E \oplus F)\big)$ and from $a_E\omega = a\omega$ in the identification $\mathcal{F}(E) \subset \mathcal{F}(E \oplus F)$. ■

6.3.6 Corollary. *Let $a, b \in \mathcal{B}_1$ both be adapted to E and let T, T' be in $\mathcal{B}^a(F)$. Then*

$$p(T)ab\,p(T') = p\big(T\mathbb{E}_0(ab)T'\big).$$

Proof. By Corollary 6.2.6 we may assume that $a \in \mathcal{B}^{(n)}$ and $b \in \mathcal{B}^{(m)}$. First, suppose $-n \neq m$. Then $\mathbb{E}_0(ab) = 0$. Without loss of generality we may assume $-n < m$. From Proposition 6.2.5 and Lemma 6.3.5 we find

$$p(T)ab\,p(T') = \widehat{\ell}^*\big(p(T)\widehat{\ell}(a^*\omega)b\omega\big)\,p(T') = 0,$$

because $\widehat{\ell}(a^*\omega)b\omega$ is an element of $E^{\odot(n+m)}$ and T vanishes on E. If $n = m$, we find $p(T)ab\,p(T') = p(T)\widehat{\ell}(a^*\omega)\widehat{\ell}^*(b\omega)p(T') = p(T)\mathbb{E}_0(ab)p(T')$. Therefore, in both cases we obtain our claimed result. ■

6.3.7 Corollary. *Suppose $a \in \mathcal{B}^{(0)}$ is adapted to E and $T \in \mathcal{B}^a(F)$. Then*

$$a\,p(T) = \mathbb{E}_0(a)p(T).$$

6.3.8 Observation. Let F be a Hilbert $\mathcal{B}$–$\mathcal{B}$–module and denote $E_t = L^2((-\infty, t], F)$, $E = E_\infty = L^2(\mathbb{R}, F)$, and, more generally, $E_K = L^2(K, F)$ for measurable subsets K of $\mathbb{R}$. Suppose $a \in \mathcal{B}^a(\mathcal{F}(E))$ is adapted to $E_{\mathbb{R}_+}$. Then for all $t \in \mathbb{R}_+$ also $\mathcal{S}_t(a)$ is adapted to $E_{\mathbb{R}_+}$. This follows from the factorization

$$\begin{aligned}
\underline{\mathcal{F}}(E) &= \underline{\mathcal{F}}(E_{\mathbb{R}_+}) \odot \Big(\mathcal{B}\omega \oplus E_{\mathbb{R}_-} \odot \underline{\mathcal{F}}(E)\Big) \\
&= \underline{\mathcal{F}}(E_{[t,\infty)}) \odot \Big(\mathcal{B}\omega \oplus E_{[0,t]} \odot \underline{\mathcal{F}}(E_{\mathbb{R}_+})\Big) \odot \Big(\mathcal{B}\omega \oplus E_{\mathbb{R}_-} \odot \underline{\mathcal{F}}(E)\Big),
\end{aligned}$$

which shows that after time shift a acts only on the very first factor. We conclude that $\big(\mathcal{S}_t\big)_{t \in \mathbb{R}_+}$ restricts to a unital endomorphism semigroup, i.e. an E_0*–semigroup* (see Section 10.1), on $\mathcal{B}^a(\mathcal{F}(E_{\mathbb{R}_+})) \odot \mathsf{id} \cong \mathcal{B}^a(\mathcal{F}(E_{\mathbb{R}_+}))$. Since by Example 6.1.5 $\mathcal{S}$ is continuous in the strong topology of $\mathcal{B}^a(\mathcal{F}(E))$, it is *a fortiori* continuous in the strong topology of $\mathcal{B}^a(\mathcal{F}(E_{\mathbb{R}_+}))$. Clearly, also here the extension to an E_0–semigroup on the von Neumann algebra $\mathcal{B}^a(\mathcal{F}^s(E_{\mathbb{R}_+}))$ consists of normal mappings.

Chapter 7

The time ordered Fock module

The time ordered Fock modules were introduced in Bhat and Skeide [BS00] as the basic examples for product systems of Hilbert modules, paralleling the fact that symmetric Fock spaces are the basic examples of Arveson's tensor product systems of Hilbert spaces [Arv89a].

After defining time ordered Fock modules and establishing their basic properties like the factorization into tensor products (Section 7.1), we investigate exponential vectors (Section 7.2). In Section 7.3 we show that the continuous units for the time ordered Fock module are precisely the exponential units and their renormalizations (Liebscher and Skeide [LS00b]). This parallels completely the case of symmetric Fock spaces, except that here the renormalization is considerably more complicated. In Section 7.4 we throw a bridge to CPD-semigroups. In Section 7.5 we present an example from Skeide [Ske99c] where we look at modules over $\mathbb{C}^2$, the diagonal subalgebra of M_2.

7.1 Basic properties

As the name tells us, the construction of the *time ordered Fock module* is connected with the time structure of its one-particle sector $L^2(\mathbb{R}, F)$. We take this into account by speaking of the time ordered Fock module over F rather than over $L^2(\mathbb{R}, F)$. Additionally, we are interested mainly in the real half-line $\mathbb{R}_+$ and include also this in the definition.

7.1.1 Definition. By Δ_n we denote the indicator function of the subset $\{(t_n, \ldots, t_1) : t_n > \ldots > t_1\}$ of $\mathbb{R}^n$. Let $\mathcal{B}$ be a unital C^*–algebra, let F be a Hilbert $\mathcal{B}$–$\mathcal{B}$–module and set $E = L^2(\mathbb{R}, F)$. (We use also the other notations from Observation 6.3.8.) By Observation 4.3.11 Δ_n acts as a projection on $E^{\bar{\odot}\, n} = L^2(\mathbb{R}^n, F^{\bar{\odot}\, n})$. We call the range of Δ_n applied $E^{\bar{\odot}\, n}$ (or some submodule) the *time ordered part* of $E^{\bar{\odot}\, n}$ (or of this submodule).

The *time ordered Fock module* over F is $\mathrm{I\Gamma}(F) = \Delta\mathcal{F}(E_{\mathbb{R}_+}) \subset \mathcal{F}(E_{\mathbb{R}_+})$ where $\Delta = \bigoplus_{n=0}^{\infty} \Delta_n$ is the projection onto the *time ordered part* of $\mathcal{F}(E)$. The *extended time ordered Fock module* is

$\breve{\mathbb{I}\Gamma}(F) = \Delta\mathcal{F}(E)$. We use the notations $\mathbb{I}\Gamma_t(F) = \Delta\mathcal{F}(E_{[0,t)})$ $(t \geq 0)$ and $\mathbb{I}\Gamma_K(F) = \Delta\mathcal{F}(E_K)$ (K a measurable subset of $\mathbb{R}$). If $\mathcal{B}$ is a von Neumann algebra on a Hilbert space G, then we indicate the strong closure by $\mathbb{I}\Gamma^s$, etc. .

The *algebraic time ordered Fock module* is $\underline{\mathbb{I}\Gamma}(F) = \Delta\underline{\mathcal{F}}(\mathfrak{S}(\mathbb{R}_+, F))$ (where here F maybe only a pre-Hilbert module). Observe that $\underline{\mathbb{I}\Gamma}(F)$ is not a subset of $\underline{\mathcal{F}}(\mathfrak{S}(\mathbb{R}_+, F))$ (unless $F^{\odot 2}$ is trivial).

Definition 7.1.1 and the factorization in Theorem 7.1.3 are due to [BS00]. The time ordered Fock module is a straightforward generalization to Hilbert modules of the Guichardet picture of symmetric Fock space [Gui72] and the generalization to the higher-dimensional case discussed by Schürmann [Sch93] and Bhat [Bha98].

7.1.2 Observation. The time shift $\mathcal{S}$ leaves invariant the projection $\Delta \in \mathcal{B}^a(\mathcal{F}(E))$. It follows that $\mathcal{S}$ restricts to an automorphism group on $\mathcal{B}^a(\breve{\mathbb{I}\Gamma}(F))$ and further to an E_0–semigroup $\mathcal{B}^a(\mathbb{I}\Gamma(F))$ (of course, both strongly continuous and normal in the case of von Neuman modules).

The following theorem is the analogue of the well-known factorization $\Gamma(L^2([0, s+t])) = \Gamma(L^2([t, s+t])) \otimes \Gamma(L^2([0, t]))$ of the symmetric Fock space. However, in the theory of product systems, be it of Hilbert spaces in the sense of Arveson [Arv89a] or of Hilbert modules in the sense of Part III (of which the time ordered Fock modules show to be the most fundamental examples), we put emphasis on the length of intervals rather than on their absolute position on the half line. (We comment on this crucial difference in Observation B.3.4.) Therefore, we are more interested to write the above factorization in the form $\Gamma(L^2([0, s+t])) = \Gamma(L^2([0, s])) \otimes \Gamma(L^2([0, t]))$, where the first factor has first to be time shifted by t. Adopting this way of thinking (where the time shift is *encoded* in the tensor product) has enormous advantages in many formulae. We will use it consequently throughout. Observe that, contrary to all good manners, we write the future in the first place and the past in the second. This order is forced upon us and, in fact, we will see soon that for Hilbert modules the order is no longer arbitrary.

7.1.3 Theorem. *The mapping u_{st}, defined by setting*

$$\begin{aligned}[u_{st}(X_s \odot Y_t)](s_m, \ldots, s_1, t_n, \ldots, t_1) &= [\mathcal{F}(\mathcal{s}_t)X_s](s_m, \ldots, s_1) \odot Y_t(t_n, \ldots, t_1) \\ &= X_s(s_m - t, \ldots, s_1 - t) \odot Y_t(t_n, \ldots, t_1), \qquad (7.1.1)\end{aligned}$$

($s + t \geq s_m \geq \cdots \geq s_1 \geq t \geq t_n \geq \cdots \geq t_1 \geq 0, X_s \in \Delta_m E^{\odot m}_{[0,s]}, Y_t \in \Delta_n E^{\odot n}_{[0,t]}$ extends as a two-sided isomorphism $\underline{\mathbb{I}\Gamma}_s(F) \odot \underline{\mathbb{I}\Gamma}_t(F) \to \underline{\mathbb{I}\Gamma}_{s+t}(F)$. It extends to two-sided isomorphisms $\mathbb{I}\Gamma_s(F) \bar{\odot} \mathbb{I}\Gamma_t(F) \to \mathbb{I}\Gamma_{s+t}(F)$ and $\mathbb{I}\Gamma^s_s(F) \bar{\odot}^s \mathbb{I}\Gamma^s_t(F) \to \mathbb{I}\Gamma^s_{s+t}(F)$, respectively. Moreover,

$$u_{r(s+t)}(\mathsf{id} \odot u_{st}) \;=\; u_{(r+s)t}(u_{rs} \odot \mathsf{id}).$$

PROOF. The extension properties are obvious. So let us show (7.1.1).

First, recall that the time shift $\mathcal{F}(\mathfrak{s}_t)$ sends $\underline{\mathbb{I}\Gamma}_s$ onto $\underline{\mathbb{I}\Gamma}_{[t,t+s)}$. Therefore, the function $u_{st}(X_s \odot Y_t)$ is in $\Delta_{n+m} E^{\odot m+n}_{[0,s+t)} \subset \underline{\mathbb{I}\Gamma}_{s+t}$. A simple computation (doing the integrations over $s_m, \ldots, s_1$ first and involving a time shift by $-t$) shows that u_{st} is an isometric mapping. Of course, all u_{st} are $\mathcal{B}$–$\mathcal{B}$–linear and fulfill the associativity condition.

It remains to show that u_{st} is surjective. $E^{\odot n}_{[s+t)}$ is spanned by functions of the form $X = \mathbb{I}_{[s_n,t_n)}\zeta_n \odot \ldots \odot \mathbb{I}_{[s_1,t_1)}\zeta_1$ $(\zeta_1, \ldots, \zeta_n \in F)$. Since we are interested in $\Delta_n E^{\odot n}_{[s+t)}$ only, (splitting an interval into two, if necessary) we may restrict to the case where for each $i = 1, \ldots, n-1$ either $s_{i+1} \geq t_i$ or $s_{i+1} = s_i, t_{i+1} = t_i$. Furthermore, (by the same argument) we may assume, that $s_1 \geq t$, or that there exists m $(1 \leq m < n)$ such that $t_m \leq t$ and $s_{m+1} \geq t$, or that $t_n \leq t$. In the first case we have $\Delta_n X \in \underline{\mathbb{I}\Gamma}_{[t,t+s)}$ so that $\Delta_n X = u_{st}(\mathcal{F}(\mathfrak{s}_t^{-1})\Delta_n X \odot \omega)$ is in the range of u_{st}. Similarly, in the third case $\Delta_n X \in \underline{\mathbb{I}\Gamma}_t$ so that $\Delta_n X = u_{st}(\omega \odot \Delta_n X)$ is in the range of u_{st}. In the second case we set $Y_2 = \Delta_{n-m}\mathbb{I}_{[s_n,t_n)}\zeta_n \odot \ldots \odot \mathbb{I}_{[s_{m+1},t_{m+1})}\zeta_{m+1} \in \underline{\mathbb{I}\Gamma}_{[t,t+s)}$ and $Y_1 = \Delta_m \mathbb{I}_{[s_m,t_m)}\zeta_m \odot \ldots \odot \mathbb{I}_{[s_1,t_1)}\zeta_1 \in \underline{\mathbb{I}\Gamma}_t$. Again, we see that $\Delta_n X = u_{st}(\mathcal{F}(\mathfrak{s}_t^{-1})Y_2 \odot Y_1)$ is in the range of u_{st}. ■

7.1.4 Observation. Letting in the preceding computation formally $s \to \infty$, we see that (7.1.1) defines a two-sided isomorphism $u_t \colon \underline{\mathbb{I}\Gamma}(F) \odot \underline{\mathbb{I}\Gamma}_t(F) \to \underline{\mathbb{I}\Gamma}(F)$. We have $u_{s+t}(\mathsf{id} \odot u_{st}) = u_t(u_s \odot \mathsf{id})$. In the sequel, we no longer write u_{st} nor u_t and just use the identifications $\underline{\mathbb{I}\Gamma}_s(F) \odot \underline{\mathbb{I}\Gamma}_t(F) = \underline{\mathbb{I}\Gamma}_{s+t}(F)$ and $\underline{\mathbb{I}\Gamma}(F) \odot \underline{\mathbb{I}\Gamma}_t(F) = \underline{\mathbb{I}\Gamma}(F)$. Notice that in the second identification $\mathcal{S}_t(a) = a \odot \mathsf{id}_{\underline{\mathbb{I}\Gamma}_t(F)} \in \mathcal{B}^a(\underline{\mathbb{I}\Gamma}(F) \odot \underline{\mathbb{I}\Gamma}_t(F)) = \mathcal{B}^a(\underline{\mathbb{I}\Gamma}(F))$. We explain this more detailed in a more general context in Sections 11.4 and 14.1.

7.2 Exponential vectors

In the symmetric Fock space we may define an *exponential vector* to any element in the one-particle sector. In the time ordered Fock module we must be more careful.

7.2.1 Definition. Let $x \in \mathfrak{S}(\mathbb{R}_+, F)$. We define the *exponential vector* $\psi(x) \in \mathbb{I}\Gamma(F)$ as

$$\psi(x) \;=\; \sum_{n=0}^{\infty} \Delta_n x^{\odot n}$$

with $x^{\odot 0} = \omega$. (Observe that if x has support $[0,t]$ and $\|x(s)\| \leq c \in \mathbb{R}_+$, then $\|\Delta_n x^{\odot n}\|^2 \leq \frac{t^n c^{2n}}{n!}$ where $\frac{t^n}{n!}$ is the volume of the set $\{(t_n, \ldots, t_1) \colon t \geq t_n \geq \ldots \geq t_1 \geq 0\}$ so that $\|\psi(x)\|^2 \leq e^{tc^2} < \infty$.)

Let $\mathfrak{t} = (t_n, \ldots, t_1) \in \mathbb{I}_t$ as defined in Appendix B.3, put $t_0 = 0$, and let $x = \sum_{i=1}^{n} \zeta_i 1\!\!1_{[t_{i-1}, t_i)}$. Then we easily check

$$\psi(x) \;=\; \psi(\zeta_n 1\!\!1_{[0, t_n - t_{n-1})}) \odot \ldots \odot \psi(\zeta_1 1\!\!1_{[0, t_1 - t_0)}). \tag{7.2.1}$$

7.2.2 Theorem. *For all $t \in [0, \infty]$ the exponential vectors to elements $x \in \mathfrak{S}([0,t], F)$ form a total subset of $\mathrm{I\Gamma}_t(F)$.*

Proof. We are done, if we show that the span of all $\psi(x)$ contains all $\Delta_n X$ to functions $X \in L^2([0,t]^n, F^{\bar{\odot}\, n})$ chosen as in the proof of Theorem 7.1.3.

In the case when all intervals $[s_i, t_i)$ appear precisely once, we may assume (possibly after having inserted additional intervals where $\zeta_i = 0$) that $t_n = t$ and that $s_i = t_{i-1}$. In other words, $(t_n, \ldots, t_1) \in \mathbb{I}_t$ and we are in the situation of (7.2.1). Replacing ζ_i with $\lambda_i \zeta_i$ and differentiating with respect to all λ_i (what, of course, is possible) and evaluating at $\lambda_1 = \lambda_n = 0$ we find $\Delta_n X$.

For the remaing cases it is sufficient to understand $X = \Delta_n \zeta_n \odot \ldots \odot \zeta_1 1\!\!1_{[0,t)^n}$ separately. Making use of $L^2(\mathbb{R}_+^n) = L^2(\mathbb{R}_+)^{\bar{\otimes}\, n}$, we write

$$1\!\!1_{[0,t)^n} \;=\; 1\!\!1_{[0,t)}^{\otimes n} \;=\; \Big(\sum_{i=1}^{N} 1\!\!1_{\left[t\frac{i-1}{N}, t\frac{i}{N}\right)}\Big)^{\otimes n} \;=\; \sum_{i_n, \ldots, i_1 = 1}^{N} 1\!\!1_{\left[t\frac{i_n-1}{N}, t\frac{i_n}{N}\right)} \otimes \ldots \otimes 1\!\!1_{\left[t\frac{i_1-1}{N}, t\frac{i_1}{N}\right)}.$$

Applying Δ_n, only summands with $N \geq i_n \geq \ldots \geq i_1 \geq 1$ contribute. For all summands where $N \geq i_n > \ldots > i_1 \geq 1$, we are reduced to the preceding case (including the factor $\zeta_n \odot \ldots \odot \zeta_1$). Let us show that the remaining summands are negligible, if N tends to ∞. For simplicity, we assume $\|\zeta_n \odot \ldots \odot \zeta_1\| = 1$. Since Δ_n is a projection, we have

$$\|X\|^2 \;\leq \sum_{N \geq i_n \geq \ldots \geq i_1 \geq 1} \left(\tfrac{t}{N}\right)^n \;=\; \tbinom{N+n-1}{n}\left(\tfrac{t}{N}\right)^n \;\leq\; \sum_{i_n, \ldots, i_1 = 1}^{N} \left(\tfrac{t}{N}\right)^n \;=\; t^n.$$

If we sum only over $N \geq i_n > \ldots > i_1 \geq 1$, then we miss all cases where $i_k = i_{k+1}$ for at least one $k \leq n-1$. If we fix k, then the remaining sum has $\binom{N+n-2}{n-1}$ summands. We have $n-1$ possibilities to choose k. So the number $(n-1)\binom{N+n-2}{n-1}$ is, maybe, bigger (because we have counted some summands more than once), but certainly not smaller than the number of omitted summands. We find

$$\Big\|\Big[\sum_{N \geq i_n \geq \ldots \geq i_1 \geq 1} - \sum_{N \geq i_n > \ldots > i_1 \geq 1}\Big] \zeta_n \odot \ldots \odot \zeta_1 1\!\!1_{\left[t\frac{i_n-1}{N}, t\frac{i_n}{N}\right)} \otimes \ldots \otimes 1\!\!1_{\left[t\frac{i_1-1}{N}, t\frac{i_1}{N}\right)}\Big\|^2$$
$$\leq\; (n-1)\tbinom{N+n-2}{n-1}\left(\tfrac{t}{N}\right)^n \;=\; \tfrac{n(n-1)}{N+n-1}\tbinom{N+n-1}{n}\left(\tfrac{t}{N}\right)^n \;\leq\; \tfrac{n(n-1)}{N+n-1} t^n.$$

Clearly, this tends to 0 for $N \to \infty$. ■

7.2.3 Corollary. *Denote by p_t^{01} the projection onto the $0-$ and $1-$particle sector of $\mathbb{\Gamma}_t$. For $\mathfrak{t} \in \mathbb{I}_t$ set*

$$p_{\mathfrak{t}}^{01} \;=\; p_{t_n - t_{n-1}}^{01} \odot \ldots \odot p_{t_2 - t_1}^{01} \odot p_{t_1}^{01}.$$

Then $p_{\mathfrak{t}}^{01} \to \mathsf{id}_{\mathbb{\Gamma}_t}$ strongly over the increasing net $\mathbb{I}_t$.

Proof. Observe that the net $p_{\mathfrak{t}}^{01}$ of projections is increasing (and, of course, bounded by $\mathsf{id}_{\mathbb{\Gamma}_t}$). Therfore, it is sufficient to check strong convergence on the total subset of exponential vectors $\psi(x)$. By (7.2.1) we may even restrict to $x = \zeta I\!I_{[0,s)}$. Splitting the interval $[0, s)$ into n and taking into account that the contributions orthogonal to the 0– and 1–particle sector are of order s^2 (cf. (7.3.4)), our claim follows by a similar estimate as in the proof of Proposition A.5.4. ■

7.2.4 Remark. Obviously, the definition of the exponential vectors extends to elements $x \in L^\infty(\mathbb{R}_+, F) \cap L^2(\mathbb{R}_+, F)$. It is also not difficult to see that it makes sense for Bochner square integrable functions $x \in L^2_B(\mathbb{R}_+, F)$. ($\psi(x)$ depends continuously on x in L^2_B–norm.) It is, however, unclear, whether it is possible to define $\psi(x)$ for arbitrary $x \in L^2(\mathbb{R}_+, F)$. We can only say that if $x \in E_{[0,s]}$, $y \in E_{[0,t]}$ are such that $\psi(x)$, $\psi(y)$ exist, then $\psi(\mathfrak{s}_t x \oplus y) = \psi(x) \odot \psi(y)$ exists, too.

7.3 Exponential units and their renormalizations

We have seen, for instance, in the proof of Corollary 7.2.3, that the exponential vectors $\xi_t = \psi(\zeta I\!I_{[0,t)})$ play a distinguished role. They fulfill the factorization

$$\xi_s \odot \xi_t \;=\; \xi_{s+t} \tag{7.3.1}$$

and $\xi_0 = \omega$. In accordance with Definition 11.2.1 we call such a family $\xi^\odot = \big(\xi_t\big)_{t \in \mathbb{R}_+}$ a ***unit***. Notice that $T_t(b) = \langle \xi_t, b\xi_t \rangle$ defines a CP-semigroup on $\mathcal{B}$ (see Proposition 11.2.3). Additionally, $\psi(\zeta I\!I_{[0,t)})$ depends continuously on t so that the corresponding semigroup is uniformly continuous (cf. Theorem 11.6.7). We ask, whether there are other continuous units $\xi^\odot$ than these ***exponential units***. The answer is given by the following two theorems from Liebscher and Skeide [LS00b].

7.3.1 Theorem. *Let $\beta \in \mathcal{B}$, $\zeta \in F$, and let $\xi^0 = \big(\xi_t^0\big)_{t \in \mathbb{R}_+}$ with $\xi_t^0 = e^{t\beta}$ be the uniformly continuous semigroup in $\mathcal{B}$ with generator β. Then $\xi^\odot(\beta, \zeta) = \big(\xi_t(\beta, \zeta)\big)_{t \in \mathbb{R}_+}$ with the component ξ_t^n of $\xi_t(\beta, \zeta) \in \mathbb{\Gamma}_t$ in the n–particle sector defined as*

$$\xi_t^n(t_n, \ldots, t_1) = \xi_{t-t_n}^0 \zeta \odot \xi_{t_n - t_{n-1}}^0 \zeta \odot \ldots \odot \xi_{t_2 - t_1}^0 \zeta \xi_{t_1}^0 \tag{7.3.2}$$

(and, of course, ξ_t^0 for $n = 0$), is a unit. Moreover, both functions $t \mapsto \xi_t \in \mathrm{I\Gamma}(F)$ and the CP-semigroup $T^{(\beta,\xi)}$ with $T_t^{(\beta,\xi)} = \langle \xi_t(\beta,\xi), \bullet \xi_t(\beta,\xi)\rangle$ are uniformly continuous and the generator of $T^{(\beta,\xi)}$ is

$$b \longmapsto \langle \zeta, b\zeta\rangle + b\beta + \beta^* b. \tag{7.3.3}$$

PROOF. For simplicity we write $\xi_t(\beta,\xi) = \xi_t$. ξ_t^0 is bounded by $e^{t\|\beta\|}$ so that

$$\|\xi_t^n\| \leq e^{t\|\beta\|}\sqrt{\frac{t^n \|\zeta\|^{2n}}{n!}}. \tag{7.3.4}$$

In other words, the components ξ_t^n are summable to a vector $\xi_t \in \mathrm{I\Gamma}_t$ with norm not bigger than $e^{t\left(\|\beta\|+\frac{\|\zeta\|^2}{2}\right)}$.

Of course, $(\xi_s \odot \xi_t)^k = \sum_{\ell=0}^{k} \xi_s^{k-\ell} \odot \xi_t^\ell$. Evaluating at a concrete tuple $(r_k, \ldots, r_1)$, there remains only one summand, namely, that where $r_\ell < t$ and $r_{\ell+1} \geq t$. (If $r_1 > t$, then there is nothing to show.) By (7.3.2), this remaining summand equals $\xi_{s+t}^k(r_k, \ldots, r_1)$, so that ξ_t fulfills (7.3.1). We conclude further that $\|\xi_{t+\varepsilon} - \xi_t\| = \|(\xi_\varepsilon - \omega) \odot \xi_t\| \leq \|\xi_\varepsilon - \omega\| \, \|\xi_t\|$ so that $t \mapsto \xi_t$ (and, therefore, also $T^{(\beta,\xi)}$) is continuous, because

$$\|\xi_\varepsilon - \omega\| \;\leq\; \left\|\xi_\varepsilon - \xi_\varepsilon^0\right\| + \left\|\xi_\varepsilon^0 - \omega\right\| \;\leq\; e^{\varepsilon\left(\|\beta\|+\frac{\|\zeta\|^2}{2}\right)} - 1 + \left\|e^{\varepsilon\beta} - \mathbf{1}\right\|.$$

For the generator we have to compute $\frac{d}{dt}\big|_{t=0}\langle \xi_t, b\xi_t\rangle$. It is easy to see from (7.3.4) that the components ξ^n for $n \geq 2$ do not contribute. For the component along the vacuum we have $\frac{d}{dt}\big|_{t=0}\langle \xi_t^0, b\xi_t^0\rangle = b\beta + \beta^* b$. For the component in the one-particle sector we find (after a substitution $s \to t - s$)

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0}\langle \xi_t^1, b\xi_t^1\rangle \;&=\; \frac{d}{dt}\Big|_{t=0}\int_0^t \langle e^{(t-s)\beta}\zeta e^{s\beta}, b e^{(t-s)\beta}\zeta e^{s\beta}\rangle \, ds \\ &=\; \frac{d}{dt}\Big|_{t=0}\left[e^{t\beta^*}\Big(\int_0^t \langle e^{s\beta}\zeta e^{-s\beta}, b e^{s\beta}\zeta e^{-s\beta}\rangle \, ds\Big)e^{t\beta}\right] \;=\; \langle \zeta, b\zeta\rangle. \end{aligned}$$

From this the form of the generator follows. ■

7.3.2 Remark. We see that the generator of $T^{(\beta,\zeta)}$ has Christensen-Evans form.

7.3.3 Remark. The exponential unit $\psi(\zeta I\!I_{[0,t)})$ correspond to $\xi_t(0,\zeta)$. We may consider $\xi_t(\beta,\zeta)$ as $\xi(0,\zeta)$ *renormalized* by the semigroup $e^{t\beta}$. This is motivated by the observation that for $\mathcal{B} = \mathbb{C}$ all factors $e^{(t_i - t_{i-1})\beta}$ in (7.3.2) come together and give $e^{t\beta}$. The other way round, in the noncommutative context we have to distribute the normalizing factor $e^{t\beta}$ over the time intervals $[t_{i-1}, t_i)$.

Now we show the converse of Theorem 7.3.1.

7.3.4 Theorem. *Let $\xi^\odot$ be a unit such that $t \mapsto \xi_t \in \mathbb{I}\Gamma(F)$ is a continuous function. Then there exist unique $\beta \in \mathcal{B}$ and $\zeta \in F$ such that $\xi_t = \xi_t(\beta, \zeta)$ as defined by* (7.3.2).

Proof. (i) ξ_t is continuous, hence, so is $\xi_t^0 = \langle \omega, \xi_t \rangle$. Moreover, $\xi_s^0 \xi_t^0 = \xi_s^0 \odot \xi_t^0 = \xi_{s+t}^0$ so that $\xi_t^0 = e^{t\beta}$ is a uniformly continuous semigroup in $\mathcal{B}$ with a unique generator $\beta \in \mathcal{B}$.

(ii) As observed in a special case in [Lie00b], any unit is determined by its components ξ^0 and ξ^1. (This follows from Corollary 7.2.3.) So we are done, if we show that ξ^1 has the desired form.

(iii) For each $f \in L^2(\mathbb{R}_+)$ we define the mapping $f^* \otimes \mathsf{id}$ as in Example 4.3.9. Recall from (4.3.3) that $((\mathfrak{s}_t f)^* \otimes \mathsf{id})\mathfrak{s}_t x = (f^* \otimes \mathsf{id})x$. Therefore, defining the function $\lambda(t) = (I\!I_{[0,t]}^* \otimes \mathsf{id})\xi_t^1$ and taking into account $\xi_{s+t}^1 = \xi_s^0 \odot \xi_t^1 + \xi_s^1 \odot \xi_t^0 = e^{s\beta}\xi_t^1 + \mathfrak{s}_t \xi_s^1 e^{t\beta}$ we find

$$\lambda(s+t) \;=\; e^{s\beta}\lambda(t) + \lambda(s)e^{t\beta}.$$

Observe that $t \mapsto \lambda(t)$ is continuous, because ξ_t^1 is continuous, $I\!I_{[0,t]}^* \otimes \mathsf{id}$ is bounded and $(I\!I_{[0,t]}^* \otimes \mathsf{id})\xi_t^1 = (I\!I_{[0,t+s]}^* \otimes \mathsf{id})\xi_t^1$ for all $s \geq 0$. Differentiating

$$e^{T\beta} \int_T^{T+t} e^{-s\beta}\lambda(s)\,ds \;=\; \int_0^t e^{-s\beta}\lambda(s+T)\,ds \;=\; t\lambda(T) + \int_0^t e^{-s\beta}\lambda(s)e^{T\beta}\,ds$$

with respect to T, we see that λ is (abitrarily) continuously differentiable.

(iv) For $T \geq 0$ we have

$$\sum_{i=1}^{2^N} I\!I_{[T\frac{i-1}{2^N},T\frac{i}{2^N}]} \tfrac{2^N}{T} (I\!I_{[T\frac{i-1}{2^N},T\frac{i}{2^N}]}^* \otimes \mathsf{id})x \;\longrightarrow\; I\!I_{[0,T]}x$$

as $N \to \infty$ for all $x \in E_{\mathbb{R}_+}$. ($\sum\limits_{i=1}^{2^N} I\!I_{[T\frac{i-1}{2^N},T\frac{i}{2^N}]} \frac{2^N}{T} I\!I_{[T\frac{i-1}{2^N},T\frac{i}{2^N}]}^*$ is an increasing net of projections on $L^2(\mathbb{R}_+)$ converging strongly to the projection $I\!I_{[0,T]}$ onto $L^2([0,T])$. Now the statement follows by Corollary 4.3.5.) Applying this to ξ_T^1 and taking into account that $(I\!I_{[s,t]}^* \otimes \mathsf{id})\xi_T^1 = e^{(T-t)\beta}\lambda(t-s)e^{s\beta}$ for $s \leq t \leq T$, we find

$$\begin{aligned}
\xi_T^1 &= \lim_{N\to\infty} \sum_{i=1}^{2^N} I\!I_{[T\frac{i-1}{2^N},T\frac{i}{2^N}]} \tfrac{2^N}{T} (I\!I_{[T\frac{i-1}{2^N},T\frac{i}{2^N}]}^* \otimes \mathsf{id})\xi_T^1 \\
&= \lim_{N\to\infty} \sum_{i=1}^{2^N} I\!I_{[T\frac{i-1}{2^N},T\frac{i}{2^N}]} \tfrac{2^N}{T} e^{T\frac{2^N-i}{2^N}\beta} \lambda\big(\tfrac{T}{2^N}\big) e^{T\frac{i-1}{2^N}\beta} = \lim_{N\to\infty} \sum_{i=1}^{2^N} I\!I_{[T\frac{i-1}{2^N},T\frac{i}{2^N}]} e^{T\frac{2^N-i}{2^N}\beta} \lambda'(0) e^{T\frac{i-1}{2^N}\beta}.
\end{aligned}$$

Putting $\zeta = \lambda'(0)$, by Proposition B.4.2 the sum converges in L^∞–norm, hence, *a fortiori* in L^2–norm of E_T, to the function $t \mapsto e^{(T-t)\beta}\zeta e^{t\beta}$, which is stated for ξ_T^1 in (7.3.2). ∎

7.3.5 Observation. In the case of a von Neumann module F, Theorem 7.3.4 remains true also, if we allow ξ_t to be in the bigger space $\mathbb{I}\Gamma^s_t(F)$. This is so, because the construction of the function λ in step 7.3 of the proof together with all its properties, in particular, the construction of $\zeta = \lambda'(0)$, works as before.

7.3.6 Remark. Fixing a semigroup ξ^0 and an element ζ in F, Equation (7.3.2) gives more general units. For that it is sufficient, if ξ^0 is bounded by Ce^{ct} for suitable constants C, c (so that ξ^n_t are summable). The following example shows that we may not hope to generalize Theorem 7.3.4 to units which are continuous in a weaker topology only.

7.3.7 Example. Let $F = \mathcal{B} = \mathcal{B}\big(\Gamma_-\big)$ where we set $\Gamma_- = \Gamma(L^2(\mathbb{R}_-))$. Then, $\mathbb{I}\Gamma(\mathcal{B}) = \mathcal{B} \otimes \mathbb{I}\Gamma(\mathbb{C}) = \mathcal{B} \otimes \Gamma(L^2(\mathbb{R}_+))$, because $\mathcal{B} \odot \mathcal{B} = \mathcal{B}$ and $\mathbb{I}\Gamma(\mathbb{C}) \cong \Gamma(L^2(\mathbb{R}_+))$. We consider elements $b \otimes f$ in $\mathbb{I}\Gamma(\mathcal{B})$ as operators $g \mapsto bg \otimes f$ in $\mathbb{I}\Gamma^s(\mathcal{B}) := \mathcal{B}(\Gamma_-, \Gamma_- \otimes \mathbb{I}\Gamma(\mathbb{C}))$. Clearly, $\mathbb{I}\Gamma^s(\mathcal{B})$ is the strong closure of $\mathbb{I}\Gamma(\mathcal{B})$. Similarly, let $\mathbb{I}\Gamma^s_t(\mathcal{B})$ denote the strong closures $\mathcal{B}(\Gamma_-, \Gamma_- \otimes \mathbb{I}\Gamma_t(\mathbb{C}))$ of $\mathbb{I}\Gamma_t(\mathcal{B})$. It is noteworthy that the $\mathbb{I}\Gamma^s_t$ form a product system even in the algebraic sense. To see this, observe that by Example 4.2.13 for $x \in \mathbb{I}\Gamma^s_s(\mathcal{B}), y \in \mathbb{I}\Gamma^s_t(\mathcal{B})$ we have $x \odot y = (x \otimes \mathsf{id})y \in \mathbb{I}\Gamma^s_{s+t}$. So let $z \in \mathbb{I}\Gamma^s_{s+t}$, and let u be a unitary in $\mathbb{I}\Gamma^s_s(\mathcal{B})$. Then $(u^* \otimes \mathsf{id})z$ is in $\mathbb{I}\Gamma^s_t(\mathcal{B})$ such that $z = (u \otimes \mathsf{id})(u^* \otimes \mathsf{id})z = u \odot ((u^* \otimes \mathsf{id})z)$.

Replacing everywhere the members $\mathbb{I}\Gamma_t(\mathbb{C})$ of the product system $\mathbb{I}\Gamma^\odot(\mathbb{C})$ by the members Γ_t of the isomorphic product system $\big(\Gamma_t\big)$ (and keeping the notation $\mathbb{I}\Gamma^s_t(\mathcal{B})$), we see that units correspond to cocycles of type (H) as introduced in Liebscher [Lie00b]. An example is the second quantized mirrored shift $\xi_t = \Gamma(\tau_t) \in \mathcal{B}\big(\Gamma(L^2(\mathbb{R}_-), \Gamma(L^2(\mathbb{R}_- \cup [0,t]))\big) = \mathbb{I}\Gamma^s_t(\mathcal{B})$, where

$$\tau_t f(s) = \begin{cases} f(s-t) & s < 0 \\ f(-s) & 0 \le s \le t. \end{cases}$$

(Observe that time reflection on the level of time ordered Fock space is a quite terrible operation which is most incompatible with the projection Δ and, therefore, cannot easily be second quantized. Here we mean exclusively second quantization on the symmteric Fock space. See Chapter 8.) A calculation shows that for an exponential vector $\psi(f)$ to a continuous function $f \in L^2(\mathbb{R}_+)$

$$\zeta\psi(f) \;=\; \lim_{r \searrow 0} \xi^1_r(0)\psi(f) \;=\; \psi(f)f(0)$$

is only a distribution (a nonclosable operator, a boundary condition). Clearly, $\zeta \notin \mathcal{B}$.

7.4 CPD-semigroups and totalizing sets of units

Let

$$\mathbb{I}\Gamma_t^{\mathfrak{U}_c}(F) = \mathsf{span}\Big\{b_n\xi_{t_n}(\beta_n,\zeta_n)\odot\ldots\odot b_1\xi_{t_1}(\beta_1,\zeta_1)b_0\Big| \\ \mathfrak{t}\in\mathbb{J}_t; b_0,\ldots,b_n,\beta_1,\ldots,\beta_n\in\mathcal{B};\zeta_1,\ldots,\zeta_n\in F\Big\}.$$

Then $\mathbb{I}\Gamma_s^{\mathfrak{U}_c}(F)\odot\mathbb{I}\Gamma_t^{\mathfrak{U}_c}(F) = \mathbb{I}\Gamma_{s+t}^{\mathfrak{U}_c}(F)$ by restriction of u_{st}. (Cf. also Proposition 11.2.4.) By $\mathfrak{U}_c(F)$ we denote the set of all contiuous units of $\mathbb{I}\Gamma(F)$. Theorem 7.3.4 tells us that $\mathfrak{U}_c(F) = \mathcal{B}\times F$.

Let $\xi^\odot,\xi'^\odot$ be two units. Obviously, also the mappings $b\mapsto\langle\xi_t, b\xi'_t\rangle$ form a semigroup on $\mathcal{B}$ (of course, in general not CP; cf. again Proposition 11.2.3). If ξ_t,ξ'_t are continuous, then so is the semigroup. Another way to say this is that the kernels

$$\mathfrak{T}_t\colon\mathfrak{U}_c(F)\times\mathfrak{U}_c(F) \longrightarrow \mathfrak{T}_t^{(\beta,\zeta),(\beta,'\zeta')} = \langle\xi_t(\beta,\zeta),\bullet\xi_t(\beta',\zeta')\rangle$$

form a uniformly continuous CPD-semigroup $\mathfrak{T}$ of kernels on $\mathfrak{U}_c(F)$ from $\mathcal{B}$ to $\mathcal{B}$ in the sense of Definition 5.4.4. Similar to the proof of (7.3.3) we show that the generator $\mathfrak{L}$ of $\mathfrak{T}$ is given by

$$\mathfrak{L}^{(\beta,\zeta),(\beta,'\zeta')}(b) = \langle\zeta, b\zeta'\rangle + b\beta' + \beta^*b \tag{7.4.1}$$

i.e. it $\mathfrak{T}$ has a CE-generator. By Theorem 5.4.7 $\mathfrak{L}$ is a conditionally completely positive definite kernel. Of course, it is an easy exercise to check this directly.

Theorem 7.2.2 tells us that the tensor products

$$\xi_{t_n}(0,\zeta_n)\odot\ldots\odot\xi_{t_1}(0,\zeta_1) \tag{7.4.2}$$

$(t_1+\ldots+t_n = t)$ form a total subset of $\mathbb{I}\Gamma_t(F)$. Therefore, the closed linear span of such vectors contains also the units $\xi^\odot(\beta,\zeta)$. But, we can specify the approximation much better

7.4.1 Lemma. *Let $\xi^\odot(\beta,\xi),\xi^\odot(\beta',\xi')$ be two continuous units.*

1. *For all $\varkappa,\varkappa'\in[0,1]$, $\varkappa+\varkappa' = 1$ we have*

$$\lim_{n\to\infty}\Big(\xi_{\frac{\varkappa t}{n}}(\beta,\zeta)\odot\xi_{\frac{\varkappa' t}{n}}(\beta',\zeta')\Big)^{\odot n} = \xi_t(\varkappa\beta+\varkappa'\beta',\varkappa\zeta+\varkappa'\zeta')$$

in the $\mathcal{B}$–weak topology.

2. *For all $b\in\mathcal{B}$ we have*

$$\lim_{n\to\infty}\Big(e^{b\frac{t}{n}}\xi_{\frac{t}{n}}(\beta,\zeta)\Big)^{\odot n} = \lim_{n\to\infty}\Big(\xi_{\frac{t}{n}}(\beta,\zeta)e^{b\frac{t}{n}}\Big)^{\odot n} = \xi_t(\beta+b,\zeta)$$

in the $\mathcal{B}$–weak topology.

3. *For all $\varkappa, \varkappa' \in \mathbb{C}$, $\varkappa + \varkappa' = 1$ we have*

$$\lim_{n\to\infty} \left(\varkappa\xi_{\frac{t}{n}}(\beta,\zeta) + \varkappa'\xi_{\frac{t}{n}}(\beta',\zeta')\right)^{\odot n} = \xi_t(\varkappa\beta + \varkappa'\beta', \varkappa\zeta + \varkappa'\zeta')$$

in norm.

7.4.2 Remark. Part 1 is a generalization from an observation by Arveson [Arv89a]. Part 2 is trivial in the case $\mathcal{B} = \mathbb{C}$. We used it first together with part 1 in Skeide [Ske99c] for $\mathcal{B} = \mathbb{C}^2$; see Section 7.5. Part 3 is the generalization of an observation by Liebscher [Lie00a].

Proof of Lemma 7.4.1. First we observe that the sequences in all three parts are bounded. Therefore, to check $\mathcal{B}$–weak convergence it is sufficient to check it with vectors of the form (7.4.2). Now Parts 1 and 2 are a simple consequence of the Trotter formula (Corollary A.5.6), although it is a little bit tedious to write down all steps explicitly.

For Part 3 we observe that $\|\xi_t(\varkappa\beta + \varkappa'\beta', \varkappa\zeta + \varkappa'\zeta') - \varkappa\xi_t(\beta,\zeta) - \varkappa'\xi_t(\beta',\zeta')\|^2$ is of order t^2. Now the statement follows as in the proof of Proposition A.5.4. ∎

7.4.3 Theorem. *Let S be a total subset of F containing 0. Then the exponential vectors to S–valued step functions are total in $\mathrm{I\Gamma}(F)$.*

Proof. It is sufficient to show the statement for $\mathrm{I\Gamma}_t(F)$ for some fixed t. By Lemma 7.4.1(3) the closure of the span of exponentials to S–valued step functions contains the exponentials to step functions with values in the affine hull of S (i.e. all linear combinations $\sum_i \varkappa_i\zeta_i$ from S with $\sum_i \varkappa_i = 1$). Since $0 \in S$, the affine hull coincides with the span of S which is dense in F. Now the statement follows, because the units depend continuously on their parameters and from totalitiy of (7.4.2). ∎

We find the following result on the exponential vectors of $\Gamma(L^2(\mathbb{R}_+))$ $(= \mathrm{I\Gamma}(\mathbb{C}))$. It was obtained first by Parthasarathy and Sunder [PS98] and later by Bhat [Bha01]. The proof in Skeide [Ske00c] arises by restricting the methods in this section to the bare essentials for the special case $\mathcal{B} = \mathbb{C}$ and fits into half a page.

7.4.4 Corollary. *Exponential vectors to indicator functions of intervals are total in $\mathrm{I\Gamma}(\mathbb{C}) = \Gamma(L^2(\mathbb{R}_+))$.*

Proof. The set $S = \{0, 1\}$ is total in $\mathbb{C}$ and contains 0. ∎

In accordance with Definition 11.2.5 we may say that the set $\xi^{\odot}(0, S)$ of units is totalizing. Recall, however, that totalizing is a weaker property. Lemma 7.4.1(2) asserts, for instance, that what a sinlge unit $\xi^{\odot}(\beta,\zeta)$ generates $\mathcal{B}$–weakly via expressions as in (11.2.3), contains the units $\xi^{\odot}(\beta + b, \zeta)$ for all $b \in \mathcal{B}$, in particular, the unit $\xi^{\odot}(0,\zeta)$.

7.4.5 Corollary. *Let S be a total subset of F containing 0 and for each $\zeta \in S$ choose $\beta_\zeta \in \mathcal{B}$. Then the set $\{\xi^\odot(\beta_\zeta, \zeta) \colon \zeta \in S\}$ is $\mathcal{B}$–weakly totalizing for $\mathrm{I\Gamma}(F)$.*

7.5 Example $\mathcal{B} = \mathbb{C}^2$

In this section we follow Skeide [Ske99c] and study in detail how the unital CP-semigroups on the diagonal subalgebra of M_2 and the associated time ordered Fock modules look like. The diagonal subalgebra is the unique unital 2–dimensional $*$–algebra. We find it convenient to identify it with the vector space $\mathbb{C}^2$ (equipped with componentwise multiplication and conjugation), rather than the diagonal matrices. In addition to the canonical basis $e_1 = \binom{1}{0}$, $e_2 = \binom{0}{1}$ we also use the basis $e_+ = \mathbf{1} = \binom{1}{1}$, $e_- = \binom{1}{-1}$. In the first basis it is easy to say when an element of $\mathbb{C}^2$ is positive (namely, if and only if both coordinates are positive), whereas in the second basis unital mappings have a particularly simple (triangular) matrix representation. We use also the notations from Example 1.6.9.

Let us start with an arbitrary Hilbert $\mathbb{C}^2$–$\mathbb{C}^2$–module F. Choose $\beta \in \mathbb{C}^2$, $\zeta \in F$ and consider the unit $\xi^\odot(\beta, \zeta)$. For short, we write $\xi_t(\beta, \zeta) = \xi_t$. By commutativity the form of the generator of the CP-semigroup $T_t(b) = \langle \xi_t, b\xi_t \rangle$ simplyfies to $\mathcal{L}(b) = \langle \zeta, b\zeta \rangle + b\frac{\beta+\beta^*}{2}$. As usual, the form of the generator is not determined uniquely by T. On the one hand, only the sum $\beta^* + \beta$ contributes so that the imaginary part of β is arbitrary. On the other hand, a positive part in $\beta^* + \beta$ can easily be included into the inner product, by adding a component $\sqrt{\beta}$ in a direct summand $\mathbb{C}^2_+ = \mathbb{C}^2$ (i.e. the simplest Hilbert $\mathbb{C}^2$–$\mathbb{C}^2$–module possible) orthogonal to ζ.

It is possible to give the explicit form of the semigroup $e^{t\mathcal{L}}$, in general, because $\mathcal{L}$ as an operator on $\mathbb{C}^2$ is similar either to a matrix $\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ or to a matrix $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$ whose exponentials can easily be computed. We do not work this out, because we are interested rather in what a single unital unit $\xi^\odot$ with parameters ζ, β generates in $\mathrm{I\Gamma}(F)$ via (11.2.3). Here T is unital and computations are much more handy.

Since we are interested in what is generated by a certain unit, we assume that F is generated by ζ. (If F is not generated by ζ, then the unit can never generate $\mathrm{I\Gamma}(F)$; cf. the proof of Theorem 13.2.7.) By Example 1.6.9, F is a submodule of $\mathbb{C}^2_+ \oplus \mathbb{C}^2_-$. The discussion about ambiguity in the choice of the parameters shows that we may include the component ζ_+ of ζ in $\mathbb{C}^2_+$ into β by adding $\frac{1}{2}\langle \zeta_+, \zeta_+ \rangle$ to β without changing the semigroup T; cf. Corollary 13.2.13. We may, therefore, assume that $F = \mathbb{C}^2_-$. Observe that this choice corresponds to say that the completely positive part $\langle \zeta, \bullet\zeta \rangle$ of $\mathcal{L}$ is the smallest possible.

We use the abbreviation $b_- = \alpha(b)$ where α is the flip automorphism of $\mathbb{C}^2$. We have $\mathbb{C}^2_- \odot \mathbb{C}^2_- = \mathbb{C}^2_+$ where the canonical isomorphism is $b \odot b' \mapsto b_- b'$ and, of course, $\mathbb{C}^2_+ \odot E = E = E \odot \mathbb{C}^2_+$ for all Hilbert $\mathbb{C}^2$–$\mathbb{C}^2$–modules E. Therefore, $\mathrm{I\Gamma}_t(\mathbb{C}^2_-) = \mathbb{C}^2 \otimes \mathrm{I\Gamma}_t(\mathbb{C})$ as Hilbert

$\mathbb{C}^2$–module. However, the left multiplication is that of $\mathbb{C}^2_+$ on $2n$–particle sectors and that of $\mathbb{C}^2_-$ on $2n+1$–particle sectors.

Now it is very easy to write down the units for $\mathrm{I\Gamma}(\mathbb{C}^2_-)$ explicitly.

7.5.1 Theorem. *Let $\xi^\odot$ be the unit for $\mathrm{I\Gamma}(\mathbb{C}^2_-)$ with parameters $\beta, \zeta = \binom{\zeta_1}{\zeta_2}$ and set $|\zeta| = \zeta_1\zeta_2$. Then*

$$\begin{aligned} \xi^{2n}(t_{2n},\dots,t_1) &= |\zeta|^n\, e^{t\beta} e^{(t_{2n}-t_{2n-1}+\ldots+t_2-t_1)(\beta_- - \beta)} \\ \xi^{2n+1}(t_{2n+1},\dots,t_1) &= \zeta\, |\zeta|^n\, e^{t\beta_-} e^{(t_{2n+1}-t_{2n}+\ldots+t_1)(\beta-\beta_-)}. \end{aligned}$$

7.5.2 Remark. The corresponding unit for $\mathrm{I\Gamma}(\mathbb{C}^2_+)$ would be given by $\xi^n(t_n,\dots,t_1) = \zeta^n e^{t\beta}$. In other words, we obtain just the exponential vectors $\psi(\zeta \mathbb{I}_{[0,t)}) = \binom{\psi(\zeta_1 \mathbb{I}_{[0,t)})}{\psi(\zeta_2 \mathbb{I}_{[0,t)})}$ rescaled by $e^{t\beta}$. Moreover, with units in this time ordered Fock module we recover only CP-semigroups of the form $T_t(b) = be^{tc}$ for some self-adjoint element $c \in \mathcal{B}$. In particular, the only unital CP-semigroup among these is the trivial one.

Let us return to $\mathrm{I\Gamma}_t(\mathbb{C}^2_-)$ and see which unital CP-semigroups are generated by which unital unit. Recall that $\xi^\odot$ is unital, if and only if $\beta+\beta^* = -\langle\zeta,\zeta\rangle$ and that the imaginary part of β does not influence the CP-semigroup.

Let T be a unital mapping. In the basis e_+, e_- it has the matrix representation

$$\widehat{T} = \begin{pmatrix} 1 & p \\ 0 & q \end{pmatrix}.$$

From $\binom{z_1}{z_2} = \frac{z_1+z_2}{2}\binom{1}{1} + \frac{z_1-z_2}{2}\binom{1}{-1}$ we find

$$\begin{aligned} T\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \frac{z_1+z_2}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{z_1-z_2}{2}\left[p\begin{pmatrix} 1 \\ 1 \end{pmatrix} + q\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] \\ &= \frac{z_1}{2}\begin{pmatrix} 1+p+q \\ 1+p-q \end{pmatrix} + \frac{z_2}{2}\begin{pmatrix} 1-p-q \\ 1-p+q \end{pmatrix}. \end{aligned}$$

Hence, T is positive (which is the same as completely positive, as $\mathbb{C}^2$ is commutative), if and only if $\binom{p}{q}$ is in the square (including boundaries) in the $\mathbb{R}^2$–plane with corner points $(1,0), (0,1), (-1,0), (0,-1)$.

Now let T_t be a family of mappings on $\mathbb{C}^2$ having matrices $\widehat{T}_t = \begin{pmatrix} 1 & p_t \\ 0 & q_t \end{pmatrix}$ with respect to the basis e_+, e_-. In order that $T = (T_t)$ be a semigroup, p_t and q_t must solve the functional equations $p_t + p_s q_t = p_{s+t}$ and $q_s q_t = q_{s+t}$. Requiring that T_t be continuous implies, as usual, differentialbility of p_t and q_t. Using this, we find $q_t = e^{-ct}$ and $p_t = \alpha(1-e^{-ct})$ with complex constants c and α. In order that T_t be positive we find $c \geq 0$ and $-1 \leq \alpha \leq 1$. These conditions are necessary and sufficient. The corresponding CP-semigroup is

$$T_t \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{z_1 + z_2}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{z_1 - z_2}{2} \left[\alpha(1 - e^{-ct}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-ct} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right].$$

The generator is

$$\mathcal{L} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{z_1 - z_2}{2} c \begin{pmatrix} \alpha - 1 \\ \alpha + 1 \end{pmatrix}. \tag{7.5.1}$$

On the other hand, the generator of the CP-semigroup generated by the unital unit $\xi^\odot$ is

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = b \longmapsto \langle \zeta, b\zeta \rangle - \langle \zeta, \zeta \rangle b = \langle \zeta, \zeta \rangle (b_- - b) = (z_1 - z_2) \begin{pmatrix} -\left|\zeta_1\right|^2 \\ \left|\zeta_2\right|^2 \end{pmatrix}.$$

Equating this to (7.5.1), we find $c = \left|\zeta_1\right|^2 + \left|\zeta_2\right|^2$ and $\alpha = \frac{|\zeta_2|^2 - |\zeta_1|^2}{|\zeta_1|^2 + |\zeta_2|^2}$.

Of course, $c = 0$ (i.e. $\zeta = 0$) yields the trivial CP-semigroup independently of α. Different choices for $c > 0$ correspond to a time scaling. Here we have to distinguish the two essentially different cases where $|\zeta| = 0$ (i.e. $\alpha = \pm 1$) and where $|\zeta| \neq 0$ (i.e. $|\alpha| < 1$).

In the case $|\zeta| = 0$ only the components ξ^0 and ξ^1 of the unit $\xi^\odot$ are different from 0. The case $\alpha = -1$ we analyze in detail in Example 12.3.7 and the case $\alpha = 1$ follows from this, because, in general, a sign change of α just corresponds to flip of the components in $\mathbb{C}^2$. Since $\zeta \odot \zeta = 0$ in this case, we find that the time ordered Fock module over the (one-dimensional) module $\mathbb{C}^2 \zeta \mathbb{C}^2 = \mathbb{C}\zeta$ consists only of its 0– and 1–particle sector and, indeed, is generated by the unit. The CP-semigroup may be considered as the *unitization* of a contractive CP-semigroup on $\mathbb{C}$; cf. also Section 17.2.

Now we come to the case $|\zeta| \neq 0$, whence ζ is invertible. It is our goal to show that also in this case any unit is totalizing for $\mathrm{I\Gamma}(\mathbb{C}^2_-)$ at least in the $\mathbb{C}^2$–weak topology (which coincides with the weak topology, for $\mathbb{C}^2$ is finite dimensional). Define $p_t \colon \mathrm{I\Gamma}_t(\mathbb{C}^2_-) \to \mathrm{I\Gamma}_t(\mathbb{C}^2_-)$ by setting $p_t x = e_1 x e_1 + e_2 x e_2$. Notice that p_t is the projection onto the direct sum over all $2n$–particle sectors.

7.5.3 Lemma. *Let $\xi^\odot$ be an arbitrary unit for $\mathrm{I\Gamma}(\mathbb{C}^2_-)$. Then*

$$\lim_{n \to \infty} \left(p_{\frac{t}{n}} \xi_{\frac{t}{n}} \right)^{\odot n} = \xi^0_t.$$

Proof. By Corollary 7.2.3 we have $\lim_{n \to \infty} \left(p^{01}_{\frac{t}{n}} \right)^{\odot n} = \mathsf{id}_{\mathrm{I\Gamma}^0_t(\mathbb{C}^2_-)}$ in the strong topology. Now the result follows from the fact that $p_t p^{01}_t$ is the projection onto the vaccum. ■

7.5.4 Corollary. *Any unit $\xi^\odot$ for $\mathrm{I\Gamma}(\mathbb{C}^2_-)$ with parameters β, ζ ($|\zeta| \neq 0$) is totalizing on its own.*

PROOF. By Lemma 7.5.3 elements of the form (11.2.3) for that unit generate the vacuum. Now the statement follows as in Corollary 7.4.4. ■

Chapter 8

The symmetric Fock module

The full Fock module can be constructed for an arbitrary (pre-)Hilbert $\mathcal{B}$–$\mathcal{B}$–module E as one-particle sector. For the time ordered Fock module we started with $E = L^2(\mathbb{R}_+, F)$. Here, at least, F can be arbitrary. In this Chapter we want to construct the *symmetric Fock module* in analogy to the symmetric Fock space. To that goal we need a self-inverse *flip isomorphism* $\mathcal{F}$ on $E \odot E$ and in order to define higher permutations on $E^{\odot n}$ this flip should also be $\mathcal{B}$–$\mathcal{B}$–linear. It turns out that these requirments cannot be fulfilled for arbitrary E. Also for $E = L^2(\mathbb{R}_+, F)$ additional requirements for F are necessary.

One way out is to consider centered modules E. This was proposed in Skeide [Ske98a] and applied to the example which we discuss in Appendix D. Goswami, Sinha [GS99] constructed a calculus on a special symmetric Fock module which is contained in our set-up. We discuss this in Section 8.1.

We meat another very special (noncentered) one-particle module, which allows for a flip, in our construction of the *square of white noise* from Accardi and Skeide [AS00a, AS00b]. We describe the construction in the remainder of this chapter starting from Section 8.2.

8.1 The centered symmetric Fock module

In this section we return to a more algebraic level. On the one hand, this is necessary for Section 8.2 and because the most impostant operators are unbounded. On the other hand, we want to point out that the tensor factorization of the symmetric Fock space can also be shown by using only number vectors as simple as by using exponential vectors.

Let E be a centered pre-Hilbert $\mathcal{B}$–module. The basis for the construction of a (unitary) representation the *symmetric group* S_n (i.e. the set of bijections σ on $\{1, \ldots, n\}$) is the flip $\mathcal{F}$ on $E \odot E$ (Proposition 4.2.15) which *flips* $x \odot y$ to $y \odot x$, but only, if $x, y \in C_{\mathcal{B}}(E)$.

8.1.1 Proposition. *Let $\sigma \in S_n$. The mapping*

$$x_n \odot \ldots \odot x_1 \longmapsto x_{\sigma(n)} \odot \ldots \odot x_{\sigma(1)} \tag{8.1.1}$$

$(x_1, \ldots, x_n \in C_{\mathcal{B}}(E))$ extends to a unitary on $E^{\odot n}$ (and further to a unitary on $\overline{E^{\odot n}}$ and $\overline{E^{\odot n}}^s$, respectively). In this way we define a unitary representation of S_n on $E^{\odot n}$.

Proof. Any permutation may be expressed in terms of transpositions of next neighbours. It follows that (8.1.1) may be decomposed into compositions of tensor products of the ($\mathcal{B}$–$\mathcal{B}$–linear) flip with the identities and, therefore, is well-defined and unitary. The representation property follows from standard results about the symmetric group. ■

8.1.2 Corollary. *By setting*

$$p_n \colon x_n \odot \ldots \odot x_1 \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(n)} \odot \ldots \odot x_{\sigma(1)} \tag{8.1.2}$$

$(x_1, \ldots, x_n \in C_{\mathcal{B}}(E))$, we define a projection in $\mathcal{B}^{a,bil}(E^{\odot n})$.

8.1.3 Definition. For a centered pre-Hilbert $\mathcal{B}$–module E the *n–fold symmetric tensor product* is $E^{\odot_s n} = p_n E^{\odot n}$. By Observation 1.6.4 $E^{\odot_s n}$ is a $\mathcal{B}$–$\mathcal{B}$–submodule of $E^{\odot n}$ and, obviously, $E^{\odot_s n}$ is centered. For all $x_1, \ldots, x_n \in E$ we set

$$x_n \odot_s \ldots \odot_s x_1 = p_n(x_n \odot \ldots \odot x_1).$$

We use similar notions for centered Hilbert modules and centered von Neumann modules.

8.1.4 Proposition. *$E^{\odot_s n}$ is generated by symmetric tensors $x \odot \ldots \odot x$ to centered elements $x \in C_{\mathcal{B}}(E)$.*

Proof. This follows from the *multiple polarization formula*

$$\sum_{\sigma \in S_n} x_{\sigma(n)} \odot \ldots \odot x_{\sigma(1)} = \frac{1}{2^n} \sum_{\varepsilon_n, \ldots, \varepsilon_1 = \pm 1} \varepsilon_n \ldots \varepsilon_1 (\varepsilon_n x_n + \ldots + \varepsilon_1 x_1)^{\odot n}. \tag{8.1.3}$$

For convenience we sketch a proof. Expanding the product $(\varepsilon_n x_n + \ldots + \varepsilon_1 x_1)^{\odot n}$ into a sum, only those summands contribute where each $\varepsilon_k x_k$ $(k = 1, \ldots, n)$ appears at least once (otherwise, the sum over the corresponding ε outside the product gives $\pm$ the identical term, hence 0) and, therefore, precisely once. In the remaining terms all ε dissapear, because $\varepsilon^2 = 1$. We obtain precisely the sum over all permutations where, however, each permutation appears 2^n times (one factor 2 for each possibility to choose ε). ■

8.1.5 Definition. Let $p = \bigoplus_{n=0}^{\infty} p_n \in \mathcal{B}^{a,bil}(\underline{\mathcal{F}}(E))$. The *(algebraic) symmetric Fock module over* E is $\underline{\Gamma}(E) = p\underline{\mathcal{F}}(E)$. (As usual, we denote $\Gamma(E) = \overline{\underline{\Gamma}(E)}$ and $\Gamma(E)^s = \overline{\underline{\Gamma}(E)}^s$.)

Let $N \in \mathcal{L}^{a,bil}(\underline{\mathcal{F}}(E))$ be *the* number operator, defined by $N \restriction E^{\odot n} = n\,\mathsf{id}$. Observe that N is self-adjoint and that $pN = Np$. For any function $f \colon \mathbb{N}_0 \to \mathbb{C}$ we define $f(N)$ in the sense of *functional calculus*, i.e. $f(N) \restriction E^{\odot n} = f(n)\,\mathsf{id}$. For $x \in E$ we define the *symmetric creator* as $a^*(x) = \sqrt{N}p\ell^*(x)$ and the *symmetric annihilator* $a(x) = \ell(x)p\sqrt{N}$ as adjoint of $a^*(x)$.

Obviously, $a^*(x)$, $a(x)$ leave invariant $\underline{\Gamma}(E)$. On $\underline{\Gamma}(E)$ we can forget about the projection p in $a(x)$. For $x, y \in C_{\mathcal{B}}(E)$ we find

$$a^*(x)y^{\odot n} \;=\; \frac{1}{\sqrt{n+1}}\sum_{i=0}^{n} y^{\odot i} \odot x \odot y^{\odot(n-i)} \qquad a(x)y^{\odot n} \;=\; \sqrt{n}\langle x, y\rangle y^{\odot(n-1)}. \tag{8.1.4}$$

Further, we have

$$\begin{aligned} a(x)a^*(x')y^{\odot n} \;&=\; \frac{1}{\sqrt{n+1}}a(x)\sum_{i=0}^{n} y^{\odot i} \odot x' \odot y^{\odot(n-i)} \\ &=\; \langle x, x'\rangle y^{\odot n} \;+\; \langle x, y\rangle \sum_{i=0}^{n-1} y^{\odot i} \odot x' \odot y^{\odot(n-1-i)} \end{aligned} \tag{8.1.5}$$

and

$$a^*(x')a(x)y^{\odot n} \;=\; \sqrt{n}a^*(x')\langle x, y\rangle y^{\odot(n-1)} \;=\; \langle x, y\rangle \sum_{i=0}^{n-1} y^{\odot i} \odot x' \odot y^{\odot(n-1-i)}.$$

Taking the difference, the sums over i dissappear. Taking into account that $y^{\odot n}$ is arbitrary, and that by Proposition 8.1.4, $\underline{\Gamma}(E)$ is spanned by vectors of the form $a^*(x)^n\omega = \sqrt{n!}x^{\odot n}$ ($x \in C_{\mathcal{B}}(E)$), we find the *CCR* (canonical commutation relations)

$$[a(x), a^*(x')] \;=\; \langle x, x'\rangle. \tag{8.1.6}$$

The CCR remain valid also, if only one of the elements x, x' is in the center. In this case, we also have $a^*(x)a^*(x') = a^*(x')a^*(x)$. However, nothing like this is true for more general elements in $x, y \in E$.

8.1.6 Theorem. *The mapping*

$$a^*(x)^n\omega \odot a^*(y)^m\omega \;\longmapsto\; a^*(x)^n a^*(y)^m\omega$$

($x \in C_{\mathcal{B}}(E), y \in C_{\mathcal{B}}(F)$) *establishes a two-sided isomorphism* $\underline{\Gamma}(E) \odot \underline{\Gamma}(F) \to \underline{\Gamma}(E \oplus F)$.

Proof. The vectors on both sides are generating. Therefore, it is sufficient to show isometry. We have

$$\big\langle a^*(x)^n a^*(y)^m\omega, a^*(x')^{n'}a^*(y')^{m'}\omega\big\rangle = \big\langle a^*(y)^m\omega, a(x)^n a^*(x')^{n'}a^*(y')^{m'}\omega\big\rangle.$$

Without loss of generality suppose that $n \geq n'$. We have

$$a(x)^n a^*(x')^{n'} = a(x)^{n-n'}a(x)^{n'}a^*(x')^{n'} = a(x)^{n-n'}\sum_{k=0}^{n'} b_k a^*(x')^k a(x)^k$$

where $b_k \in \mathcal{B}$. As $a(x)$ commutes with $a^*(y')$ and $a(x)\omega = 0$, the only non-zero contribution comes from $a(x)^{n-n'}b_0$. On the other hand, also b_0 commutes with $a^*(y')$ so that also $a(x)^{n-n'}$ (commuting with $a^*(y)$) comes to act directly on ω and gives 0, unless $n = n'$. Henceforth, the only non-zero contributions appear for $n = n'$ and $m = m'$. Taking into account that in this case $b_0 = \langle a^*(x)^n\omega, a^*(x')^n\omega\rangle$, we find

$$\begin{aligned}\big\langle a^*(x)^n a^*(y)^m\omega, a^*(x')^n a^*(y')^m\omega\big\rangle &= \big\langle a^*(y)^m\omega, \langle a^*(x)^n\omega, a^*(x')^n\omega\rangle\; a^*(y')^m\omega\big\rangle \\ &= \big\langle a^*(x)^n\omega \odot a^*(y)^m\omega, a^*(x')^n\omega \odot a^*(y')^m\omega\big\rangle. \blacksquare\end{aligned}$$

8.1.7 Remark. In the proof of Theorem 7.2.2 we have seen that the elements $\Delta_n x^{\odot n}$ ($x \in E_{\mathbb{R}_+} = L^2(\mathbb{R}_+, F)$) form a total subset of $\Delta_n \overline{E_{\mathbb{R}_+}^{\odot n}}$. For centered F it is even sufficient to consider only $x \in C_{\mathcal{B}}(E_{\mathbb{R}_+})$. It is clear that

$$\Delta_n x^{\odot n} \;\longmapsto\; \frac{p_n x^{\odot n}}{\sqrt{n!}}$$

establishes a two-sided isomorphism $\mathrm{I\!\Gamma}(F) \to \Gamma(E_{\mathbb{R}_+})$. (Observe that this is not true for $\underline{\mathrm{I\!\Gamma}}(F)$ and $\underline{\Gamma}(E_{\mathbb{R}_+})$.) This allows also to retract the time shifts on $\Gamma(E)$ and $\Gamma(E_{\mathbb{R}_+})$ from the time shift on $\breve{\mathrm{I\!\Gamma}}(F)$ and $\mathrm{I\!\Gamma}(F)$, respectively.

Under the isomorphism the exponential vectors transform into

$$\psi(x) \;=\; \sum_{n=0}^{\infty}\frac{a^*(x)^n\omega}{n!}.$$

This is true for all $x \in E_{\mathbb{R}_+}$ for which $\psi(x)$ exists in $\mathrm{I\!\Gamma}(F)$, not only for centered elements. As there is no problem in defining the exponential vectors in $\Gamma(E_{\mathbb{R}_+})$ for all $x \in E_{\mathbb{R}_+}$, the same is true for $\mathrm{I\!\Gamma}(F)$, at least in the case, when F is centered.

Finally, we mention that for centered modules the symmetric (hence, also the time ordered) Fock module is generated already by its exponential units to central elements, the *central exponential units*; cf. Definition 11.2.1. These particular exponential units generate a CPD-semigroup (in the sense of Proposition 11.2.3) which was used in Accardi and Kozyrev

[AK99] to classify certain cocycles on the symmetric Fock space (with the help of special versions of Lemmata 5.2.7 and 5.3.5). Restrictions of the completely positive mappings on $M_n(\mathcal{B})$ used in [AK99] to completely positive mappings $\mathcal{B} \to M_n(\mathcal{B})$ have been used before by Fagnola and Sinha [FS93] and Lindsay and Wills [LW00].

Now we want to define also *conservation operators*. With the help of Example 4.2.17 we may define the representation λ_n^i $(i = 1, \ldots, n)$ of $\mathcal{B}^a(E)$ acting on the i–th tensor site of $E^{\odot n}$. For $x_j \in C_{\mathcal{B}}(E)$ we obtain

$$\lambda_n^i(T) x_1 \odot \ldots \odot x_n \;=\; x_1 \odot \ldots \odot T x_i \odot \ldots \odot x_n$$

(and Example 4.2.17 shows that the extension is well-defined).

8.1.8 Definition. For $T \in \mathcal{B}^a(E)$ we define the *conservation operator* on $\underline{\Gamma}(E)$ as

$$\lambda(T) \;=\; \bigoplus_{n=0}^{\infty} \sum_{i=1}^{n} \lambda_n^i(T).$$

Formally, we may consider (and, sometimes, will do so) $\lambda(T)$ also as an operator on $\underline{\mathcal{F}}(E)$. However, it is clear that $\lambda(T)$ leaves invariant $\underline{\Gamma}(E)$. We conclude that $p\lambda(T)p = \lambda(T)p$ and (since $\lambda(T)^* = \lambda(T^*)$) also $p\lambda(T)p = p\lambda(T)$.

By computations similar to those leading to (8.1.6), (on $\underline{\Gamma}(E)$) we find the analogues

$$[\lambda(T), a^*(x)] \;=\; a^*(Tx) \qquad [a(x), \lambda(T^*)] \;=\; a(Tx) \qquad [\lambda(T), \lambda(T')] \;=\; \lambda([T, T']) \tag{8.1.7}$$

$(x \in C_{\mathcal{B}}(E); T, T' \in \mathcal{B}^{a,bil}(E) = C_{\mathcal{B}}(\mathcal{B}^a(E)))$ of the relations well-known on the symmetric Fock space.

The definition of the symmetric Fock module and of the creators and annihilators is from [Ske98a]. The definition of the conservation operators (although already mentioned in [Ske98a]) is a generalization from that given in [GS99] for the special symmetric Fock module $\Gamma^s(\mathfrak{H}_{\mathcal{B}})) = \mathcal{B} \bar{\otimes}^s \Gamma(\mathfrak{H})$ for some Hilbert space $\mathfrak{H}$ and a von Neumann algebra $\mathcal{B}$ on a Hilbert space G. In the specialization $\mathcal{B} = \mathcal{B}(G)$, the same discussion as for the full Fock module in Example 6.1.6 shows that the operators are a condensed way to rewrite the sums over components in a coordinate based approach to calculus on $G \bar{\otimes} \Gamma(\mathfrak{H})$ as described, for instance, in Parsarathy's monograph [Par92].

8.2 The square of white noise

Accardi, Lu, and Volovich introduced in [ALV99] the *square of white noise*. In the remaining sections of this chapter we discuss the realization of the square of white noise which we found in Accardi and Skeide [AS00a, AS00b]. In the final Section 8.7 we discuss the systematic construction of representations from Accardi, Franz and Skeide [AFS00] and some results on calculus with respect to square of white noise from Accardi, Hida and Kuo [AHK00]. We start with a short introduction.

Following [ALV99], in this section we understand by a *white noise* operator-valued distributions a_t^* and a_t (indexed by $t \in \mathbb{R}$) which fulfill the CCR $[a_t, a_s^*] = \delta(t-s)$. It is possible to give precise definitions of how to deal with such objects, but here they serve only as motivation. The connection with the CCR on the symmetric Fock space $\underline{\Gamma}(L^2)$ is made by the observation that, doing some formal computations, the integrals $a_f^* = \int f(t) a_t^* \, dt$ ($f \in L^2$) fulfill (8.1.6).

Therefore, formally the square of white noise should be the operator-valued distributions $B_t^* = a_t^{*2}$ and $B_t = a_t^2$ fulfilling commutation relations which follow from the CCR. Introducing also the distribution $N_t = a_t^* a_t$, we find the result

$$[B_t, B_s^*] \;=\; 2\delta^2(t-s) + 4\delta(t-s)N_t \qquad\qquad [N_t, B_s^*] \;=\; 2\delta(t-s)B_t^*$$

(and all other commutators, not following by adjoint, being 0) from [ALV99]. Unfortunately, the objects B_t^*, B_s are too singular as is manifest in the fact that their formal commutator has a factor $\delta^2(t-s)$, which *a priori* does not make sense. To overcome this trouble it was proposed in [ALV99] to consider a *renormalization* of the singular object δ^2 in which δ^2 is replaced by $2c\delta$ where $c > 0$. This choice is motivated by a *regularization procedure* where δ is approximated by functions δ_ε such that $\delta_\varepsilon^2 \to 2c\delta$ in a suitable sense (where c might be even complex).

After this renormalization, again *smearing out* the densities by setting $B_f^* = \int f(t) B_t^* \, dt$ and $N_a^* = \int a(t) N_t^* \, dt$, and computing the formal commutators, we find the following relations.

$$[B_f, B_g^*] = 2c \operatorname{\mathsf{Tr}}(\overline{f}g) + 4N_{\overline{f}g} \qquad f, g \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \tag{8.2.1a}$$

$$[N_a, B_f^*] = 2B_{af}^* \qquad a \in L^\infty(\mathbb{R}), f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \tag{8.2.1b}$$

and $[B_f^*, B_g^*] = [N_a, N_{a'}] = 0$ where we set $\operatorname{\mathsf{Tr}} f = \int f(t) \, dt$. Our goal is to find a representation of the $*$–algebra generated by these relations.

In [ALV99] a representation was constructed with the help of a Kolmogorov decomposition for a certain positive definite kernel. It was not so difficult to define the correct kernel, but as usual, it was difficult to show that it is, indeed, positive definite. In [Sni00] Sniady found an explicit form of the kernel from which positivity is evident. Here we proceed in a different way, motivated by the following observations. Relation (8.2.1a), looks like the usual CCR, except that the *inner product* on the right-hand side takes values in the algebra generated by the conservation operators N_a. It is, therefore, natural to try a realization on a Hilbert module over the algebra of conservation operators. Additionally, on a Hilbert module we have a chance to realize also Relation (8.2.1b), by putting explicitly a suitable left multiplication by conservation operators. Once again, this reconfirms how important it is to have the freedom to choose a left multiplication different from obvious ones.

Since the conservation operators are unbounded, we cannot use the theory of Hilbert modules over C^*–algebras, but, we need the theory of pre-Hilbert modules over more general $*$–algebras as described in [AS98]. We postpone the discussion of these to Appendix C. In Section 8.3 we show that the generators N_a of the algebra of conservation operators are free. This allows to define representations just by fixing the values of the representation on N_a, and is essential for the definition of the left multiplication. It allows to identify the algebra generated by N_a not just as an abstract algebra generated by relations, but, concretely as an algebra of conservation operators.

The main part is Section 8.4. Here we construct a two-sided pre-Hilbert module E, and we show that it is possible to construct a symmetric Fock module $\Gamma(E)$ over E. We see that the natural creation operators $a^*(f)$ on this Fock module and the natural left multiplication by N_a fulfill Relation (8.2.1a) up to an additive term and Relation (8.2.1b). By a tensor product construction we obtain a pre-Hilbert space where Relations (8.2.1a,8.2.1b) are realized. This representation coincides with the one constructed in [ALV99].

In Section 8.5 we show that our representation space is isomorphic to the usual symmetric Fock space over $L^2(\mathbb{R}, \ell^2)$. In Section 8.6 we show that our representation may be considered as an extension of Boukas' representation of Feinsilver's *finite difference algebra* [Fei87] on the *finite difference Fock space.* In Section 8.7 we discuss results from Accardi, Franz and Skeide [AFS00]. We point out that the calculus based on the square of white noise generalizes Boukas' calculus in [Bou91b]. In [PS91] Parthasarathy and Sinha realized the finite difference algebra by operators on a symmetric Fock space. They do, however, not consider the question, whether this representation is equivalent to Boukas' representation. We point out that this is not the case.

8.3 The algebra of conservation operators of $L^\infty(\mathbb{R})$

8.3.1 Definition. Let us denote the vacuum of $\underline{\Gamma}(L^2(\mathbb{R}))$ by Ω. For $a \in L^\infty(\mathbb{R})$ we define the conservation operator $N_a = \lambda(a) \in \mathcal{L}^a\big(\underline{\Gamma}(L^2(\mathbb{R}))\big)$. By

$$\mathcal{N} = \mathsf{alg}\{N_a \ (a \in L^\infty(\mathbb{R}))\}$$

we denote the unital algebra generated by all N_a.

Cearly, $N_a^* = N_{a^*}$ and $[N_a, N_{a'}] = N_{[a,a']} = 0$, so that $\mathcal{N}$ is a commutative $*$–algebra.

Let I_1, I_2 be two disjoint measurable subsets of $\mathbb{R}$. By Theorem 8.1.6 we have

$$\underline{\Gamma}(L^2(S_1)) \otimes \underline{\Gamma}(L^2(S_2)) \;=\; \underline{\Gamma}(L^2(S_1 \cup S_2)).$$

In this identification we have $N_{\mathbb{I}_{S_1}} = N_{\mathbb{I}_{S_1}} \otimes \mathsf{id}$ and $N_{\mathbb{I}_{S_2}} = \mathsf{id} \otimes N_{\mathbb{I}_{S_2}}$. Similar statements are true for factorization into more than two disjoint subsets.

Since $N_a\Omega = 0$ for any $a \in L^\infty(\mathbb{R})$, the vacuum state $\varphi_\Omega(\bullet) = \langle \Omega, \bullet\Omega \rangle$ is a character for $\mathcal{N}$. Its kernel consists of the span of all monomials with at least one factor N_a and its GNS-pre-Hilbert space is just $\mathbb{C}\Omega$.

8.3.2 Definition. See Appendix C. As positivity defining subset of $\mathcal{N}$ we choose

$$S = \bigl\{ N_{\mathbb{I}_{I_1}} \ldots N_{\mathbb{I}_{I_n}} \colon I_i \text{ bounded intervals in } \mathbb{R} \ (n \in \mathbb{N}_0; i = 1, \ldots, n) \bigr\}.$$

Obviously, $\mathcal{N}$ is a P^*–algebra.

8.3.3 Proposition. *The defining representation* id *of* $\mathcal{N}$ *on* $\underline{\Gamma}(L^2(\mathbb{R}))$ *is an* S*–representation.*

Proof. By Remark C.1.4 it is sufficient to show that $N_{\mathbb{I}_I}$ is of the form $\sum\limits_i b_i^* b_i$ where b_i are taken (for all I) from a commutative subalgebra of $\mathcal{L}^a\bigl(\underline{\mathcal{F}}(L^2(\mathbb{R}))\bigr)$. But this follows from

$$N_{\mathbb{I}_I} = \sum_{1 \le i \le n < \infty} \lambda_i^n(\mathbb{I}_I) = \sum_{1 \le i \le n < \infty} \lambda_i^n(\mathbb{I}_I)^* \lambda_i^n(\mathbb{I}_I). \blacksquare$$

8.3.4 Corollary. *By Observation C.1.6* φ_Ω *is* S*–positive and its GNS-representation is an* S*–representation.*

In the following section we intend to define a representation of $\mathcal{N}$ by assigning to each N_a an operator and extension as algebra homomorphism. The goal of the remainder of the present section is to show that this is possible, at least, if we restrict to the subalgebra $\mathfrak{S}(\mathbb{R})$ of step functions, which is dense in $L^\infty(\mathbb{R})$ in a suitable weak topology.

Clearly, $\mathsf{alg}\{N\}$ (where $N = N_{\mathbf{1}}$ is the number operator) is isomorphic to the algebra of polynomials in one self-adjoint indeterminate. Moreover, for each measurable non-null-set $S \subset \mathbb{R}$ the algebra $\mathsf{alg}\{N_{\mathbb{I}_S}\}$ is isomorphic to $\mathsf{alg}\{N\}$. Therefore, for any self-adjoint element a in a $*$–algebra $\mathcal{A}$ the mapping $N \mapsto a$ extends to a homomorphism $\mathsf{alg}\{N\} \to \mathcal{A}$.

Let $\mathsf{t} = (t_0, \ldots, t_m)$ be a tuple with $t_0 < \ldots < t_m$. Then by the factorization $\underline{\Gamma}(L^2(t_0, t_m)) = \underline{\Gamma}(L^2(t_0, t_1)) \otimes \ldots \otimes \underline{\Gamma}(L^2(t_{m-1}, t_m))$ we find

$$\mathcal{N}_{\mathsf{t}} := \mathsf{alg}\{N_{\mathbb{I}_{[t_{k-1},t_k]}} \ (k = 1, \ldots, m)\} = \mathsf{alg}\{N_{\mathbb{I}_{[t_0,t_1]}}\} \otimes \ldots \otimes \mathsf{alg}\{N_{\mathbb{I}_{[t_{m-1},t_m]}}\}.$$

Therefore, any involutive mapping

$$T_{\mathfrak{t}}\colon\ \mathfrak{S}_{\mathfrak{t}}(\mathbb{R}) := \mathsf{span}\{I\!I_{[t_{k-1},t_k]}\ (k=1,\ldots,m)\} \longrightarrow \mathcal{A}$$

with commutative range defines a unique homomorphism $\rho_{\mathfrak{t}}\colon \mathcal{N}_{\mathfrak{t}} \to \mathcal{A}$ fulfilling $\rho_{\mathfrak{t}}(N_a) = T_{\mathfrak{t}}(a)$.

Now we are ready to prove the universal property of the algebra $\mathcal{N}_{\mathfrak{S}} := \bigcup\limits_{\mathfrak{t}} \mathcal{N}_{\mathfrak{t}}$ which shows that $\mathcal{N}_{\mathfrak{S}}$ is nothing but the symmetric tensor algebra over the involutive vector space $\mathfrak{S}(\mathbb{R})$.

8.3.5 Theorem. *Let $T\colon \mathfrak{S}(\mathbb{R}) \to \mathcal{A}$ be an involutive mapping with commutative range. Then there exists a unique homomorphism $\rho\colon \mathcal{N}_{\mathfrak{S}} \to \mathcal{A}$ fulfilling $\rho(N_a) = T(a)$.*

PROOF. It suffices to remark that $\mathcal{N}_{\mathfrak{S}}$ is the inductive limit of $\mathcal{N}_{\mathfrak{t}}$ over the set of all tuples $\mathfrak{t}$ directed increasingly by "inclusion" of tuples. Denoting by $\beta_{\mathfrak{t}\mathfrak{s}}$ the canonical embedding $\mathcal{N}_{\mathfrak{s}} \to \mathcal{N}_{\mathfrak{t}}$ ($\mathfrak{s} \le \mathfrak{t}$) we easily check that $\rho_{\mathfrak{t}} \circ \beta_{\mathfrak{t}\mathfrak{s}} = \rho_{\mathfrak{s}}$. In other words, the family $\rho_{\mathfrak{t}}$ extends as a unique homomorphism ρ to all of $\mathcal{N}_{\mathfrak{S}}$. ■

8.4 Realization of square of white noise

The idea to realize Relations (8.2.1a) and (8.2.1b) on a symmetric Fock module is to take the right-hand side of (8.2.1a) as the definition of an $\mathcal{N}$–valued inner product on a module E generated by the elements $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and then to define a left multiplication by elements of $\mathcal{N}$ such that the generating elements f fulfill (8.2.1b). In Corollary 8.4.2 we explain that such a proceeding would have immediate success, if we tried to realize Sniady's *free square of white noise* [Sni00] (where the CCR-part (8.2.1a) of the relations is replaced by the generalized Cuntz relation) on a full Fock module. However, in our context the direct attempt with an inner product determined by (8.2.1a) fails. Therefore, we start with the linear ansatz in (8.4.1) and adjust the constants later suitably.

In view of Theorem 8.3.5, for the time being, we restrict to elements in $\mathcal{N}_{\mathfrak{S}}$. By (8.2.1a) this makes it necessary also to restrict to elements $f \in \mathfrak{S}(\mathbb{R})$.

On $\mathfrak{S}(\mathbb{R}) \otimes \mathcal{N}_{\mathfrak{S}}$ with its natural right $\mathcal{N}_{\mathfrak{S}}$–module structure we define for arbitrary positive constants β and γ a sesquilinear mapping $\langle\bullet,\bullet\rangle$, by setting

$$\langle f \otimes \mathbf{1}, g \otimes \mathbf{1}\rangle = M_{\overline{f}g} \quad \text{where} \quad M_a = \beta\, \mathsf{Tr}\, a + \gamma N_a \tag{8.4.1}$$

and by right linear and left anti-linear extension.

We define a left action of M_a, by setting

$$M_a(f \otimes \mathbf{1}) = f \otimes M_a + \alpha a f \otimes \mathbf{1}$$

and right linear extension to all elements of $\mathfrak{S}(\mathbb{R})\otimes\mathcal{N}_{\mathfrak{S}}$. Here α is an arbitrary real constant. Observe that the scalar term in M_a does not change this commutation relation. Therefore, N_a fulfills the same commutation relations with α replaced by $\frac{\alpha}{\gamma}$. In view of (8.2.1b) this fraction should be equal to 2.

By definition, mutliplication by M_a from the left is a right linear mapping on $\mathfrak{S}(\mathbb{R})\otimes\mathcal{N}_{\mathfrak{S}}$. One easily checks that $M_{a^*} = M_a^*$ is an adjoint with respect to the sesquilinear mapping (8.4.1) and the actions of $M_aM_{a'}$ and $M_{a'}M_a$ coincide. By Theorem 8.3.5 this left action extends to a left action of all elements of $\mathcal{N}_{\mathfrak{S}}$.

8.4.1 Proposition. (8.4.1) *is a semi-inner product so that* $\mathfrak{S}(\mathbb{R})\otimes\mathcal{N}_{\mathfrak{S}}$ *is a semi-Hilbert* $\mathcal{N}_{\mathfrak{S}}$*–module.*

Proof. We have to check only the positivity condition, because the remaining properties are obvious. By Observation C.2.3 it is sufficient to check positivity for elements of the form $I\!I_{I_i}\otimes\mathbf{1}$, because these elements generate $\mathfrak{S}(\mathbb{R})\otimes\mathcal{N}_{\mathfrak{S}}$ as a right module. Additionally, we may assume that $I_i\cap I_j=\emptyset$ for $i\neq j$. Then $\langle I\!I_{I_i}\otimes\mathbf{1}, b(I\!I_{I_j}\otimes\mathbf{1})\rangle = 0$ for $i\neq j$, whatever $b\in\mathcal{N}_{\mathfrak{S}}$ might be. Now let b be in S. We may assume (possibly after modifying the I_i suitably) that b has the form $\prod\limits_i N_{I\!I_{I_i}}^{n_i}$ where $n_i\in\mathbb{N}_0$. Observe that $N_{I\!I_{I_i}}(I\!I_{I_j}\otimes\mathbf{1}) = (I\!I_{I_j}\otimes\mathbf{1})N_{I\!I_{I_i}}$ for $i\neq j$, and that (proof by induction)

$$N_{I\!I_I}^n(I\!I_I\otimes\mathbf{1}) \;=\; \sum_{k=0}^n \binom{n}{k}\left(\tfrac{\alpha}{\gamma}\right)^{(n-k)}(I\!I_I\otimes N_{I\!I_I}^k).$$

It follows that

$$\langle I\!I_{I_i}\otimes\mathbf{1}, b(I\!I_{I_j}\otimes\mathbf{1})\rangle \;=\; \delta_{ij}M_{I\!I_{I_i}}\sum_{k=0}^{n_i}\binom{n_i}{k}\left(\tfrac{\alpha}{\gamma}\right)^{(n_i-k)}N_{I\!I_{I_i}}^k\prod_{\ell\neq i}N_{I\!I_{I_\ell}}^{n_\ell}.$$

Let us define $b_k = \langle I\!I_{I_k}\otimes\mathbf{1}, b(I\!I_{I_k}\otimes\mathbf{1})\rangle$ and $b_{ki}=\delta_{ki}\mathbf{1}$. Then $\langle I\!I_{I_i}\otimes\mathbf{1}, b(I\!I_{I_j}\otimes\mathbf{1})\rangle = \sum\limits_k b_{ki}^*b_kb_{kj}$ (and, of course, $b_k\in P(S)$). ■

We may divide out the length-zero elements so that

$$E := \mathfrak{S}(\mathbb{R})\otimes\mathcal{N}_{\mathfrak{S}}/\mathcal{N}_{\mathfrak{S}(\mathbb{R})\otimes\mathcal{N}_{\mathfrak{S}}}$$

is a two-sided pre-Hilbert $\mathcal{N}_{\mathfrak{S}}$–module. We are now in a position to use all results from Appendix C and, in particular, to define the full Fock module $\underline{\mathcal{F}}(E)$ in the sense of Appendix C.3. We use the notation $f\otimes b+\mathcal{N}_{\mathfrak{S}(\mathbb{R})\otimes\mathcal{N}_{\mathfrak{S}}} = fb$. Clearly, we have

$$M_af = fM_a + \alpha af. \tag{8.4.2}$$

8.4.2 Corollary. *The creators $\ell^*(f)$ fulfill the relations $\ell(f)\ell^*(g) = M_{\overline{f}g}$ and $[M_a, \ell^*(f)] = \alpha\ell^*(af)$. In other words, we realized (modulo appropriate choice of the constants α, β, γ) the free square of white noise in the sense of Sniady [Sni00].*

On the generating subset $f_n \odot \ldots \odot f_1$ of $E^{\odot n}$ we find by repeated application of (8.4.2)

$$M_a f_n \odot \ldots \odot f_1 \;=\; f_n \odot \ldots \odot f_1 M_a \;+\; \alpha\lambda(a) f_n \odot \ldots \odot f_1.$$

Therefore, on the full Fock module $\underline{\mathcal{F}}(E)$ we have the relation

$$M_a = M_a^r + \alpha\lambda(a) \tag{8.4.3}$$

where M_a^r denotes multiplication by M_a from the right in the sense of Example 1.6.8.

Now we try to define the symmetric Fock module over E analogously to Definition 8.1.5. The basis for the symmetrization is the bilinear unitary flip. As we already remarked, we may not hope to define a flip on $E \odot E$ by just sending $x \odot y$ to $y \odot x$ for all $x, y \in E$. We may, however, hope to succeed, if we, as for centered modules, define such a flip only on such x, y which come from a suitable generating subset of E.

8.4.3 Proposition. *The mapping*

$$\mathcal{F}\colon f \odot g \longmapsto g \odot f \qquad \textit{for all } f, g \in \mathfrak{S}(\mathbb{R}) \subset E$$

extends to a unique two-sided isomorphism $E \odot E \to E \odot E$.

Proof. We find

$$\begin{aligned}\langle f \odot g, f' \odot g'\rangle = \langle g, \langle f, f'\rangle g'\rangle = \langle g, M_{\overline{f}f'} g'\rangle = \langle g, g' M_{\overline{f}f'} + \alpha\overline{f}f'g'\rangle \\ = M_{\overline{f}f'} M_{\overline{g}g'} + \alpha M_{\overline{g}\overline{f}f'g'} = M_{\overline{g}g'} M_{\overline{f}f'} + \alpha M_{\overline{f}\overline{g}g'f'} = \langle g \odot f, g' \odot f'\rangle.\end{aligned}$$

The elements $f \odot g$ form a (right) generating subset of $E \odot E$. Therefore, $\mathcal{F}$ extends as a well-defined isometric (i.e. inner product preserving) mapping to $E \odot E$. Clearly, this extension is surjective so that $\mathcal{F}$ is unitary.

It remains to show that $\mathcal{F}$ is bilinear. Again it is sufficient to show this on a generating subset and, of course, to show it only for the generators M_a of $\mathcal{N}_{\mathfrak{S}}$. We find

$$\begin{aligned}\mathcal{F}(M_a f \odot g) = \mathcal{F}(f \odot g M_a + \alpha(af \odot g + f \odot ag)) \\ = g \odot f M_a + \alpha(g \odot af + ag \odot f) = M_a g \odot f = M_a \mathcal{F}(f \odot g).\end{aligned} \blacksquare$$

Now we are in a position to define the symmetric Fock module $\underline{\Gamma}(E)$ precisely as in Definition 8.1.5. We have $PM_a = M_aP$, i.e. P and $P\lambda(a) = \lambda(a)P = P\lambda(a)P$. Consequently, (8.4.3) remains true also on our symmetric Fock module. Again, we do not distinguish carefully between $\lambda(a)$ and its restriction to $\underline{\Gamma}(E)$ and denote the number operator by $N = \lambda(\mathbf{1})$. As $\lambda(a)$ is bilinear, so is N and, of course, $NP = PN$. Also here we find by (8.1.3) that the symmetric tensors form a generating subset.

For $x \in E$ we define the creation operator on $\underline{\Gamma}(E)$ as $a^*(x) = \sqrt{N} P \ell^*(x)$. Clearly, $x \mapsto a^*(x)$ is a bilinear mapping, because $x \mapsto \ell^*(x)$ is. We find the commutation relation

$$M_a a^*(f) = a^*(M_a f) = a^*(f M_a + \alpha a f) = a^*(f) M_a + \alpha a^*(af).$$

As before, $a^*(x)$ has an adjoint, namely, $a(x) = \ell(x) P \sqrt{N}$.

Now we restrict our attention to creators $a^*(f)$ and annihilators $a(f)$ to elements f in $\mathfrak{S}(\mathbb{R}) \subset E$. Their actions on symmetric tensors $g^{\odot n}$ $(g \in \mathfrak{S}(\mathbb{R}))$ are the same as in (8.1.4) where now $\langle f, g \rangle = M_{\overline{f}g}$. Again, $a^*(f)a^*(g) = a^*(g)a^*(f)$, but nothing like this is true for $a^*(x)$ and $a^*(y)$ for more general elements in $x, y \in E$.

For the CCR we have to compute $a(f)a^*(f')$ and $a^*(f')a(f)$. For the first product we find the same expression as in (8.1.5), but for the second an additional commutation must be done and we have to exploit our special structure. We find

$$\begin{aligned} a^*(f')a(f)g^{\odot n} =& \sqrt{n} a^*(f') M_{\overline{f}g} g^{\odot(n-1)} = \sqrt{n}\big(M_{\overline{f}g} a^*(f') - \alpha a^*(\overline{f}gf')\big) g^{\odot(n-1)} \\ =& M_{\overline{f}g} \sum_{i=0}^{n-1} g^{\odot i} \odot f' \odot g^{\odot(n-1-i)} \;-\; \alpha\lambda(\overline{f}f')g^{\odot n}. \end{aligned}$$

Taking the difference, the sums over i dissappear. Taking into account (8.4.3), we find

$$[a(f), a^*(f')] \;=\; M_{\overline{f}f'} + \alpha\lambda(\overline{f}f') \;=\; 2M_{\overline{f}f'} - M^r_{\overline{f}f'} \;=\; \beta \operatorname{\mathsf{Tr}}(\overline{f}f') + 2\gamma N_{\overline{f}f'} - \gamma N^r_{\overline{f}f'}.$$

In other words, putting $\beta = 2c$ and $\gamma = 2$ we have realized (8.2.1a) by operators $a^*(f)$, however, only modulo some right multiplication by certain elements of $\mathcal{N}_{\mathfrak{S}}$. (Notice that this is independent of the choice of α. Putting $\alpha = 4$ we realize also (8.2.1b).)

So we have to do two things. Firstly, we must get rid of contributions of N^r_a in the above relation. Secondly, in order to compare with the construction in [ALV99] we must interpret our construction in terms of pre-Hilbert spaces. Both goals can be achieved at once by the following construction. We consider the tensor product $H = \underline{\Gamma}(E) \odot \mathbb{C}\Omega$ of $\underline{\Gamma}(E)$ with the pre-Hilbert $\mathcal{N}_{\mathfrak{S}}$–$\mathbb{C}$–module $\mathbb{C}\Omega$ which is the pre-Hilbert space carrying the GNS-representation of the vacuum state φ_Ω on $\mathcal{N}_{\mathfrak{S}}$. This tensor product is possible by Proposition 8.3.3 and its Corollary. Thus, H is a pre-Hilbert space and carries a representation of $\mathcal{L}^a(\underline{\Gamma}(E))$. In this representation all operators N^r_a are represented by 0. Indeed, by Example 1.6.8 N^r_a commutes with everything, so that we put it on the right, and

$$N^r_a g^{\odot n} \odot \Omega = g^{\odot n} \odot N^r_a \Omega = 0.$$

By B^*_f we denote the image of $a^*(f)$ in $\mathcal{L}^a(H)$. The image of N_a coincides with the image of $4\lambda(a)$. We denote both by the same symbol N_a. By $\Phi = \omega \odot \Omega$ we denote the vacuum in H.

8.4.4 Theorem. *The operators $B^*_f, N_a \in \mathcal{L}^a(H)$ $(f, a \in \mathfrak{S}(\mathbb{R}))$ fulfill Relations* (8.2.1a), (8.2.1b), *and $[B^*_f, B^*_g] = [N_a, N_{a'}] = 0$. Moreover, the vectors $B^{*n}_f\Phi$ $(f \in \mathfrak{S}(\mathbb{R}), n \in \mathbb{N}_0)$ span H.*

8.4.5 Remark. As the representation is determined uniquely by existence of the cyclic vacuum Φ, it follows that H is precisely the pre-Hilbert space as constructed in [ALV99]. However, in [ALV99] the inner product on the total set of vectors was defined *a priori* and it was quite tedious to show that it is positive. Here positivity and also well-definedness of the representation are automatic.

8.4.6 Remark. Putting $H_n = \mathsf{span}\{B^{*n}_f\Phi \ (f \in \mathfrak{S}(\mathbb{R}))\}$, we see that $H = \bigoplus_{n=0}^{\infty} H_n$ is an *interacting Fock space* with creation operators B^*_f as introduced in [ALV97] in the notations of [AS98]. We discuss these in Chapter 9.

8.4.7 Theorem. *The realization of Relations* (8.2.1a) *and* (8.2.1b) *extends to elements $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $a \in L^\infty(\mathbb{R})$ as a representation by operators on $\bigoplus_{n=0}^{\infty} \overline{H_n}$.*

Proof. We extend the definition of the operators B^*_f and N_a formally to $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $a \in L^\infty(\mathbb{R})$, considering them as operators on vectors of the form $B^{*n}_f\Phi$ $(f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), n \in \mathbb{N}_0)$. The inner product of such vectors we define by continuous extension of the inner product of those vectors where $f \in \mathfrak{S}$ in the σ–weak topology of $L^\infty(\mathbb{R})$ (which, clearly, is possible and unique). Positivity of this inner product follows by approximation with inner products, and well-definedness of our operators follows, because all operators have formal adjoints. ■

8.5 H as symmetric Fock space

Let $I \subset \mathbb{R}$ be a finite union of intervals. Denote by H_I the subspace of H spanned by vectors of the form $B^{*n}_f\Phi$ $(f \in \mathfrak{S}(I), n \in \mathbb{N}_0)$. In particular, for $0 \le t \le \infty$ set $H_t := H_{[0,t)}$. (This means that $H_0 = H_\emptyset = \mathbb{C}\Phi$.) Notice that H_I does not depend on whether the intervals in I are open, half-open, or closed.

Denote by $I + t$ the time shifted set I. Denote by f_t the time shifted function $\mathfrak{s}_t f$. Obviously, by sending $B^{*n}_f\Phi$ to $B^{*}_{f_t}{}^{n}\Phi$ we define an isomorphism $H_I \to H_{I+t}$.

Observe that by Relation (8.2.1a) the operators B_f and B^*_g to functions $f \in \mathfrak{S}(I)$ and $g \in \mathfrak{S}(\mathbb{R}\backslash I)$ commute. Define $\mathcal{N}_I := \mathsf{alg}\{N_a \ (a \in \mathfrak{S}(I))\}$. Then by Relation (8.2.1b) also the elements of $\mathcal{N}_I$ commute with all B_g to functions $g \in \mathfrak{S}(\mathbb{R}\backslash I)$.

8.5.1 Theorem. *Let $I, J \subset \mathbb{R}$ be finite unions of intervals such that $I \cap J$ is a null-set. Then*

$$U_{IJ}\colon B^{*n}_f B^{*m}_g \Phi \longmapsto B^{*n}_f \Phi \ \otimes \ B^{*m}_g \Phi \qquad \textit{for} \qquad f \in H_I, g \in H_J$$

extends as an isomorphism $H_{I \cup J} \to H_I \otimes H_J$. Of course, the composition of these isomorphisms is associative in the sense that $(U_{IJ} \otimes \mathsf{id}) \circ U_{(I\cup J)K} = (\mathsf{id} \otimes U_{JK}) \circ U_{I(J\cup K)}$.

Proof. Precisely, as in Theorem 8.1.6. The only tricky point is to observe that also here b_0 commutes with $a^*(g')$, because $I \cap J$ is a null-set. ■

8.5.2 Corollary. *We have $H_s \otimes H_t \cong H_{[0,s)+t} \otimes H_t \cong H_{s+t}$. Also here the isomorphisms $U_{st}\colon H_s \otimes H_t \to H_{s+t}$ compose associatively.*

Now we turn to the completions $\overline{H_t}$ and show that they are symmetric Fock spaces $\Gamma(L^2([0,t),\ell^2))$. We do this by finding the analogue of exponential units for $\overline{H_t}$ and show that their inner products are those of a exponential units for $\Gamma(L^2([0,t),\ell^2))$ to a totalizing subset of ℓ^2.

Good canditates for exponential units are

$$\psi_\rho(t) = \sum_{n=0}^{\infty} \frac{{B^*_{\rho 1\!\!1_t}}^n \Phi}{n!} = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} {B^*_{1\!\!1_t}}^n \Phi$$

where $\rho \in \mathbb{C}$. Whenever $\psi_{\rho_0}(t)$ exists, then it is an analytic vector-valued function of ρ with $|\rho| < |\rho_0|$. It is not difficult to check that whenever $\psi_\rho(s)$ and $\psi_\rho(t)$ exist, then also $\psi_\rho(s+t)$ exists and equals (in the factorization as in Corollary 8.5.2) $\psi_\rho(s) \otimes \psi_\rho(t)$. Moreover, as $\psi_\rho(t)$ is analytic in ρ, we may differentiate. It follows that ${B^*_{1\!\!1_t}}^n \Phi = \frac{d^n}{d\rho^n}\big|_{\rho=0} \psi_\rho(t)$ is in the closed linear span of $\psi_\rho(t)$ ($|\rho| < |\rho_0|$). Therefore, if for each $t > 0$ there exists $\rho_0 > 0$ such that $\psi_{\rho_0}(t)$ exists, then the vectors $\psi_\rho(t)$ form a totalizing set of units.

8.5.3 Lemma. *$\psi_\rho(t)$ exists, whenever $|\rho| < \frac{1}{2}$. Moreover, we have*

$$\langle \psi_\rho(t), \psi_\sigma(t) \rangle = e^{-\frac{ct}{2} \ln(1-4\overline{\rho}\sigma)} \tag{8.5.1}$$

where the function

$$\varkappa\colon (\rho,\sigma) \mapsto -\frac{c}{2} \ln(1 - 4\overline{\rho}\sigma)$$

is a positive definite kernel on $U_{\frac{1}{2}}(0) \times U_{\frac{1}{2}}(0)$.

Proof. First, we show that the left-hand side of (8.5.1) exists in the simpler case $\sigma = \rho$ showing, thus, existence of $\psi_\rho(t)$.

Set $f = \rho I\!I_t$. Then $(\overline{f}f)f = |\rho|^2 f$. This yields the commutation relation $N_{\overline{f}f}B_f^* = B_f^* N_{\overline{f}f} + 2|\rho|^2 B_f^*$. Moreover, $2c\,\mathsf{Tr}(\overline{f}f) = 2c|\rho|^2 t$. We find

$$\begin{aligned}
B_f B_f^{*n} =& B_f^* B_f B_f^{*n-1} \;+\; (2c|\rho|^2 t + 4N_{\overline{f}f})B_f^{*n-1} \\
=& B_f^* B_f B_f^{*n-1} \;+\; B_f^{*n-1}(2c|\rho|^2 t + 8|\rho|^2 (n-1)) \;+\; B_f^{*n-1} 4N_{\overline{f}f} \\
=& B_f^{*n} B_f \;+\; nB_f^{*n-1} 4N_{\overline{f}f} \\
&+\; B_f^{*n-1} 2|\rho|^2 \big((ct + 4(n-1)) + (ct + 4(n-2)) + \ldots + (ct + 0)\big) \\
=& B_f^{*n} B_f \;+\; nB_f^{*n-1} 4N_{\overline{f}f} \;+\; B_f^{*n-1} 2n|\rho|^2 (ct + 2(n-1)).
\end{aligned}$$

If we apply this to the vacuum Φ, then the first two summands dissappear. We find the recursion formula

$$\frac{\langle B_f^{*n}\Phi, B_f^{*n}\Phi\rangle}{(n!)^2} = 4|\rho|^2 \Big(\frac{ct}{2n} + \frac{n-1}{n}\Big) \frac{\langle B_f^{*n-1}\Phi, B_f^{*n-1}\Phi\rangle}{((n-1)!)^2}.$$

It is clear that $\sum\limits_{n=0}^{\infty} \frac{\langle B_f^{*n}\Phi, B_f^{*n}\Phi\rangle}{(n!)^2}$ converges, if and only if $4|\rho|^2 < 1$ or $|\rho| < \frac{1}{2}$.

For fixed $\rho \in U_{\frac{1}{2}}(0)$ the function $\langle \psi_\rho(t), \psi_\rho(t)\rangle$ is the uniform limit of entire functions on t and, therefore, itself an entire function on t. In particular, since $\psi_\rho(s+t) = \psi_\rho(s) \otimes \psi_\rho(t)$, there must exist a number $\varkappa \in \mathbb{R}$ (actually, in $\mathbb{R}_+$, because $\langle \psi_\rho(t), \psi_\rho(t)\rangle \geq 1$) such that $\langle \psi_\rho(t), \psi_\rho(t)\rangle = e^{\varkappa t}$. We find $\varkappa$ by differentiating at $t = 0$. The only contribution in

$$\frac{d}{dt}\Big|_{t=0} 4|\rho|^2 \Big(\frac{ct}{2n} + \frac{n-1}{n}\Big) \cdot \ldots \cdot 4|\rho|^2 \Big(\frac{ct}{2} + 0\Big)$$

comes by the *Leibniz rule*, if we differentiate the last factor and put $t = 0$ in the remaining ones. We find

$$\frac{d}{dt}\Big|_{t=0} \langle \psi_\rho(t), \psi_\rho(t)\rangle = \sum_{n=1}^{\infty} (4|\rho|^2)^n \frac{1}{n}\frac{c}{2} = -\frac{c}{2}\ln(1 - 4|\rho|^2).$$

The remaining statements follow essentially by the same computations, replacing $|\rho|^2$ with $\overline{\rho}\sigma$. Cleary, $\overline{\rho}\sigma$ is a positive definite kernel. Then by *Schur's lemma* also the function $\varkappa(\rho, \sigma)$ as a limit of positive linear combinations of powers of $\overline{\rho}\sigma$ is positive definite. ■

8.5.4 Remark. The function $\varkappa$ is nothing but the covariance function of the product system in the sense of Arveson [Arv89a], which is defined on the set of all units, restricted to the set of special units $\psi_\rho(t)$. In the set of all units we must take into account also multiples e^{ct} of our units, and the covariance function on this two parameter set is only a conditionally positive kernel.

Let

$$v_\rho = \sqrt{\frac{c}{2}}\Big(2\rho\,,\ \frac{(2\rho)^2}{\sqrt{2}}\,,\ \dots\,,\ \frac{(2\rho)^n}{\sqrt{n}}\,,\ \dots\Big)\ \in\ \ell^2.$$

Then $\langle v_\rho, v_\sigma\rangle = -\frac{c}{2}\ln(1-4\overline{\rho}\sigma)$ and the vectors v_ρ are total in ℓ^2 and $v_0 = 0$. In other words, the Kolmogorov decomposition for the covariance function is the pair $(\ell^2, \rho \mapsto v_\rho)$ and the vectors v_ρ form a totalizing set. The following theorem is a simple corollary of Lemma 8.5.3.

8.5.5 Theorem. *There is a unique isomorphism* $\overline{H_\infty} \to \Gamma(L^2(\mathbb{R}_+, \ell^2))$*, invariant under time shift and fulfilling*

$$\psi_\rho(t) \longmapsto \psi(v_\rho I\!I_t).$$

8.5.6 Remark. Defining E_I as the submodule of E generated by $\mathfrak{S}(I)$, we find (for disjoint I and J) $\underline{\Gamma}(E_{I\cup J}) = \underline{\Gamma}(E_I)\odot\underline{\Gamma}(E_J)$ precisely as in the proof of Theorem 8.1.6. Clearly, setting $E_t = E_{[0,t)}$, we find a *tensor product system* $\big(\underline{\Gamma}(E_t)\big)$ of pre-Hilbert $\mathcal{N}_\mathfrak{S}$–$\mathcal{N}_\mathfrak{S}$–modules in the sense of Definition 11.1.1 (generalized suitably to Hilbert modules over P^*–algebras).

8.6 Connections with the finite difference algebra

After the rescaling $c \to 2$ and $\rho \to \frac{\varrho}{2}$, the right-hand side of (8.5.1), extended as in (7.2.1) from indicator functions to step functions, is the kernel used by Boukas [Bou88, Bou91a] to define a representation space for Feinsilver's *finite difference algebra* [Fei87]. Therefore, Boukas' space and ours coincide.

Once established that the representation spaces coincide, it is natural to ask, whether the algebra of square of white noise contains elements fulfiling the relations of the finite difference algebra. Indeed, setting $c = 2$ and defining

$$Q_f = \frac{1}{2}(B^*_f + N_f) \qquad P_f = \frac{1}{2}(B_{\overline{f}} + N_f) \qquad T_f = \mathbf{1}\,\mathsf{Tr}\,f + P_f + Q_f \tag{8.6.1}$$

for $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we find

$$[P_f, Q_g] = [T_f, Q_g] = [P_f, T_g] = T_{fg}. \tag{8.6.2}$$

Specializing to $f = \overline{f} \in \mathfrak{S}$ these are precisely the relations of the finite difference algebra. In fact, the operators Q_f, P_f, T_f are precisely those found by Boukas. However, it is not clear whether the relation $T_f = \mathbf{1}\,\mathsf{Tr}\,f + P_f + Q_f$ follows already from (8.6.2), or is independent. (In the second case, Boukas' representation has no chance to be faithful.)

In all cases, the operators Q_f, P_f, T_f are not sufficient to recover B_f, B^*_f, N_f. (We can only recover the operators $B^*_f - B_f$ and $B^*_f + B_f + 2N_f$.) Whereas the algebra of square of white noise is generated by creation, annihilation, and conservation operators, the representation of the finite difference algebra is generated by certain linear combinations of such.

8.7 Square of white noise as current algebra

Without going into detail, we review some results from Accardi, Franz and Skeide [AFS00] which show how representations of algebras like the square of white noise can be studied systematically. Let $\mathfrak{g}$ be an involutive complex Lie algebra algebra. Suppose we have three linear mappings (π, η, L) where $\pi\colon \mathfrak{g} \to \mathcal{L}^a(\mathfrak{H})$ is a *representation* of $\mathfrak{g}$ on some pre-Hilbert space $\mathfrak{H}$, i.e. $\pi(X^*) = \pi(X)^*$ and $\pi([X,Y]) = [\pi(X), \pi(Y)]$, $\eta\colon \mathfrak{g} \to \mathfrak{H}$ is a *cocycle* with respect to this representation, i.e. $\eta([X,Y]) = \pi(X)\eta(Y) - \pi(Y)\eta(X)$, and $L\colon \mathfrak{g} \to \mathbb{C}$ is a *coboundary* of η, i.e. $L([X,Y]) = \langle \eta(X^*), \eta(Y)\rangle - \langle \eta(Y^*), \eta(X)\rangle$ and L is hermitian. To any $X \in \mathfrak{g}$ and any $f \in \mathfrak{S}(\mathbb{R}_+)$ we assign the operator

$$X_f \;=\; \lambda(f\pi(X)) + a^*(f\eta(X)) + a(\overline{f}\eta(X^*)) + \mathbf{1}L(X)\operatorname{Tr} f$$

on the symmetric Fock space $\underline{\Gamma}(L^2(\mathbb{R}_+, \mathfrak{H}))$. Simple applications of the CCR (8.1.6) and Relations (8.1.7) show that the operators X_f, Y_g fulfill the commutations relations

$$\begin{aligned}[X_f, Y_g] \;&=\; \lambda\big(fg\pi([X,Y])\big) + a^*\big(fg\eta([X,Y])\big) + a\big(\overline{fg}\eta([X,Y]^*)\big) + \mathbf{1}L([X,Y])\operatorname{Tr}(fg)\\ &=\; [X,Y]_{fg}.\end{aligned}$$

In other words, $f\colon X \mapsto X_f$ is a representation of the *current Lie algebra* $\mathfrak{S}(\mathbb{R}_+, \mathfrak{g})$ over $\mathfrak{g}$ (equipped with the pointwise operation). This fact is well-known from the theory of current representations as discussed (among many others) in Parthsarathy and Schmidt [PS72]. Not so well-known is, maybe, the fact that the current Lie algebra (or better the current algebra, i.e. the *universal enveloping algebra* of the current Lie algebra which, actually, is a $*$–bialgebra) is closely related to the theory of *white noises* on bialgebras, so-called *quantum Lévy processes* by Schürmann [Sch93]. Schürmann's results assert that (under a mild continuity condition and under the assumption that there exists a *vacuum vector* distinguished by certain properties, like to be factorizing on products of elements from subalgebras to different time intervals) all representations of the current algebra arise in the above way. It is even sufficient to know only the functional L. Moreover, Schürmann provides us with a quantum stochastic calculus for the integrators $X_t := X_{\mathbb{I}_{[0,t]}}$ and shows that the products of these processes fulfill a (highly entangled in the case of general $*$–bialgebras) system of quantum stochastic differential equations. Also an *Ito formula* drops out.

The relation of these considerations with the square of white noise becomes immediate from the following observation. Let $\mathfrak{swn}$ be the Lie algebra generated by elements b, b^*, m with relations $[b, b^*] = m$ and $[m, b^*] = 2b^*$. (As real Lie algebra this is $\mathfrak{sl}_2$.) Then for all operators B^*_f, B_f, N_f fulfilling the square of white noise relations, $b^*_f \mapsto \frac{B^*_f}{2}$, $b_f \mapsto \frac{B_{\overline{f}}}{2}$, $m_f \mapsto \frac{c}{2}\operatorname{Tr} f + N_f$ establishes a one-to-one correspondence between representations of the current algebra over $\mathfrak{swn}$ and representations of the square of white noise.

The obvious idea acted out in [AFS00] is to study representations of the square of white noise by means of *Schürmann triples* (π, η, L) for $\mathfrak{swn}$. Refering the reader to [AFS00] for any further detail, we only outline some results important in connection with other representations of the finite difference algebra. It is possible to choose the Schürmann triple such that the representation on the symmetric Fock space $\underline{\Gamma}(L^2(\mathbb{R}_+, \mathfrak{H}))$ is that comming from the unitary isomorphism in Theorem 8.5.5. The cocyle η is *non-trivial* in the sense that it is not of the form $\eta(X) = \pi(X)\eta_0$ for some fixed vector $\eta_0 \in \mathfrak{H} = \ell^2$ (i.e. η is not a *coboundary*).

By taking the direct sum of the one-dimensional Lie algebra $\mathbb{C}\mathbf{1}$ and $\mathfrak{swn}$ we obtain $\mathfrak{gl}_2$. Also the finite difference algebra is, actually, the current algebra over the finite difference Lie algebra $\mathfrak{fd}$ generated by elements p, q, t fulfilling the relations coming from (8.6.2) by specializing to $p = P_{\mathbb{I}_{[0,1]}}$, $q = Q_{\mathbb{I}_{[0,1]}}$, $t = T_{\mathbb{I}_{[0,1]}}$. Also $\mathfrak{fd}$ is a Lie subalgebra of $\mathfrak{gl}_2$. Any representation of $\mathfrak{S}(\mathbb{R}_+, \mathfrak{swn})$ first extends to a representation of $\mathfrak{S}(\mathbb{R}_+, \mathfrak{gl}_2)$ (by $\mathbf{1}_f \mapsto \mathbf{1} \operatorname{\mathsf{Tr}} f$), and then restricts to a representation of $\mathfrak{S}(\mathbb{R}_+, \mathfrak{fd})$. For our representation we know this result from Section 8.6. It is Boukas' representation.

Already Parthasarathy and Sinha [PS91] noticed that Boukas' representation space is isomorphic to a symmetric Fock space. They also constructed a representation of the finite difference algebra. Their construction starts, however, from a trivial cocycle. This cocycle extends to a trivial cocycle for the same representation of $\mathfrak{swn}$ as before. As the cocycle for our representation is non-trivial, the functionals L in these two cases are certainly different, and Schürmann's uniqueness results reconfirm the result by Accardi and Boukas [AB00] that the representation of the finite difference algebras obtained by Boukas is not unitarily equivalent to that obtained in [PS91].

Chapter 9

Interacting Fock spaces

In [ALV97] Accardi, Lu, and Volovich proposed the following definition. An *interacting Fock space* over a Hilbert space is the usual *full* (or *boltzmanian*) *Fock space* $\underline{\mathcal{F}}(H) = \bigoplus_{n\in\mathbb{N}_0} H^{\otimes n}$ over a Hilbert space H where, however, direct sum and tensor products are understood algebraically, and where the (semi-)inner product on the n–particle sector $H^{\otimes n}$ is rather arbitrary. The creators $a^*(f)$ $(f \in H)$ are the usual ones. Restrictions to the semiinner product arise by the requirement that each creator $a^*(f)$ should have an adjoint $a(f)$ with respect to the new inner product. This implies that the creators (and also the annihilators) respect the kernel of the semiinner product; see Corollary 1.4.3.

This definition was suggested by the observation that in the *stochastic limit* of an electron coupled to the electro magnetic field as computed in Accardi, Lu [AL96] the limit distribution of the field operators in the vacuum state of the field and some state on the system space of the electron can be understood as the vacuum expectation of creators and annihilators on an interacting Fock space. In the meantime, we know many other examples of interacting Fock spaces; see e.g. Section 9.1. We mention the representation space of the *renormalized square of white noise* (see Section 8.4). The list of examples can be continued *ad infinitum*. However, it is not our goal to give an account of the history of interacting Fock space.

In this chapter we present the results from Accardi and Skeide [AS98] which assert that, in a certain sense, interacting Fock spaces and full Fock modules are, more or less, two ways to look at the same thing. Already in the QED-example (see Appendix D) from Accardi and Lu [AL96] the idea arose to use the language of Hilbert modules to understand better the underlying structure. In fact, the idea is very natural. The limit computed in [AL96], actually, is the limit of the *vacuum conditional expectation* (see Example 4.4.12) from the algebra of operators on $S\otimes\Gamma(L^2(\mathbb{R}^d))$ onto the algebra of operators on S, where S denotes the Hilbert space of the electron. Therefore, the GNS-construction of the limit should provide us with a Hilbert module. However, in [AL96] the limit of the vacuum conditional expectation was computed only weakly. In [Ske98a] we showed that the limit conditional expectation exists. We pointed out that that GNS-module of the vacuum conditional expectation is a full Fock module and the moments of the limits of the field operators are those of creators and annihilators in the vacuum conditional expectation of this Fock module; see Appendix D.

Motivated by the examples we ask, whether it is possible in general to represent operators on an interacting Fock space by operators on a full Fock module and, thus, to glue together the theory of interacting Fock spaces and the theory of full Fock modules. In Section 9.3 we answer this question in the affirmative sense by an explicit construction (Theorems 9.3.1, 9.3.2, and 9.3.6). We obtain in full algebraic generality that the algebra generated by creators and annihilators on an interacting Fock space is determined by the module generalization of the Cuntz relations (6.1.1). In Section 9.2 we show that it is also possible to associate with a given Fock module an interacting Fock space. In Example 9.3.7 we explain that the construction in Section 9.2 reverses the construction in Section 9.3.

We obtain a clearer picture of what the construction actually does, if we restrict to the subcategory of interacting Fock spaces which are *embeddable* (via an isometry) into a usual full Fock space. In Section 9.4 we show that a creator $a^*(f)$ on an embeddable interacting Fock space may be represented as a modified creator $\varkappa\ell^*(f)$ on a full Fock space (Theorem 9.4.5). Here $\varkappa$ is in the relative commutant of the number operator, in other words, $\varkappa$ leaves invariant the number of particles. In the module picture the one-particle sector of the Fock space is replaced by a two-sided module, precisely, over the algebra of such operators. Therefore, in the module picture it is possible to 'absorb' the operator $\varkappa$ into the creator on the full Fock module over the one-particle module (Theorem 9.4.9). We also provide two criteria which show that there are plenty of embeddable interacting Fock spaces (Theorems 9.4.2, and 9.4.3).

In Section 9.1 we define what we understand by interacting Fock space. The definition differs slightly from the definition in [ALV97]. The difference consists, however, only in that we divided out the kernel of the semiinner product of [ALV97] in order to have an inner product. Then we describe some examples of interacting Fock spaces. The generalization of the notion of Hilbert module and full Fock module to Hilbert modules over P^*–algebras are discussed in Appendix C. These are necessary in view of Example 9.1.5 due to Accardi and Bozejko [AB98] where a relation between orthogonal polynomials and interacting Fock spaces is pointed out, and also for our realization of the square of white noise in Chapter 8. In Section 9.2 we show for some examples how distributions of creators and annihilators on an interacting Fock space may be realized as distributions of creators and annihilators on a suitable full Fock module.

In Section 9.5 we explain all aspects from the first sections in the example of the symmetric Fock space. We point out the origin of the complications and explain why the symmetric Fock space is a "bad" example for an interacting Fock space.

9.1 Basic definitions and examples

The definition of interacting Fock space used here differs slightly from the definition in [ALV97]. The difference is that we divide out the kernel of the inner product. The benefits from this approach are a positive definite inner product and absence of the condition that the operators have to respect some kernel (cf. the introduction). Of course, we loose the tensor product structure of the n–particle sector. Instead of a tensor product $H^{\otimes n}$ we are concerned with rather arbitrary pre-Hilbert spaces H_n. However, the H_n are required to be spanned by the range of all creators. Let us introduce some notation.

9.1.1 Definition. Let $\big(H_n\big)_{n\in\mathbb{N}_0}$ be a family of pre-Hilbert spaces. Denote by $\mathcal{H} = \bigoplus\limits_{n\in\mathbb{N}_0} H_n$ their algebraic direct sum. Similar to Definition 6.2.2 (setting $H_n = \{0\}$ for $n < 0$), we define for each $m \in \mathbb{Z}$ the space

$$\mathcal{L}^a_m(\mathcal{H}) = \big\{A \in \mathcal{L}^a(\mathcal{H})\colon AH_n \subset H_{n+m}\ (n \in \mathbb{N}_0)\big\}.$$

9.1.2 Definition. Let $\big(H_n\big)_{n\in\mathbb{N}_0}$ be a family of pre-Hilbert spaces with $H_0 = \mathbb{C}\Omega$. Let H be another pre-Hilbert space. We say $\mathcal{I} = \bigoplus\limits_{n\in\mathbb{N}_0} H_n$ is an *interacting Fock space* (based on H), if there exists a mapping $a^*\colon H \to \mathcal{L}^a_1(\mathcal{I})$, fulfilling $\mathsf{span}\big(a^*(H)H_n\big) = H_{n+1}$.

The operators $a^*(f)$ are called *creators*. Their adjoints $a(f)$ are called *annihilators*. Observe that the linear span of all $a^*(f_n)\ldots a^*(f_1)\Omega$ $(f_i \in H)$ is H_n. By $\mathcal{A}(\mathcal{I})$ we denote the $*$–algebra generated by all $a^*(f)$ $(f \in H)$.

9.1.3 Remark. Usually, we identify H and H_1 by the additional requirement $a^*(f)\Omega = f$ for all $f \in H$. However, in Example 9.1.5 we will have $a^*(f)\Omega = \sqrt{\omega_1}f$ for a fixed real number $\omega_1 > 0$. Another problem may appear, if $a^*(f)\Omega = 0$ although $f \neq 0$. Therefore, it is important to keep the freedom to choose H and H_1 differently.

The definition of interacting Fock space is very flexible. Of course, the usual full Fock space $\underline{\mathcal{F}}(H)$ with $H_n = H^{\otimes n}$ and its natural inner product is an example. But also the symmetric Fock space $\underline{\Gamma}(H)$ fits into the description; see Secion 9.5. Although isomorphic to a symmetric Fock space (Section 8.5), it is better to look also at the representation space of the square of white noise (Section 8.4) as an interacting Fock space, because the square of white noise creators B_f^* are precisely the creators in the interacting Fock space picture, whereas this is not true in the picture of the symmetric Fock space; see Section 8.7.

In the following examples we construct several interacting Fock spaces in the way as described in [ALV97]. In other words, we start with a full Fock space and then change the inner product on the n–particle sector and divide out the kernel. In these cases we always choose the creators $a^*(f)$ to be the images of the usual ones $\ell^*(f)$ on the quotient. Necessarily, the $\ell^*(f)$ have to respect the kernel of the new inner product. By Corollary 1.4.3, giving an explicit adjoint of $\ell^*(f)$, this condition is fulfilled automatically, and the image of this adjoint on the quotient is the the unique adjoint $a(f)$ of $a^*(f)$.

9.1.4 Example. The Lu-Ruggeri interacting Fock space. In [LR98] the n–particle sectors of the full Fock space $\underline{\mathcal{F}}(L^2(\mathbb{R}))$ had been equipped with a new inner product by setting

$$\langle f_n \otimes \ldots \otimes f_1, g_n \otimes \ldots \otimes g_1 \rangle = \int_0^\infty dx_1 \int_{x_1}^\infty dx_2 \ldots \int_{x_{n-1}}^\infty dx_n \, \overline{f_1(x_1)} \ldots \overline{f_n(x_n)} g_n(x_n) \ldots g_1(x_1).$$

Notice that this is nothing but the integral over the n–simplex $\{x_n \geq \ldots \geq x_1 \geq 0\}$. An adjoint of the creator $\ell^*(f)$ is given by

$$[\ell(f) g \otimes g_n \otimes \ldots \otimes g_1](x_n, \ldots, x_1) = \int_{x_n}^\infty dx \, \overline{f(x)} g(x) g_n(x_n) \ldots g_1(x_1) \quad \text{and} \quad \ell(f)\Omega = 0.$$

Choosing H_n as the pre-Hilbert space obtained from $L^2(\mathbb{R})^{\otimes n}$ by dividing out the length-zero elements of the new semiinner product, we get an interacting Fock space. Of course, what we obtain is precisely the time ordered Fock space $\underline{\amalg\Gamma}(\mathbb{C})$. New is the definition of creators and annihilators on this space; see also Muraki [Mur00].

9.1.5 Example. The one-mode interacting Fock space and orthogonal polynomials. Let μ be a symmetric probability measure on $\mathbb{R}$ with compact support so that all moments $\int x^n \mu(dx)$ $(n \in \mathbb{N}_0)$ exist. It is well-known that there exists a sequence $\big(\omega_n\big)_{n\in\mathbb{N}}$ of non-negative real numbers and a sequence $\big(P_n\big)_{n\in\mathbb{N}_0}$ of (real) polynomials, such that $P_0 = 1$, $P_1 = x$,

$$xP_n = P_{n+1} + \omega_n P_{n-1} \quad (n \geq 1),$$

and

$$\langle P_n, P_m \rangle := \int P_m(x) P_n(x) \mu(dx) = \delta_{mn} \omega_n \ldots \omega_1.$$

Let us consider the ***one-mode Fock space*** $\underline{\mathcal{F}}(\mathbb{C})$. Denote by e_n *the* basis vector of $\mathbb{C}^{\otimes n}$ and equip the n–particle sector with a new (semi-)inner product by setting $\langle e_n, e_n \rangle = \omega_n \ldots \omega_1$. Of course, $\ell^*(e_1)$ has an adjoint. Dividing out the kernel of the new inner product (which is non-trivial, if and only if some of the ω_n are 0) we obtain the ***one-mode interacting Fock space*** $\mathcal{I}_\omega$. In [AB98] Accardi and Bozejko showed that the mapping $e_n \mapsto P_n$ establishes a unitary U from the completion of $\mathcal{I}_\omega$ onto $L^2(\mathbb{R}, \mu)$. Moreover, denoting $a^* = a^*(e_1)$, one obtains $Ua^*U^*P_n = P_{n+1}$ and $U(a^* + a)U^* = x$. The last equation means that the operator of multiplication by x on $L^2(\mathbb{R}, \mu)$ is represented on the one-mode interacting Fock space by the sum $a^* + a$.

For later use in Example 9.3.8 and as a motivation for Section 9.4 we present a variant of the preceding discussion. Assume that all ω_n are different from 0. (This means that the support of μ contains infinitely many points.) Let us use the normalized polynomials $Q_n = \frac{1}{\sqrt{\omega_n \ldots \omega_1}} P_n$. The recursion formula becomes

$$xQ_n = \sqrt{\omega_{n+1}} Q_{n+1} + \sqrt{\omega_n} Q_{n-1} \quad (n \geq 1),$$

with $Q_0 = 1$ and $Q_1 = \frac{x}{\sqrt{\omega_1}}$. Then the mapping $e_n \mapsto Q_n$ establishes a unitary V from the usual full Fock space $\underline{\mathcal{F}}(\mathbb{C})$ onto $L^2(\mathbb{R}, \mu)$. Moreover, denoting by $\ell^* = \ell^*(e_1)$ the usual creator, one obtains $V\ell^*V^*Q_n = Q_{n+1}$ and $V(\sqrt{\omega_N}\ell^* + \ell\sqrt{\omega_N})V^* = x$. By $\sqrt{\omega_N}$ we mean the function $n \mapsto \omega_n$ of the number operator $N \colon e_n \mapsto ne_n$. In other words, instead of the real part of the creator a^* on the interacting Fock space, we obtain the real part of the modified creator $\sqrt{\omega_N}\ell^*$ on the usual full Fock space. It is easy to see that $a^* \mapsto \sqrt{\omega_N}\ell^*$ still defines a $*$–algebra monomorphism $\mathcal{A}(\mathcal{I}) \to \mathcal{L}^a(\underline{\mathcal{F}}(\mathbb{C})$, if some ω_n are 0. In this case one just has to use the partial isometry V defined as above as long as $\omega_n \neq 0$, and mapping e_n to 0 for all $n \geq n_0$ where n_0 is the smallest n for which $\omega_n = 0$. It is noteworthy, that V^* always is an isometry.

9.2 Interacting Fock space from full Fock module

In this section we look at full Fock modules in the sense of Appendix C.3 and show how to obtain interacting Fock spaces from full Fock modules. The we illustrate this in an Example. Recall Convention C.1.8.

Let E be pre-Hilbert $\mathcal{B}$–module and suppose that $\mathcal{B} \subset \mathcal{B}^a(G)$ acts (S–positively) on a pre-Hilbert space G. Let Ω be a fixed unit vector in G and suppose that the state $\langle \Omega, \bullet \Omega \rangle$ ***separates*** the elements of E in the sense that $\langle \Omega, \langle x, x \rangle \Omega \rangle = 0$ implies $x = 0$. We set $H_0 = \mathbb{C}\Omega$. Refering again to the Stinespring construction, we denote by $H_n = E^{\odot n} \odot \Omega$ ($n \in \mathbb{N}$) the subspaces of $E^{\odot n} \odot G$ consisting of all elements $L_x\Omega$ ($x \in E^{\odot n}$).

Then $\mathcal{I} = \bigoplus_{n\in\mathbb{N}_0} H_n$ is an interacting Fock space based on H_1. Let $G_\Omega = \mathcal{B}\Omega$. It is easy to see that $H_n = E^{\odot n} \odot G_\Omega$ $(n \in \mathbb{N})$. Thus, $\mathcal{I}$ is just $\underline{\mathcal{F}}(E) \odot G_\Omega \ominus (\mathbf{1} - |\Omega\rangle\langle\Omega|)G_\Omega$. The creators are given by $a^*(h) = \ell^*(x) \odot \mathsf{id} \upharpoonright \mathcal{I}$, where x is the unique element in E, fulfilling $L_x\Omega = h \in H_1$. By construction, $\ell^*(x) \odot \mathsf{id}$ leaves invariant the subspace $\mathcal{I}$ of $\underline{\mathcal{F}}(E) \odot G_\Omega$. Defining the projection $p_\Omega = |\Omega\rangle\langle\Omega| \oplus \bigoplus_{n\in\mathbb{N}} \mathsf{id}_{H_n}$ onto $\underline{\mathcal{F}}(E) \odot G_\Omega$, we have $a^*(h) = (\ell^*(x) \odot \mathsf{id})p_\Omega$, so that the adjoint of $a^*(h)$ is given by $a(h) = p_\Omega(\ell(x) \odot \mathsf{id}) \upharpoonright \mathcal{I}$.

In Example 9.3.7 we will see that by this construction an arbitrary interacting Fock space based on H_1 can be recovered from a full Fock module. If $a^*(h)\Omega = 0$ implies $a^*(h) = 0$, then the whole construction also works for interacting Fock spaces based on more general pre-Hilbert spaces H.

9.2.1 Example. The full Fock module for the Lu-Ruggeri interacting Fock space. Our goal is to recover the inner product of elements in the interacting Fock space from Example 9.1.4 by the inner product of suitable elements in a full Fock module. Let

$$E = \mathsf{span}\big\{ f \boxtimes z \colon (s,t) \mapsto f(s)\mathbb{I}_{[0,s]}(t)z(t) \;\big|\; f \in L^2(\mathbb{R}^+), z \in \mathcal{C}_b(\mathbb{R}^+)\big\}$$

where $\mathbb{R}^+ = [0,\infty)$. One may understand the $\boxtimes$–sign as a time ordered tensor product. Observe that not one of the non-zero functions in E is simple. Clearly, E is invariant under the left multiplication $z(f \boxtimes z') = (zf) \boxtimes z'$ and the right multiplication $(f \boxtimes z')z = f \boxtimes (z'z)$ by elements $z \in \mathcal{C}_b(\mathbb{R}^+)$. Moreover, the inner product

$$\langle f \boxtimes z, f' \boxtimes z'\rangle(t) = \int ds\, \overline{(f \boxtimes z)(s,t)}(f' \boxtimes z')(s,t) = \int_t^\infty ds\, \overline{f(s)z(t)} f'(s)z'(t)$$

maps into the continuous bounded functions on $\mathbb{R}^+$ so that E becomes a pre-Hilbert $\mathcal{C}_b(\mathbb{R}^+)$–module.

Define the state $\varphi(z) = z(0)$ on $\mathcal{C}_b(\mathbb{R}^+)$. One easily checks that

$$\varphi\big(\langle (f_n \boxtimes \mathbf{1}) \odot \ldots \odot (f_1 \boxtimes \mathbf{1}), (g_n \boxtimes \mathbf{1}) \odot \ldots \odot (g_1 \boxtimes \mathbf{1})\rangle\big) = \langle f_n \otimes \ldots \otimes f_1, g_n \otimes \ldots \otimes g_1\rangle$$

where the right-hand side is the inner product from Example 9.1.4.

9.3 Full Fock module from interacting Fock space

Our goal is to associate with an arbitrary interacting Fock space $\mathcal{I}$ a full Fock module in such a way that certain $*$–algebras of operators on $\mathcal{I}$ may represented as operators on that Fock module. In particular, we want to express the moments of operators on $\mathcal{I}$ in the vacuum expectation $\langle\Omega, \bullet\Omega\rangle$ by moments of the corresponding operators on the Fock module in a state of the form

$$\varphi \circ \mathbb{E}_0,$$

where $\mathbb{E}_0 = \langle \omega, \bullet \omega \rangle$ denotes the vaccum conditional expectation on the Fock module, and where φ is a state. We will see that we can achieve our goal by a simple reinterpretation of the *graduation* of $\mathcal{L}^a(\mathcal{I})$ in Definition 9.1.1. Since we work in a purely algebraic framework, we cannot consider the full $*$–algebra $\mathcal{L}^a(\mathcal{I})$. It is necessary to restrict to the $*$–algebra $\mathcal{A}^0(\mathcal{I}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}^a_n(\mathcal{I})$; see Remark 9.3.4 below. Clearly, $\mathcal{A}^0(\mathcal{I})$ is a graded $*$–algebra.

Let $\mathcal{I} = \bigoplus_{n \in \mathbb{N}_0} H_n$ be an interacting Fock space and let S be the subset of $\mathcal{L}^a_0(\mathcal{I})$ consisting of all elements of the form a^*a where, however, a may stem from the bigger algebra $\mathcal{L}^a(\mathcal{I})$; cf. Remark C.1.4. As $\mathcal{L}^a_k(\mathcal{I})\mathcal{L}^a_\ell(\mathcal{I}) \subset \mathcal{L}^a_{k+\ell}(\mathcal{I})$ we find that all spaces $\mathcal{L}^a_m(\mathcal{I})$ are $\mathcal{L}^a_0(\mathcal{I})$–$\mathcal{L}^a_0(\mathcal{I})$–modules. Clearly,

$$\langle x, y \rangle = x^*y$$

fulfills our positivity condition (Definition C.2.1) and all other properties of an $\mathcal{L}^a_0(\mathcal{I})$–valued inner product so that $\mathcal{L}^a_m(\mathcal{I})$ becomes a pre-Hilbert $\mathcal{L}^a_0(\mathcal{I})$–$\mathcal{L}^a_0(\mathcal{I})$–module.

One easiliy checks that $\mathcal{L}^a_k(\mathcal{I}) \odot \mathcal{L}^a_\ell(\mathcal{I}) = \mathsf{span}\big(\mathcal{L}^a_k(\mathcal{I})\mathcal{L}^a_\ell(\mathcal{I})\big)$ via the identification $x \odot y = xy$. (See also Remark 9.3.4.) We set $E^0 = \mathcal{L}^a_1(\mathcal{I})$ and define the *maximal* full Fock module $\underline{\mathcal{F}}^0(\mathcal{I})$ associated with the interacting Fock space $\mathcal{I}$ by

$$\underline{\mathcal{F}}^0(\mathcal{I}) = \underline{\mathcal{F}}(E^0) = \bigoplus_{n \in \mathbb{N}_0} (E^0)^{\odot n} \subset \bigoplus_{n \in \mathbb{N}_0} \mathcal{L}^a_n(\mathcal{I}).$$

We explain in Remark 9.3.3 in which sense this module is maximal.

Let $A \in \mathcal{L}^a_m(\mathcal{I})$. By setting

$$ax_n \odot \ldots \odot x_1 = ax_n \ldots x_1 = \begin{cases} Ax_n \ldots x_1 & \text{for } n+m \geq 0 \\ 0 & \textit{otherwise}, \end{cases}$$

we define an element a in $\mathcal{L}^a(\underline{\mathcal{F}}(E^0))$.

9.3.1 Theorem. *The linear extension of the mapping $A \mapsto a$ to all element a in $\mathcal{A}^0(\mathcal{I})$ defines a $*$–algebra monomorphism $\mathcal{A}^0(\mathcal{I}) \to \mathcal{L}^a(\underline{\mathcal{F}}(E^0))$.*

Proof. We perform the Stinespring construction. One easily checks that $\underline{\mathcal{F}}^0(\mathcal{I}) \odot \mathcal{I} = \mathcal{I}$ and that $\rho(a) = A$. Therefore, $A \mapsto a$ is injective and, clearly, a $*$–homomorphism. (Cf. also the appendix of [Ske98a].) ■

9.3.2 Theorem. *For all $A \in \mathcal{A}^0(\mathcal{I})$ we have*

$$\langle \Omega, A\Omega \rangle = \langle \Omega, \mathbb{E}_0(a) \rangle \Omega \rangle.$$

PROOF. It is sufficient to check the statement for $A \in \mathcal{L}^a_m(\mathcal{I})$. If $m \neq 0$, then both sides are 0. If $m = 0$, then $a\omega = a = A$. (Here we made the identifications $\mathcal{L}^a_0(\mathcal{I})(E^0)^{\odot 0} \subset (E^0)^{\odot 0} = \mathcal{L}^a_0(\mathcal{I})$.) Therefore, both sides coincide also for $m = 0$. ■

9.3.3 Remark. The module $\underline{\mathcal{F}}^0(\mathcal{I})$ is maximal in the sense that the vacuum ω is cyclic for $\mathcal{A}^0(\mathcal{I})$ and that $\mathcal{A}^0(\mathcal{I})$ is the biggest subalgebra of $\mathcal{L}^a(\mathcal{I})$ which may be represented on a purely algebraic full Fock module. Cf. also Remark 9.3.4.

The following somewhat lengthy remark explains to some extent why we have to restrict to $\mathcal{A}^0(\mathcal{I})$, and why $\mathcal{L}^a_k(\mathcal{I}) \odot \mathcal{L}^a_\ell(\mathcal{I})$ cannot coincide with $\mathcal{L}^a_{k+\ell}(\mathcal{I})$. The reader who is not interested in these explanations may skip the remark.

9.3.4 Remark. An excursion about duality. In our framework, where the constructions of direct sum and tensor product are understood purely algebraically, there is a strong anti-relation between spaces which arise by such constructions and spaces of operators on them. For instance, a vector space V may be understood as the direct sum $\bigoplus_{b\in B}(\mathbb{C}b)$ over all subspaces $\mathbb{C}b$ where b runs over a basis B of V. To any $b \in B$ we associate a linear functional β_b in the algebraic dual V' of V by setting $\beta_b(b') = \delta_{bb'}$. Then the direct sum $V'_B = \bigoplus_{b\in B}(\mathbb{C}\beta_b)$ over all subspaces $\mathbb{C}\beta_b$ of V' is a subspace of V' which depends on B, whereas the direct product over all $\mathbb{C}\beta_b$ may be identified with V' itself. Obviously, V'_B is dense in V' with respect to the weak* topology. Problems of this kind are weakened, when topology comes in, but they do not dissappear. For instance, also the Banach space dual V^* of a Banach space V, usually, is much "bigger" than V.

As another example let us consider the space $\mathcal{L}(V,W)$ of linear mappings between two vector spaces V and W; cf. the appendix of [Ske98a]. Clearly, $\mathcal{L}(V,W)$ is an $\mathcal{L}(W)$–$\mathcal{L}(V)$–module. Denote by $\mathcal{L}_\mathfrak{f}(V,W)$ the finite rank operators. Notice that we may identify $\mathcal{L}_\mathfrak{f}(V,W)$ with $W \otimes V'$. The elements of $W' \otimes V$ act on $\mathcal{L}(V,W)$ as linear functionals. Clearly, $\mathcal{L}_\mathfrak{f}(V,W)$ is dense in $\mathcal{L}(V,W)$ with respect to the locally convex Hausdorff topology coming from this duality. It is noteworthy that an element $a \in \mathcal{L}(W)$ acts as right module homomorphism on both, $\mathcal{L}(V,W)$ and $\mathcal{L}_\mathfrak{f}(V,W)$. Actually, a as an element of $\mathcal{L}^r(\mathcal{L}(V,W))$ is uniquely determined by its action on $\mathcal{L}_\mathfrak{f}(V,W)$ and, therefore, the algebras $\mathcal{L}^r(\mathcal{L}(V,W))$ and $\mathcal{L}^r(\mathcal{L}_\mathfrak{f}(V,W))$ are isomorphic; see [Ske98a].

Applying the preceding considerations in an appropriate way, one may show the following results. (Here $^{\overline{}}$ means closure in a space of operators between pre-Hilbert spaces with respect to the weak topology.)

$$\overline{\mathsf{span}}\big(\mathcal{L}^a_0(\mathcal{I})a^*(H)\mathcal{L}^a_0(\mathcal{I})\big) = E^0 \quad \text{and} \quad \overline{\mathcal{L}^a_k(\mathcal{I}) \odot \mathcal{L}^a_\ell(\mathcal{I})} = \mathcal{L}^a_{k+\ell}(\mathcal{I}).$$

Finally, the action of $\mathcal{A}^0(\mathcal{I})$ on $\underline{\mathcal{F}}^0(\mathcal{I})$ may be extended (uniquely) to an action of $\mathfrak{L}^a(\mathcal{I}) = \overline{\mathcal{A}^0(\mathcal{I})}$ on $\overline{\underline{\mathcal{F}}^0(\mathcal{I})}$. This suggests also to introduce the closures $\overline{E \odot F}$ and $\overline{\underline{\mathcal{F}}(E)}$ as a *dual* tensor product and a *dual* full Fock module, respectively.

Now let us return to our original subject. So far we said what we understand by the maximal full Fock module associated with $\mathcal{I}$. What could be the minimal full Fock module? The answer is simple. A minimal Fock module should contain everything within the maximal Fock module, what is *cum grano salis* generated by by $a^*(H)$, but not more.

Consequently, we restrict to the $*$–subalgebra $\mathcal{A}(\mathcal{I})$ of $\mathcal{A}^0(\mathcal{I})$ generated by $a^*(H)$. The graduation on $\mathcal{A}^0(\mathcal{I})$ gives rise to a graduation on $\mathcal{A}(\mathcal{I})$. Using the notation

$$A^\varepsilon = \begin{cases} A^* & \text{if } \varepsilon = 1 \\ A & \text{if } \varepsilon = -1 \end{cases}$$

we find

$$\begin{aligned} E_m &:= \mathcal{A}(\mathcal{I}) \cap \mathfrak{L}^a_m(\mathcal{I}) \\ &= \mathsf{span}\Big\{a^{\varepsilon_n}(f_n) \ldots a^{\varepsilon_1}(f_1) \;\Big|\; f_i \in H, (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1,1\}^n, \sum_{k=1}^n \varepsilon_k = m\Big\}. \end{aligned}$$

We set $\mathcal{B} = E_0 + \mathbb{C}\mathbf{1}$. Again all E_m are pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules. However, now we have $E_k \odot E_\ell = E_{k+\ell}$. Set $E = E_1$. Clearly, $E = \mathsf{span}(\mathcal{B}a^*(H)\mathcal{B})$.

9.3.5 Definition. By the *minimal* full Fock module associated with the interacting Fock space $\mathcal{I}$ we mean $\underline{\mathcal{F}}_0(\mathcal{I}) = \underline{\mathcal{F}}(E)$.

9.3.6 Theorem. *Theorems 9.3.1 and 9.3.2 remain true when restricted to $\mathcal{A}(\mathcal{I})$ and $\underline{\mathcal{F}}_0(\mathcal{I})$. In particular, $A \mapsto a$ defines a $*$–algebra isomorphism $\mathcal{A}(\mathcal{I}) \to \mathcal{A}(\underline{\mathcal{F}}_0(\mathcal{I}))$.*

PROOF. Clear. ■

9.3.7 Example. The converse of Section 9.2. Let $\underline{\mathcal{F}}_0(\mathcal{I})$ be the minimal Fock module associated with an interacting Fock space based on H_1; cf. Remark 9.1.3. Then the state $\langle\Omega, \bullet\Omega\rangle$ separates the elements of E. Obviously, the pre-Hilbert space $\underline{\mathcal{F}}_0(\mathcal{I}) \odot \Omega$ as constructed in Section 9.2, is nothing but $\mathcal{I}$ and the creator $(\ell^*(x) \odot \mathsf{id})p_\Omega$ on the former coincides with the creator $a^*(h)$ on the latter, where $h = x \odot \Omega$. Therefore, the construction of the minimal Fock module is reversible.

We could ask, whether also the construction in Section 9.2 is reversible, in the sense that it is possible to recover the Fock module $\underline{\mathcal{F}}(E)$ we started with. However, as the construction only involves the subspace $\underline{\mathcal{F}}_0(\mathcal{I}) \odot \Omega$ and not the whole space $\underline{\mathcal{F}}_0(\mathcal{I}) \odot G$, we definitely may loose information. For instance, if E is the direct sum of two $\mathcal{B}_i$–modules E_i $(i = 1, 2)$ with an obvious $\mathcal{B}_1 \oplus \mathcal{B}_2$–module structure, and if we choose a state $\langle \Omega_1, \bullet \Omega_1 \rangle$, which is 0 on $\mathcal{B}_2$, then we loose all information about E_2.

9.3.8 Example. Let H be a pre-Hilbert space. Then the full Fock space $\mathcal{I} = \underline{\mathcal{F}}(H)$ is itself an interacting Fock space. On the minimal Fock module $\underline{\mathcal{F}}_0(\mathcal{I})$ we may represent not much more than the $*$–algebra $\mathcal{A}(\underline{\mathcal{F}}(H))$ which is generated by all creators $a^*(f) = \ell^*(f)$ on the original Fock space. On the maximal Fock module $\underline{\mathcal{F}}^0(\mathcal{I})$ we may represent the full $*$–algebra $\mathcal{A}^0(\underline{\mathcal{F}}(H))$. In particular, operators on $\underline{\mathcal{F}}(H)$ of the form $z\ell^*(f)z'$ $(f \in H; b, b' \in \mathcal{L}^a_0(\underline{\mathcal{F}}(H)))$ are represented by creators $\ell^*(b\ell^*(f)b')$ on $\underline{\mathcal{F}}^0(\mathcal{I})$.

For instance, in Example 9.1.5 we established an isometry $\xi = V^*U \colon \mathcal{I}_\omega \to \underline{\mathcal{F}}(\mathbb{C})$ from the one-mode interacting Fock into the one-mode full Fock space. We found $\xi a^* \xi^* = \sqrt{\omega_N}\ell^*$. This *squeezed* creator on the full Fock space, immediately, becomes the creator $\ell^*(\sqrt{\omega_N}\ell^*)$ on the maximal Fock module $\underline{\mathcal{F}}^0(\underline{\mathcal{F}}(\mathbb{C}))$ associated with $\underline{\mathcal{F}}(\mathbb{C})$.

It is noteworthy that all ingredients of the construction of $\underline{\mathcal{F}}_0(\mathcal{I}_\omega)$ and $\underline{\mathcal{F}}^0(\mathcal{I}_\omega)$, being subsets of $\mathcal{A}^0(\mathcal{I}_\omega)$, may be identified isometrically with ingredients of the corresponding construction of $\underline{\mathcal{F}}_0(\underline{\mathcal{F}}(\mathbb{C}))$ and $\underline{\mathcal{F}}^0(\underline{\mathcal{F}}(\mathbb{C}))$, being subsets of $\mathcal{A}^0(\underline{\mathcal{F}}(\mathbb{C}))$, via the mapping $\Xi(\bullet) = \xi \bullet \xi^*$.

What we did in Examples 9.1.5 and 9.3.8 for the one-mode interacting Fock space consisted in two parts. Firstly, we constructed an isometry from $\mathcal{I}_\omega$ into $\underline{\mathcal{F}}(\mathbb{C})$. Under this isometry the creator a^* on $\mathcal{I}_\omega$ became the squeezed creator $\sqrt{\omega_N}\ell^*$ on $\underline{\mathcal{F}}(\mathbb{C})$. Secondly, after constructing the maximal Fock module $\underline{\mathcal{F}}_0(\underline{\mathcal{F}}(\mathbb{C}))$ the squeezed creator became a usual creator on the maximal Fock module. In the following section we will see that these two steps are possible in general for a wide class of interacting Fock spaces.

9.4 Embeddable interacting Fock spaces

9.4.1 Definition. Let $\mathcal{I} = \bigoplus\limits_{n \in \mathbb{N}_0} H_n$ be an interacting Fock space based on H. We say $\mathcal{I}$ is an *embeddable* interacting Fock space, if there exists an isometry $\xi \colon \mathcal{I} \to \overline{\underline{\mathcal{F}}(H)}$, which respects the n–particle sector, i.e.

$$\xi H_n \subset \overline{H^{\otimes n}} \quad \text{and} \quad \xi\Omega = \Omega.$$

We say $\mathcal{I}$ is *algebraically embeddable*, if ξ maps into $\underline{\mathcal{F}}(H)$.

The following two theorems show that there exist many embeddable and many algebraically embeddable interacting Fock spaces. Actually, all known examples of interacting Fock spaces fit into the assumptions of one of these two theorems.

9.4.2 Theorem. *Let $\mathcal{I}$ be an interacting Fock space based on H and define the surjective linear operator $\Lambda\colon \underline{\mathcal{F}}(H) \to \mathcal{I}$ by setting*

$$\Lambda(f_n \otimes \ldots \otimes f_1) = a^*(f_n) \ldots a^*(f_1)\Omega \quad \textit{and} \quad \Lambda\Omega = \Omega.$$

Then the following two conditions are equivalent.

(i) The operator Λ has an adjoint Λ^* in $\mathfrak{L}(\mathcal{I}, \overline{\underline{\mathcal{F}}(H)})$.

(ii) There exists an operator $L\colon \underline{\mathcal{F}}(H) \to \overline{\underline{\mathcal{F}}(H)}$ fulfilling $LH^{\otimes n} \subset \overline{H^{\otimes n}}$, such that

$$\langle a^*(f_n) \ldots a^*(f_1)\Omega, a^*(g_n) \ldots a^*(g_1)\Omega\rangle = \langle f_n \otimes \ldots \otimes f_1, Lg_n \otimes \ldots \otimes g_1\rangle.$$

Moreover, if one of the conditions is fulfilled, then $\mathcal{I}$ is embeddable.

PROOF. Clearly Condition (i) implies Condition (ii), because $L = \Lambda^*\Lambda$ has the claimed properties. So let us assume that Condition (ii) is fulfilled.

Firstly, we show that $\mathcal{I}$ is embeddable. The operator L must be positive. In particular, L is bounded below. Henceforth, by *Friedrich's theorem* L has a self-adjoint extension. Denote by λ the positive square root of this extension (whose domain, clearly, contains $\underline{\mathcal{F}}(H)$). Then the equation $\xi a^*(f_n) \ldots a^*(f_1)\Omega = \lambda f_n \otimes \ldots \otimes f_1$ defines an isometry $\xi\colon \mathcal{I} \to \overline{\underline{\mathcal{F}}(H)}$.

Secondly, we show existence of Λ^*. We have to show that for each $I \in \mathcal{I}$ there exists a constant $C_I > 0$, such that $\langle \Lambda F, I\rangle \leq \|F\|\, C_I$ for all $F \in \underline{\mathcal{F}}(H)$. We may choose $G \in \underline{\mathcal{F}}(H)$ such that $\Lambda G = I$. Then our assertion follows from

$$\langle \Lambda F, I\rangle = \langle \Lambda F, \Lambda G\rangle = \langle F, LG\rangle \leq \|F\|\, \|LG\|\,. \blacksquare$$

9.4.3 Theorem. *Let $\mathcal{I}$ be an interacting Fock space based on H and suppose that H has a countable Hamel basis. Then $\mathcal{I}$ is algebraically embeddable.*

PROOF. Let $\big(e_i\big)_{i\in\mathbb{N}}$ denote the Hamel basis for H. We may assume this basis to be orthonormal. (Otherwise, apply the *Gram-Schmidt orthonormalization procedure.*) Enumerate the vectors $e^n_k = e_{k_n} \otimes \ldots \otimes e_{k_1}$ $(k = (k_1, \ldots, k_n) \in \mathbb{N}^n)$ in a suitable way. In other words, find a bijective mapping $\sigma\colon \mathbb{N} \to \mathbb{N}^n$. Then apply the orthonormalization to the total sequence $\big(b^n_{\sigma(i)}\big)_{i\in\mathbb{N}}$ of vectors in H_n where we set $b^n_k = a^*(e_{k_n}) \ldots a^*(e_{k_1})\Omega$. The result of orthonormalization is another sequence $\big(c^n_i\big)_{i\in\mathbb{N}}$ of vectors, some of which are 0 and the remaining forming an orthonormal basis for H_n. Then

$$\xi c^n_i = \begin{cases} e^n_{\sigma(i)} & \text{for } c^n_i \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

defines the claimed isometry. ∎

We remark that ξ has an adjoint ξ^* defined on the domain $\mathcal{D}_{\xi^*} = \xi\mathcal{I} \oplus (\xi\mathcal{I})^\perp$ dense in $\overline{\underline{\mathcal{F}}(H)}$. Clearly, this domain is mapped by ξ^* onto $\mathcal{I}$.

Before we show the implications of Definition 9.4.1, we provide a simple but useful factorization Lemma about operators on tensor products of vector spaces.

9.4.4 Lemma. *Let U, V, W, and X be vector spaces and let $S \in \mathfrak{L}(W,U)$ and $T \in \mathfrak{L}(V \otimes W, X)$ be operators, such that $Sw = 0$ implies $T(v \otimes w) = 0$ for all $v \in V$. Then there exists an operator $R \in \mathfrak{L}(V \otimes U, X)$, such that*

$$T = R(\mathsf{id} \otimes S).$$

Proof. Denote $\mathcal{N} = \ker(S)$. Then there exists a subspace $\mathcal{N}^0 \subset W$, such that $W = \mathcal{N}^0 \oplus \mathcal{N}$ and $S \upharpoonright \mathcal{N}^0$ is a bijective mapping onto SW. Analgously, we may find $(SW)^0$, such that $U = SW \oplus (SW)^0$. In this way we expressed S as the mapping

$$S = (S \upharpoonright \mathcal{N}^0) \oplus 0 \colon \mathcal{N}^0 \oplus \mathcal{N} \longrightarrow SW \oplus (SW)^0.$$

Defining the mapping

$$S^{\mathrm{inv}} = (S \upharpoonright \mathcal{N}^0)^{-1} \oplus 0 \colon SW \oplus (SW)^0 \longrightarrow \mathcal{N}^0 \oplus \mathcal{N},$$

we find $S^{\mathrm{inv}} S = \mathbf{1} \oplus 0$ on $\mathcal{N}^0 \oplus \mathcal{N}$.

Set $R = T(\mathsf{id} \otimes S^{\mathrm{inv}})$. Then for all $v \in V$ and $w \in \mathcal{N}$ we have $R(\mathsf{id} \otimes S)(v \otimes w) = 0 = T(v \otimes w)$ and for $w \in \mathcal{N}^0$ we find $R(\mathsf{id} \otimes S)(v \otimes w) = T(v \otimes S^{\mathrm{inv}} S w) = T(v \otimes w)$. ∎

The basis for our application of Lemma 9.4.4 is the identification

$$\underline{\mathcal{F}}(H) = H \otimes \underline{\mathcal{F}}(H) \,\oplus\, \mathbb{C}\Omega. \tag{9.4.1}$$

If S is a mapping on $\underline{\mathcal{F}}(H)$, then by $\mathsf{id} \otimes S$ we mean the mapping $\mathsf{id} \otimes S \oplus 0$ acting on the right-hand side of (9.4.1). We have the commutation relation

$$\ell^*(f) S = (\mathsf{id} \otimes S) \ell^*(f).$$

Notice also that $\overline{\underline{\mathcal{F}}(H)} \supset H \otimes \overline{\underline{\mathcal{F}}(H)} \,\oplus\, \mathbb{C}\Omega$.

9.4.5 Theorem. *Let $\mathcal{I}$ be an embeddable interacting Fock space based on H. Then there exists a mapping $\varkappa \colon (H \otimes \mathcal{D}_{\xi^*} \,\oplus\, \mathbb{C}\Omega) \to \mathcal{D}_{\xi^*}$, respecting the n–particle sectors, such that*

$$\varkappa \ell^*(f) = \xi a^*(f) \xi^*$$

for all $f \in H$. In other words, the mapping $a^*(f) \mapsto \varkappa\ell^*(f)$ extends to a $*$–algebra monomorphism $\mathcal{A}(\mathcal{I}) \to \mathfrak{L}^a(\mathcal{D}_{\xi^*})$ and the vacuum expectation is mapped to the vacuum expectation.

Moreover, if $\mathcal{I}$ is algebraically embeddable, then $\varkappa\ell^*(f)$ is an element of $\mathfrak{L}^a(\underline{\mathcal{F}}(H))$.

9.4.6 Remark. Of course, $\varkappa\ell^*(f)$ has an adjoint (even an adjoint which leaves invariant the domain $\mathcal{D}_{\xi^*}$). However, notice that this does not imply that $\varkappa$ has an adjoint.

Proof of Theorem 9.4.5. We have $\Lambda\ell^*(f) = a^*(f)\Lambda$. In particular, if $\Lambda F = 0$ for some $F \in \underline{\mathcal{F}}(H)$, then $\Lambda(f \otimes F) = \Lambda(\ell^*(f)F) = a^*(f)\Lambda F = 0$ for all $f \in H$.

We set $V = H$, $W = \underline{\mathcal{F}}(H)$, $U = \xi\mathcal{I}$, and $X = \overline{\mathcal{F}(H)}$. Furthermore, we define $S = \xi\Lambda \in \mathfrak{L}(W,U)$ and $T = S \upharpoonright (H \otimes W)$. Clearly, the assumptions of Lemma 9.4.4 are fulfilled. Therefore, there exists a mapping $R \in \mathfrak{L}(V \otimes U, X) = \mathfrak{L}(H \otimes \mathcal{D}_{\xi^*}, \overline{\mathcal{F}(H)})$, such that $T(f \otimes F) = R(f \otimes SF)$ for all $f \in H$ and all $F \in \underline{\mathcal{F}}(H)$.

We have

$$\xi a^*(f)\xi^*(\xi\Lambda)F = \xi a^*(f)\Lambda F = \xi\Lambda\ell^*(f)F = T(f \otimes F) = R(f \otimes SF) = R\ell^*(f)(\xi\Lambda)F.$$

Since the domain of $R\ell^*(f)$ is U and $\xi\Lambda F$ ($F \in \underline{\mathcal{F}}(H)$) runs over all elements of U, we find $\xi a^*(f)\xi^* \upharpoonright U = R\ell^*(f)$. We define $\varkappa \in \mathfrak{L}(H \otimes \mathcal{D}_{\xi^*} \oplus \mathbb{C}\Omega, X)$ by setting

$$\varkappa(f \otimes F) = \begin{cases} R(f \otimes F) & \text{for } F \in \xi\mathcal{I} \\ 0 & \text{for } F \in (\xi\mathcal{I})^\perp \end{cases}$$

and $\varkappa\Omega = 0$. Then $\varkappa\ell^*(f) = \xi a^*(f)\xi^*$. Clearly, the range of $\varkappa$ is contained in $\xi\mathcal{I}$, because the range of $\xi a^*(f)\xi^*$ is. ■

We define $\lambda = \xi\Lambda$ and denote by λ_n the restriction of λ to the n–particle sector. Notice that λ_n is a mapping $H^{\otimes n} \to \overline{H^{\otimes n}}$. Denote also by $\varkappa_n$ the restriction of $\varkappa$ to the n–particle sector of $\mathcal{D}_{\xi^*}$.

9.4.7 Corollary. *λ fulfills*

$$\lambda \upharpoonright (H \otimes \underline{\mathcal{F}}(H)) = \varkappa(\mathsf{id} \otimes \lambda).$$

In terms of n–particle sectors this becomes the recursion formula

$$\lambda_{n+1} = \varkappa_{n+1}(\mathsf{id} \otimes \lambda_n) \quad \text{and} \quad \lambda_0 = \mathsf{id}_{\mathbb{C}\Omega}$$

for λ_n. The recursion formula is resolved uniquely by

$$\lambda_n = \varkappa_n(\mathsf{id} \otimes \varkappa_{n-1}) \ldots (\mathsf{id}^{\otimes(n-1)} \otimes \varkappa_1) \quad (n \geq 1).$$

PROOF. We have

$$\begin{aligned}\varkappa(\mathsf{id}\otimes\lambda)(f_n\otimes\ldots\otimes f_1) =& \varkappa\ell^*(f_n)\lambda(f_{n-1}\otimes\ldots\otimes f_1) = \xi a^*(f_n)\xi^*\xi\Lambda(f_{n-1}\otimes\ldots\otimes f_1)\\ =&\xi\Lambda\ell^*(f_n)(f_{n-1}\otimes\ldots\otimes f_1) = \lambda(f_n\otimes\ldots\otimes f_1). \blacksquare\end{aligned}$$

9.4.8 Corollary. *We have*

$$\langle a^*(f_n)\ldots a^*(f_1)\Omega, a^*(g_n)\ldots a^*(g_1)\Omega\rangle = \langle \varkappa_n\ell^*(f_n)\ldots\varkappa_1\ell^*(f_1)\Omega, \varkappa_n\ell^*(g_n)\ldots\varkappa_1\ell^*(g_1)\Omega\rangle.$$

9.4.9 Theorem. *Let $\mathcal{I}$ be an algebraically embeddable interacting Fock space based on H. Then the mapping*

$$a^*(f)\longmapsto \ell^*(\varkappa\ell^*(f))$$

extends to a $*$–algebra monomorphism from $\mathcal{A}^0(\mathcal{I})$ into the $*$–algebra of adjointable operators on the maximal full Fock module $\underline{\mathcal{F}}^0(\underline{\mathcal{F}}(H))$ associated with $\underline{\mathcal{F}}(H)$. (Here the full Fock space $\underline{\mathcal{F}}(H)$ is interpreted as an interacting Fock space.) Also Theorem 9.3.2 remains true.

PROOF. $\varkappa\ell^*(f) = \xi a^*(f)\xi^*$ is an element of $E^0 = \mathfrak{L}^a_1(\underline{\mathcal{F}}(H))$ and $\Xi(\bullet) = \xi\bullet\xi^*$ is a $*$–algebra monomorphism $\mathcal{A}^0(\mathcal{I})\to\mathcal{A}^0(\underline{\mathcal{F}}(H))$. Validity of Theorem 9.3.2 follows by $\xi\Omega = \Omega$. $\blacksquare$

9.5 The symmetric Fock space as an interacting Fock space

In this section we discuss how the symmetric Fock space (Section 8.1) fits into the set-up of interacting Fock spaces. In particular, we identify concretely several mappings which played a crucial role in the preceding section.

Let H be a pre-Hilbert space. By setting $H_n = H^{\otimes_s n}$ and $a^*(f) = \sqrt{N}p\ell^*(f)$ we turn $\underline{\Gamma}(H)$ into an interacting Fock space based on $H = H_1$.

Defining ξ as the canonical embedding of $\underline{\Gamma}(H)$ into $\underline{\mathcal{F}}(H)$, we see that $\underline{\Gamma}(H)$ is algebraically imbeddable. Notice that $\xi^* = p$. But also the stronger conditions of Theorem 9.4.2 are fulfilled (even leaving invariant the algebraic domain). Indeed, from the commutation relation $\ell^*(f)\sqrt{N} = \sqrt{N-1}\ell^*(f)$ we find that

$$\begin{aligned}a^*(f_n)\ldots a^*(f_1)\Omega &= p\sqrt{N}\ldots\sqrt{N-n+1}\ell^*(f_n)\ldots\ell^*(f_1)\Omega\\ &= p\sqrt{N}\ldots\sqrt{N-n+1}f_n\otimes\ldots\otimes f_1,\end{aligned}$$

i.e. $\Lambda = p\sqrt{N!}$. Of course, $\Lambda^* = \xi\Lambda\xi$. So, if we are sloppy in distinguishing between $\underline{\Gamma}(H)$ and the subspace $p\underline{\mathcal{F}}(H)$ of $\underline{\mathcal{F}}(H)$, then Λ is symmetric and coincides more or less with λ. Of course, $L = pN!$. The definition of $a^*(f)$ yields directly $\varkappa = p\sqrt{N}$. We may verify explicitly the recursion formula in Corollary 9.4.7.

The CCR read

$$a(f)a^*(g) = a^*(g)a(f) + \langle f, g\rangle$$

Here we see that the algebra $\mathcal{B} = E_0$, over which the minimal Fock module is a two-sided module, contains already the quite complicated operator $a^*(g)a(f) + \langle f, g\rangle$ commuting with the number operator. The complications are caused by the fact that the projection p_n on the n–particle sector acts on all tensors of its argument. This is extremely incompatible with what creators on a full Fock space can do, which only act at the first tensor. Correspondingly, the additional algebraic structure which we introduce in the module description has to do a lot to repair this 'defect'.

On the other hand, we know that the symmetric Fock space over $L^2(\mathbb{R}^+)$ is isomorphic to the time ordered Fock space. Also here we can write down the operator L. However, if $F_n(t_n, \ldots, t_1)$ is a time ordered function, and if we 'create' a function f_{n+1}, then we find $f_{n+1}(t_{n+1})F_n(t_n, \ldots, t_1)$. In order to project this function to the time ordered subspace, we need only to look for the relation between t_{n+1} and t_n. The 'deeper' time arguments are not involved by the projection. This explains why the module description of the time ordered Fock space is much more transparent and also more illuminating than the module descritpion of the symmetric Fock space $\underline{\Gamma}(L^2(\mathbb{R}^+))$. We see the difference also by looking at creators from Example 9.1.4, which are bounded.

Although a module description is in principle always possible, we must choose carefully for which of the interacting Fock spaces we try a module description. A good criterion is to look at how complicated the algebra $\mathcal{B}$ is. Fortunately, in all applications there are natural choices for $\mathcal{B}$ and the image of $\mathcal{B}$ in the algebra $\mathcal{A}(\underline{\mathcal{F}}(E))$, usually, is much 'smaller' than $\mathcal{A}(\underline{\mathcal{F}}(E))$.

Part III

Product systems

Product systems of Hilbert spaces (Arveson systems) were discovered by Arveson [Arv89a] in the study of E_0–semigroups on $\mathcal{B}(G)$. We met product systems of (pre-)Hilbert $\mathcal{B}$–$\mathcal{B}$–modules in Bhat and Skeide [BS00] in the study of CP-semigroups and their dilations and we refer the reader who whishes a complete motivation, deriving literally speaking the notion of product system from CP-semigroups, to [BS00]. Here we prefer to give a more direct treatment starting in Section 11.1 with the definition of product systems, anf then explore their properties systematically. Most results from [BS00] being specific to CP-semigroups (and some extensions) can be found in Chapter 12 which is independent of the remainder of Part III.

After the short Chapter 10 about relevant notions from dilation theory, we start directly with the definition of product systems (Section 11.1) and units for them (Section 11.2). Although the definitions are the formal analogues of Arveson's definitions, the approaches are very much different, and we comment on the relation to Arveson systems only at the end of these sections and in Chapter 15 about future directions.

Once established that a set of units for a product system gives rise to a CPD-semigroup (Proposition 11.2.3), it is natural to ask for the converse. In Section 11.3 we show (basically, by generalization of the corresponding construction for CP-semigroups in [BS00]) that each CPD-semigroup may, indeed, may be recovered as the CPD-semigroup associated with a set of units for a product system. As usual, this product system is unique, if it is generated in a suitable sense by the set of units. While the GNS-construction or the Kolmogorov decomposition may be considered as the linking step between a single mapping (be it completely positive or completely positive definite) and Hilbert modules, the construction of product systems in Section 11.3 may be considered as the GNS-construction for a whole semigroup of such mappings. Therefore, we refer to the (unique minimal) product system as the *GNS-system* of the corresponding semigroup.

Arveson systems are classified in a first step by their supply of units. Type I systems are those which are *generated* by their units. Rephrasing this in our words, we say type I product systems are (modulo serveral topological variants) those which are the GNS-system of their associated CPD-semigroup. Arveson showed that type I Arveson systems consist of time ordered Fock spaces. We are able to show in Chapter 13 the analogue statement at least for product systems of von Neumann modules. Like in the proof for Arveson systems the crucial object is the generator of the associtated CPD-semigroup, there just a semigroup of positive definite $\mathbb{C}$–valued kernels with a conditionally positive definite kernel as generator. Here the situation is considerably more involved. The generator is a conditionally completely positive definite kernel and we are able to show that it has Christenson-Evans form (as conjectured in Theorem 5.4.14) only after having shown first that the GNS-system of a CP-semigroup (i.e. a product system generated by a single unit, a trivial thing in the case of Arveson systems) consists of time ordered Fock modules. Among other results which we have to provide before we can show this, there is a characterization of the endomporphisms of time ordered product systems (utilizing the elegant and purely algebraic ideas from Bhat [Bha99]) which allows us to find a powerful criterion to decide, whether a certain subset of units for a time ordered system is generating or not. *En passant* we show also that the results by Christenson and Evans [CE79] about the generator of a CP-semigroup are equivalent to the existence of a *central* unit (a unit consisting of centered elements) in a product system of von Neumann modules which has at least one continuous unit.

In Chapter 14 we present two alternative constructions of product systems. The first one in Section 14.1 starts like Arveson from a (strict) E_0–semigroup but on $\mathcal{B}^a(E)$ for some Hilbert $\mathcal{B}$–module (with a unit vector). This is a direct generalization to Hilbert modules of Bhat's [Bha96] approach to Arveson systems. The second construction Section 14.2 is a simple generalization of the construction starting from a CP-semigroup to a construction starting from a system of *transition expectations*. In discrete time transition expectations are related to quantum Markov chains in the sense of Accardi [Acc74, Acc75]. The continuous time version is a generalization of a proposal by Liebscher [Lie00b] which we considered in Liebscher and Skeide [LS00b].

Chapter 10

Introduction to dilations

10.1 Dilations

CP-semigroups (i.e. semigroups $T = \big(T_t\big)_{t\in\mathbb{T}}$ of, usually unital, completely positive mappings T_t on a unital C^*–algebra $\mathcal{B}$, where $\mathbb{T}$ is $\mathbb{R}_+$ or $\mathbb{N}_0$) and their *dilations* to *E_0–semigroups* (i.e. semigroups $\vartheta = \big(\vartheta_t\big)_{t\in\mathbb{T}}$ of unital contractive endomorphisms ϑ_t on a unital pre–C^*–algebra $\mathcal{A}$) may be considered as the main subjects of these notes. There are almost as many notions of *dilation* as authors writing on them. The common part of all these notions may be illustrated in the following diagram.

$$\begin{tikzcd} \mathcal{B} \arrow[r, "T_t"] \arrow[d, "\mathfrak{i}"'] & \mathcal{B} \\ \mathcal{A} \arrow[r, "\vartheta_t"'] & \mathcal{A} \arrow[u, "\mathfrak{p}"'] \end{tikzcd} \tag{10.1.1}$$

10.1.1 Definition. Let $T = \big(T_t\big)_{t\in\mathbb{T}}$ be a unital CP-semigroup on a unital C^*–algebra $\mathcal{B}$. By a *dilation* of T to $\mathcal{A}$ we understand a quadruple $(\mathcal{A}, \vartheta, \mathfrak{i}, \mathfrak{p})$, consisting of a unital pre–C^*–algebra $\mathcal{A}$, an E_0–semigroup $\vartheta = \big(\vartheta_t\big)_{t\in\mathbb{T}}$, a *canonical injection* (i.e. an injective homomorphism) $\mathfrak{i}\colon \mathcal{B} \to \mathcal{A}$, and an *expectation* $\mathfrak{p}\colon \mathcal{A} \to \mathcal{B}$ (i.e. a unital completely positive mapping such that $\varphi = \mathfrak{i}\circ\mathfrak{p}$ is a conditional expectation onto $\mathcal{A}_0 = \mathfrak{i}(\mathcal{B})$), such that Diagram (10.1.1) is commutative (i.e. $\mathfrak{p} \circ \vartheta_t \circ \mathfrak{i} = T_t$) for all $t \in \mathbb{T}$.

Of course, setting $t = 0$ we find $\mathfrak{p} \circ \mathfrak{i} = \mathsf{id}_{\mathcal{B}}$. Hence, we could also identify $\mathcal{B}$ as the subalgebra $\mathcal{A}_0$ of $\mathcal{A}$. But, as $\mathbf{1}_{\mathcal{B}}$ may not coincide with $\mathbf{1}_{\mathcal{A}}$, this would complicate the definitions. Sometimes, we are dealing with different embeddings $\mathfrak{i}$. For these and other reasons we prefer to distinguish clearly between the two algebras.

10.1.2 Remark. Definition 10.1.1 is quasi the minimum a dilation should fulfill. (We could allow for non-unital $\mathcal{A}$ or ϑ, i.e. ϑ is only an *e_0–semigroup*.) Often, it is required that $\mathfrak{i}(\mathbf{1})$ is the unit of $\mathcal{A}$. In this case, we say the dilation is indexdilation!unital*unital*. Some authors, e.g. Kümmerer [Küm85], require that ϑ consists of automorphisms. Therefore, it extends to an automorphism group on the *Grothendieck group* $\widetilde{\mathbb{T}}$ of $\mathbb{T}$ (i.e. $\widetilde{\mathbb{T}} = \mathbb{R}$ or $\widetilde{\mathbb{T}} = \mathbb{Z}$. Several authors, e.g. Accardi [Acc74, Acc75, Acc78] and also [Küm85], ship $\mathcal{A}$ with a ***filtration***, i.e. a mapping $I \rightarrow \mathcal{A}_I \subset \mathcal{A}$ defined on measurable subsets (of $\mathbb{R}_+$ or $\mathbb{R}$) or (unions of) intervals such that $\mathcal{A}_I \subset \mathcal{A}_J$ whenever $I \subset J$, which is ***covariant***, i.e. $\vartheta_t(\mathcal{A}_I) = \mathcal{A}_{t+I}$. Again, the unital subalgebras $\mathcal{A}_I$ may or may not contain the unit of $\mathcal{A}$. One may or may not require that $\mathcal{A}_{I \cup J}$ be generated $\mathcal{A}_I$ and $\mathcal{A}_J$. If there is a filtration, then there should be also a family φ_I of conditional expectations onto $\mathcal{A}_I$ fulfilling $\varphi_I \circ \varphi_J = \varphi_{I \cap J}$. The setting of [Küm85] always requires existence of an ***invariant*** (i.e. $\psi \circ \vartheta_t = \psi$ and $\psi \circ \varphi = \psi$) faithful state ψ on $\mathcal{A}$. This state induces a faithful state $\psi \circ \mathfrak{i}$ on $\mathcal{B}$ which is invariant for T, because $\psi \circ \mathfrak{i} \circ T_t = \psi \circ \mathfrak{i} \circ \mathfrak{p} \circ \vartheta_t \circ \mathfrak{i} = \psi \circ \vartheta_t \circ \mathfrak{i} = \psi \circ \mathfrak{i}$.

We did not yet speak about possible topological requirements. Semigroups may be uniformly continuous, C_0–semigroups, or (in the case of von Neumann algebras) strongly continuous semigroups (see Definition A.5.1 for our conventions). Of course, if an E_0–semigroup ϑ is uniformly continuous, then it consists of automorphisms. But, even our automorphism groups will usually not be uniformly continuous. Independently, the mappings in a semigroup can be continuous in several natural topologies. If we do not say something, we do not assume more than boundedness. For T_t this is automatic, because (precisely for this convenience) we always assume that $\mathcal{B}$ is a C^*–algebra (and unital). For E_0–semigroups ϑ on a pre–C^*–algebra $\mathcal{A}$ boundedness is a requirement. But, if $\mathcal{A}$ is spanned by C^*–subalgebras (for instance, by all $\vartheta_t \circ \mathfrak{i}(\mathcal{B})$), then also ϑ is contractive automatically.

We mention that Sauvageot [Sau86] constructed a for each unital CP-semigroup T a unital dilation to an automorphism group including a unital filtration with corresponding conditional expectations. The dilating automorphism group is, however, non-continuous in any reasonable topology. This is in strong contrast with our construction of a *weak dilation* in Chapters 11 and 12 which preserves the possible maximum of topological properties of T. (Of course, an E_0–semigroup cannot be uniformly continuous, unless it consists of automorphisms, which is not the case with our construction, as soon as T itself does not consist of automorphisms.) We mention also that Sauvageot's construction is the only one, so far, whose mechanism we were not able to explain advantageously in terms of Hilbert modules. A construction of a unital dilation (preserving continuity, but without any filtration) which can be understood in terms of Hilbert modules is that by Evans and Lewis [EL77]. This dilation is closely related to the Hilbert module analogue of Arveson's spectral algebra. We discuss this in Section 12.5.

10.2 White noise

The notion of *dilation* is an attempt to understand the *evolution* T of the *observable algebra* $\mathcal{B}$ of a small system as an *expectation* $\mathfrak{p}$ from the evolution ϑ of the observable algebra $\mathcal{A}$ of a bigger system into which the small system is embedded (identification of $\mathcal{B}$ and $\mathcal{A}_0$). There is a huge amount of literature discussing the physical motivation. Among many many others we mention only Davies [Dav76], Evans and Lewis [EL77], or Arveson [Arv96].

The general idea is that an *unperturbed* or *free* system like $\mathcal{A}$ *evolves* via an endomorphism semigroup like ϑ. (Automorphism groups correspond to *reversibility*, endomorphism semigroups to *non-dissipativeness* of the evolution. We do not discuss the motivation for the second choice, but refer the reader, for instance, to the introduction of [Arv96].) The fact that T is only a CP-semigroup reflects that the small system $\mathcal{B}$ is *interacting* with the big one. The evolution is *dissipative*, energy dissipates from the small system to the environment, and cannot be understood intrinsically looking at $\mathcal{B}$ alone.

If also the evolution T of the subsystem $\mathcal{B}$ consists of endomorphisms, i.e. $\mathcal{B}$ evolves *freely*, then we can consider $\mathcal{B}$ as an *independent* subsystem of $\mathcal{A}$. If in the extreme case $T_t = \mathsf{id}$ is constant, i.e. if ϑ leaves invariant the subsystem $\mathcal{A}_0$, as illustrated in the following diagram,

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\mathsf{id}"] \arrow[d, "\mathfrak{i}"'] \arrow[dr, "\mathfrak{i}"] & \mathcal{B} \\ \mathcal{A} \arrow[r, "\vartheta_t"'] & \mathcal{A} \arrow[u, "\mathfrak{p}"'] \end{tikzcd} \tag{10.2.1}$$

then we speak of a *white noise*. (Passing from endomorphisms T_t to $T_t = \mathsf{id}$ may be interpreted as switching from the *Heisenberg picture* where the observables in the small system evolve freely to the *interaction picture* where the freely evolving observables do no longer change with time.)

10.2.1 Definition. A *white noise* is a dilation of the *trivial* semigroup $(\mathsf{id})_{t\in\mathbb{T}}$, such that $\vartheta_t \circ \mathfrak{i} = \mathfrak{i}$, i.e. Diagram (10.2.1) commutes for all $t \in \mathbb{T}$.

10.2.2 Remark. Also this is a minimal definition and, as discussed in Remark 10.1.2, many authors require additional properties. The most common extra property is *independence* of algebras $\mathcal{A}_I$ and $\mathcal{A}_J$ for disjoint sets I, J. In the (very general) sense of [Küm85] this means $\varphi(a_I a'_J) = \varphi(a_I)\varphi(a'_J)$ for $a_I \in \mathcal{A}_I$ and $a'_J \in \mathcal{A}_J$.

10.2.3 Example. The time shifts $\mathcal{S}$ on the algebra of operators on the full Fock module $\mathcal{F}(L^2(\mathbb{R}, F))$ or on the time ordered Fock module $\breve{\mathbb{I}\Gamma}(F)$ with $\mathfrak{i}$ being the canonical identification of $\mathcal{B}$ as operators on the module and with $\mathfrak{p} = \mathbb{E}_0$ are examples of a white noise with automorphism groups. The restrictions to $\mathcal{F}(L^2(\mathbb{R}_+, F))$ and $\mathrm{I\Gamma}(F)$, respectively, are examples for a white noise with endomorphism semigroups.

10.3 Cocycles

An interesting question is, whether a dilation of a CP-semigroup can be understood as a *coupling* to a white noise via a *cocycle*. Often, like in Part IV the white noises are those from Example 10.2.3 and a suitable cocycle can be obtained with the help of a quantum stochastic calculus. In these cases the cocylce is constructed directly from parameters which determine the generator of the CP-semigroup (cf. Theorem 16.7.1).

10.3.1 Definition. A *left (right) cocycle* in $\mathcal{A}$ with respect to ϑ is a family $\mathfrak{u} = \big(\mathfrak{u}_t\big)_{t\in\mathbb{T}}$ of elements $\mathfrak{u}_t$ in $\mathcal{A}$, fulfilling

$$\mathfrak{u}_{s+t} \;=\; \mathfrak{u}_t\vartheta_t(\mathfrak{u}_s) \qquad\qquad \big(\; \mathfrak{u}_{s+t} \;=\; \vartheta_t(\mathfrak{u}_s)\mathfrak{u}_t \;\big) \tag{10.3.1}$$

and $\mathfrak{u}_0 = \mathbf{1}$. If $\mathfrak{u}_0$ not necessarily $\mathbf{1}$, then we speak of a *weak* cocycle.

A cocycle $\mathfrak{u}$ in $\mathcal{A}$ is *contractive*, *positive*, *partially isometric*, *isometric*, *unitary*, etc., if $\mathfrak{u}_t$ is for all $t \in \mathbb{T}$. A cocycle is *local*, if $\mathfrak{u}_t$ commutes with $\vartheta_t(\mathcal{A})$ for all $t \in \mathbb{T}$.

We collect the following obvious properties of cocycles.

10.3.2 Proposition. $\mathfrak{u}$ *is left cocycle in* $\mathcal{A}$*, if and only if* $\mathfrak{u}^* = \big(\mathfrak{u}_t^*\big)_{t\in\mathbb{T}}$ *is a right cocycle. In this case* $\vartheta^{\mathfrak{u}} = \big(\vartheta_t^{\mathfrak{u}}\big)_{t\in\mathbb{T}}$ *with* $\vartheta_t^{\mathfrak{u}}(a) = \mathfrak{u}_t\vartheta_t(a)\mathfrak{u}_t^*$ *is a CP-semigroup on* $\mathcal{A}$*. This semigroup is unital, an endomorphism semigroup, an* E_0*–semigroup, contractive, if and only if* $\mathfrak{u}$ *is coisometric, isometric, unitary, contractive, respectively.*

10.3.3 Definition. We say the semigroup $\vartheta^{\mathfrak{u}}$ is *conjugate* to the semigroup ϑ via the cocycle $\mathfrak{u}$. We say two E_0–semigroups ϑ, ϑ' on $\mathcal{A}$ are *outer conjugate*, if ϑ' is conjugate to ϑ via a unitary cocycle $\mathfrak{u}$.

10.3.4 Observation. If $\mathfrak{u}$ is a unitary left cycocle with respect to ϑ, then $\mathfrak{u}^* = \big(\mathfrak{u}_t^*\big)_{t\in\mathbb{T}}$ is a left cocycle with respect to $\vartheta^{\mathfrak{u}}$ and $(\vartheta^{\mathfrak{u}})^{\mathfrak{u}^{-1}} = \vartheta$. Therefore, outer conjugacy is an equivalence relation among E_0–semigroups on $\mathcal{A}$.

10.4 Weak Markov flows and weak dilations

An intermediate structure is that of a *Markov flow* $J_t\colon \mathcal{B} \to \mathcal{A}$ as illustrated in the following diagram.

$$\begin{tikzcd} \mathcal{B} \arrow[r, "T_t"] \arrow[d, "\mathfrak{i}"'] \arrow[dr, "J_t"] & \mathcal{B} \\ \mathcal{A} \arrow[r, "\vartheta_t"'] & \mathcal{A} \arrow[u, "\mathfrak{p}"'] \end{tikzcd} \tag{10.4.1}$$

However, without relations to additional structures like filtrations and related conditional expectations the properties of such a Markov process cannot be discussed in this generality. The situation improves considerably, if we pass to *weak Markov flows* $j_t\colon \mathcal{B} \to \mathcal{A}$ as defined by Bhat and Parthasarathy [BP94, BP95]. These have the additional property that the conditional expectation φ and the embedding $\mathfrak{i}$ have the form as discussed in Example 4.4.6. We illustrate this.

$$\begin{tikzcd} \mathcal{B} \arrow[r, "T_t"] \arrow[d, "\mathfrak{i}=j_0"'] \arrow[dr, "j_t"] & \mathcal{B} & \mathcal{A}_0 \arrow[l, "j_0^{-1}"'] \\ \mathcal{A} \arrow[r, "\vartheta_t"'] & \mathcal{A} \arrow[u, "\mathfrak{p}"'] \arrow[ur, "\varphi=j_0(\mathbf{1})\bullet j_0(\mathbf{1})"'] & \end{tikzcd} \tag{10.4.2}$$

Here j_0^{-1} means the left inverse of j_0. Putting $p_t = j_t(\mathbf{1})$ we obtain the *Markov property*

$$p_t j_{s+t}(b) p_t \;=\; j_t \circ T_s(b) \tag{10.4.3}$$

from $p_0 j_s(b) p_0 = j_0 \circ T_s(b)$ by *time shift* ϑ_t. This property does no longer involve the dilating E_0–semigroup ϑ.

10.4.1 Definition. A pair $(\mathcal{A}, j)$ consisting of a unital pre–C^*–algebra $\mathcal{A}$ and a family $j = \big(j_t\big)_{t\in\mathbb{T}}$ of homomorphisms $j_t\colon \mathcal{B} \to \mathcal{A}$ is a *weak Markov flow* for the CP-semigroup T, if it fulfills (10.4.3) (where always $p_t = j_t(\mathbf{1})$). A *weak Markov quasiflow* is a weak Markov flow $(\mathcal{A}, j)$ except that j_0 need not be injective and $\mathcal{A}$ need not be unital.

A dilation $(\mathcal{A}, \vartheta, \mathfrak{i}, \mathfrak{p})$ of T is a *weak dilation*, if $\varphi = \mathfrak{i}(\mathbf{1}) \bullet \mathfrak{i}(\mathbf{1})$. (In this case, by the preceding discussion, the homomorphisms $j_t = \vartheta_t \circ \mathfrak{i}$ form a weak Markov flow.)

10.4.2 Remark. We need quasiflows only in Section 12.4 when we investigate universal properties of such flows.

A weak dilation gives rise to a weak Markov flow. In Section 12.4 we recover Bhat's result [Bha99] that under a certain minimality condition on a weak Markov flow $(\mathcal{A}, j)$ also the converse is true, i.e. the mapping $j_s(b) \mapsto j_{s+t}(b)$ extends to a (contractive) endomorphism ϑ_t of the subalgebra of $\mathcal{A}$ generated by the set $\vartheta_{\mathbb{T}}(\mathcal{B})$. Clearly, in this case the ϑ_t form an e_0–semigroup ϑ dilating T (except that $\mathcal{A}_\infty$, usually, is non-unital).

10.4.3 Remark. By (10.4.3) applied to $b = \mathbf{1}$ and by Proposition A.7.2(4), p_t is an increasing family of projections. In the original definition in [BP94] p_t may be an arbitrary family of increasing projections. If, however, such a family fulfills (10.4.3) with a family $j_t \colon \mathcal{B} \to \mathcal{A}$ of homomorphisms, then the j_t form already a weak Markov flow. To see this observe that $p_t j_t(\mathbf{1}) p_t = j_t(\mathbf{1})$. By Proposition A.7.2(5) we have $p_t \geq j_t(\mathbf{1})$ so that $j_t(\mathbf{1}) j_{s+t}(b) j_t(\mathbf{1}) = j_t(\mathbf{1}) p_t j_{s+t}(b) p_t j_t(\mathbf{1}) = j_t(\mathbf{1}) j_t \circ T_s(b) j_t(\mathbf{1}) = j_t \circ T_s(b)$.

10.4.4 Remark. If $p_0 = \mathbf{1}$, then $p_t = \mathbf{1}$ for all t. Therefore, if a weak dilation (or a weak Markov flow) is unital, then $\vartheta \circ \mathfrak{i} = j_0 \circ T$, i.e. T is an E_0–semigroup.

Weak flows and weak dilations appear unsatisfactory in general dilation theory as most authors are interested only in unital dilations. Nevertheless, as we will see in Theorem 14.1.8, in a huge number of cases (namely, for dilations on a pre-Hilbert module in the sense of Definition 10.5.1) a dilation has sitting inside a weak dilation. Also dilations comming from a cocycle perturbation of a *white noise* in the sense of Kümmerer are contained. Often, a good deal of the dilation is already determined by the associated weak dilation. Classifying weak dilations by product systems (what is one of the major tasks of Part III), therefore, also helps classifying dilations. Let us say it clearly: We use weak dilations as a theoretical tool to understand better also unital dilations.

Contrary to other types of dilations, among weak dilations we can single out a unique universal one, the *GNS-dilation*. For this dilation ingredients like filtrations and related conditional expectations can be constructed, and do not form a part of the defintions. We already pointed out that in some cases it is sufficient to know only parts of Diagram (10.4.2) in order to reconstruct the remaining ones. In Chapter 15 we give a more complete cross reference about the connections among the several notions and tensor product systems of Hilbert modules play the crucial linking role.

10.5 Dilation on Hilbert modules

We now approach the set-up which will be ours throughout Part III. Consider the situation in Diagram (10.4.2) and let us do the GNS-construction (E, ξ) for the expectation $\mathfrak{p}$. Assume that E is essential, i.e. $\mathcal{A}$ acts faithfully on E. Then Examples 4.4.6 and 4.4.10 tell us that Diagram (10.4.2) simplifies as follows.

$$\begin{tikzcd} \mathcal{B} \arrow[r, "T_t"] \arrow[d, "\mathfrak{i}=j_0=\xi\bullet\xi^*"'] \arrow[dr, "j_t"] & \mathcal{B} \\ \mathcal{B}^a(E) \supset \mathcal{A} \arrow[r, "\vartheta_t"'] & \mathcal{A} \subset \mathcal{B}^a(E) \arrow[u, "\mathfrak{p}=\langle\xi,\bullet\xi\rangle"'] \end{tikzcd} \tag{10.5.1}$$

Two questions arise. Firstly, what happens, if $\mathcal{A}$ does not act faithfully? In this case we could try to divide out the kernel of the canonical representation of $\mathcal{A}$ on E. However, in order that ϑ gives rise to an E_0–semigroup on the quotient, it is necessary that ϑ respects the kernel. In settings where a faithfull invariant state is required (e.g. in [Küm85]), the canonical representation is always faithful. Secondly, if E is essential, does ϑ extend to all of $\mathcal{B}^a(E)$? This is one of the main properties of our GNS-dilation, contrary, for instance, to the dilation to an e_0–semigroup on a subalgebra of $\mathcal{B}(H)$ as constructed by Bhat [Bha01].

In these notes we consider with few exceptions E_0–semigroups on $\mathcal{B}^a(E)$ for some (pre-) Hilbert module and dilations to such. The set-up of dilation, weak dilation and weak Markov flow is now illustrated in the following diagram.

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "T_t"] \arrow[d, "\mathfrak{i}"'] \arrow[dr, "J_t"] & \mathcal{B} \\
\mathcal{B}^a(E) \arrow[r, "\vartheta_t"'] & \mathcal{B}^a(E) \arrow[u, "\mathfrak{p}=\langle\xi,\bullet\xi\rangle"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B} \arrow[r, "T_t"] \arrow[d, "j_0=\xi\bullet\xi^*"'] \arrow[dr, "j_t"] & \mathcal{B} \\
\mathcal{B}^a(E) \arrow[r, "\vartheta_t"'] & \mathcal{B}^a(E) \arrow[u, "\mathfrak{p}=\langle\xi,\bullet\xi\rangle"']
\end{tikzcd}
\tag{10.5.2}$$

10.5.1 Definition. Let T be unital CP-semigroup on a unital C^*–algebra $\mathcal{B}$. A *dilation* of T on a pre-Hilbert $\mathcal{B}$–module is a quadruple $(E, \vartheta, \mathfrak{i}, \xi)$ consisting of a pre-Hilbert $\mathcal{B}$–module E, an E_0–semigroup ϑ on $\mathcal{B}^a(E)$, an embedding $\mathfrak{i}$, and a unit vector $\xi \in E$ such that the left diagram in (10.5.2) commutes.

A *weak dilation* of T on a pre-Hilbert $\mathcal{B}$–module is a triple (E, ϑ, ξ) (ϑ and ξ as before) such that the right diagram of (10.5.2) commutes.

A *weak Markov flow* of T on a pre-Hilbert $\mathcal{B}$–module is a triple (E, j, ξ) consisting of a pre-Hilbert $\mathcal{B}$–module E, a family $j = \big(j_t\big)_{t\in\mathbb{T}}$ of homomorphisms $j_t \colon \mathcal{B} \to \mathcal{B}^a(E)$ fulfilling (10.4.3) (with $p_t = j_t(\mathbf{1})$), and a unit vector $\xi \in E$ such that $j_0 = \xi \bullet \xi^*$.

The difference between a weak Markov flow $(\mathcal{B}^a(E), j)$ and a weak Markov flow (E, j, ξ) on E is the vector ξ. If the GNS-construction of the conditional expectation $p_0 \bullet p_0$ is faithful, then Example 4.4.6 tells us that we may pass from a weak Markov flow $(\mathcal{A}, j)$ to a weak Markov flow (E, j, ξ) where with the identification $\mathcal{B} = j_0(\mathcal{B})$ we have $E = \mathcal{A}p_0$ and $\xi = p_0$.

A crucial consequence of Definition 10.5.1 of dilation, compared with the more general Definition 10.1.1, is that the algebra $\mathcal{B}^a(E)$ contains the projection $p_0 = \xi\xi^*$ and, therfore, all its time shifts $p_t = \vartheta_t(p_0)$. More generally, setting $j_0(b) = \xi b \xi^*$ and $j_t = \vartheta_t \circ j_0$, from $\langle \xi, \mathfrak{i}(b)\xi \rangle = b = \langle \xi, \xi \rangle b \langle \xi, \xi \rangle = \langle \xi, j_0(b)\xi \rangle$ it follows that $p_0 \mathfrak{i}(b) p_0 = j_0(b)$ and after time shift $p_t J_t(b) p_t = j_t(b)$ (see Diagram (10.4.1)). In Theorem 14.1.8 we will see with the help of product systems that the j_t form a weak Markov flow for every dilation $(E, \vartheta, \mathfrak{i}, \xi)$. In other words, whatever $\mathfrak{i}$ might be, if $\langle \xi, \vartheta \circ \mathfrak{i}(\bullet)\xi \rangle$ is a (unital) CP-semigroup T, then it has sitting inside the weak dilation (E, ϑ, ξ) of T.

Chapter 11

Tensor product systems of Hilbert modules

11.1 Definition and basic examples

11.1.1 Definition. Let $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{T} = \mathbb{N}_0$, and let $\mathcal{B}$ be a unital C^*–algebra. A *tensor product system of pre-Hilbert modules*, or for short a *product system*, is a family $E^\odot = \bigl(E_t\bigr)_{t\in\mathbb{T}}$ of pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules E_t with a family of two-sided unitaries $u_{st}\colon E_s \odot E_t \to E_{s+t}$ ($s,t \in \mathbb{T}$), fulfilling the associativity condition

$$u_{r(s+t)}(\mathsf{id} \odot u_{st}) \;=\; u_{(r+s)t}(u_{rs} \odot \mathsf{id}) \tag{11.1.1}$$

where $E_0 = \mathcal{B}$ and u_{s0}, u_{0t} where are the identifications as in Definition 4.2.1. Once, the choice of u_{st} is fixed, we always use the identification

$$E_s \odot E_t \;=\; E_{s+t}. \tag{11.1.2}$$

We speak of tensor *product systems* of *Hilbert modules* $E^{\bar{\odot}}$ and *von Neumann modules* $E^{\bar{\odot}^s}$, if $E_s \mathbin{\bar{\odot}} E_t = E_{s+t}$ and $E_s \mathbin{\bar{\odot}^s} E_t = E_{s+t}$, respectively.

A *morphism* of product systems $E^\odot$ and $F^\odot$ is a family $w^\odot = \bigl(w_t\bigr)_{t\in\mathbb{T}}$ of mappings $w_t \in \mathcal{B}^{a,bil}(E_t, F_t)$, fulfilling

$$w_{s+t} \;=\; w_s \odot w_t \tag{11.1.3}$$

and $w_0 = \mathsf{id}_\mathcal{B}$. A morphism is *unitary*, *contractive*, and so on, if w_t is for $t \in \mathbb{T}$. An *isomorphism* of product systems is a unitary morphism.

A *product subsystem* is a family $E'^\odot = \bigl(E'_t\bigr)_{t\in\mathbb{T}}$ of $\mathcal{B}$–$\mathcal{B}$–submodules E'_t of E_t such that $E'_s \odot E'_t = E'_{s+t}$ by restriction of the identification (11.1.2).

By the *trivial* product system we mean $\bigl(\mathcal{B}\bigr)_{t\in\mathbb{T}}$ where $\mathcal{B}$ is equipped with its trivial $\mathcal{B}$–$\mathcal{B}$–module structure; see Examples 1.1.5 and 1.6.7.

11.1.2 Observation. Notice that, in general, there need not exist a projection endomorphism of $E^{\odot}$ onto a subsystem $E'^{\odot}$ of $E^{\odot}$. If, however, each projection $p_t \in \mathcal{B}^a(E_t)$ onto E'_t exists (whence, the p_t are two-sided by Observation 1.6.4), then the p_t form an endomorphism. Conversely, any projection endomorphism $p^{\odot}$ determins a product subsystem $E'_t = p_t E_t$. Therefore, in product systems of von Neumann modules there is a one-to-one correspondence between subsystems and projection endomorphisms.

11.1.3 Example. Let ϑ be an E_0–semigroup on $\mathcal{B}$, and consider the Hilbert $\mathcal{B}$–$\mathcal{B}$–modules $E_t = \mathcal{B}_{\vartheta_t}$ as in Example 1.6.7. Let us define

$$u_{st}(x_s \odot y_t) \;=\; \vartheta_t(x_s)y_t.$$

Then

$$\begin{aligned}\langle u_{st}(x_s \odot y_t), u_{st}(x'_s \odot y'_t)\rangle \;&=\; y_t^*\vartheta_t(x_s)^*\vartheta_s(x'_s)y'_t \;=\; y_t^*\vartheta_t(x_s^*x'_s)y'_t\\ &=\; \big\langle y_t, \langle x_s, x'_s\rangle.y'_t\big\rangle \;=\; \langle x_s \odot y_t, x'_s \odot y'_t\rangle,\end{aligned}$$

and

$$b.u_{st}(x_s \odot y_t) \;=\; \vartheta_{s+t}(b)\vartheta_t(x_s)y_t \;=\; \vartheta_t(b.x_s)y_t \;=\; u_{st}((b.x_s) \odot y_t) \;=\; u_{st}(b.(x_s \odot y_t)),$$

i.e. u_{st} is isometric, two-sided and, obviously, u_{st} is surjective. Also the associativity condition is fulfilled so that $E^{\odot} = \big(E_t\big)_{t\in\mathbb{T}}$ is a product system via u_{st}. We will see that this is the GNS-system which we associate with any CPD-semigroup a product system in Theorem 11.3.5 restricted to the case of E_0–semigroups.

Let ϑ' be another E_0–semigroup on $\mathcal{B}$ and denote by $E'^{\odot}$ the corresponding product system. Suppose that $u = \big(u_t\big)_{t\in\mathbb{T}}$ is an isomorphism $E \to E'$. Then by Example 1.6.7 $\vartheta'_t(b) = u_t\vartheta_t(b)u_t^*$. Moreover, by (11.1.3)

$$\begin{aligned}u_{s+t}\vartheta_t(x_s)y_t \;&=\; u_{s+t}(x_s \odot y_t) \;=\; (u_s x_s) \odot (u_t y_t)\\ &=\; \vartheta'_t(u_s x_s)u_t y_t \;=\; u_t\vartheta_t(u_s x_s)y_t \;=\; u_t\vartheta_t(u_s)\vartheta_t(x_s)y_t.\end{aligned}$$

Putting $x_s = y_t = \mathbf{1}$ we see that the $u_t \in \mathcal{B}^a(E_t, E'_t) = \mathcal{B}^a(\mathcal{B}) = \mathcal{B}$ form a right cocycle with respect to ϑ. In other words, $\vartheta' = \vartheta^u$ is outer conjugate to ϑ. Conversely, suppose that u is a unitary left cocyle with respect to ϑ, and let $\vartheta' = \vartheta^u$. Then interpreting $u_t \in \mathcal{B}$ as mapping in $\mathcal{B}^a(E_t, E'_t)$, we find

$$u_{s+t}(x_s \odot y_t) \;=\; u_t\vartheta_t(u_s)\vartheta_t(x_s)y_t \;=\; u_t\vartheta'_t(u_s x_s)y_t \;=\; \vartheta'_t(u_s x_s)u_t y_t \;=\; (u_s x_s) \odot (u_t y_t)$$

so that $u^{\odot}\colon E^{\odot} \to E'^{\odot}$ is an isomorphism of product systems. In other words, two E_0–semigroups ϑ and ϑ' on $\mathcal{B}$ are outer conjugate, if and only if their product systems are isomorphic. An E_0–semigroup consists of inner automorphisms, if and only if its products system is the trivial one.

11.1.4 Example. Consider the case when $\mathcal{B} = \mathcal{B}(G)$ for some Hilbert space G, and where $E^{\overline{\odot}^s}$ is a product system of von Neumann $\mathcal{B}(G)$–$\mathcal{B}(G)$–modules. By Example 3.3.4 we have $E_t = \mathcal{B}(G, G \bar{\otimes} \mathfrak{H}_t)$ where $\mathfrak{H}_t$ is canonically identified as the $\mathcal{B}(G)$–center $C_{\mathcal{B}(G)}(E_t)$ of E_t. By Example 4.2.13 the isomorphisms u_{st} restrict to isomorphisms $\mathfrak{H}_s \bar{\otimes} \mathfrak{H}_t \to \mathfrak{H}_{s+t}$ of the centers which, therefore, form a products system $\mathfrak{H}^{\overline{\otimes}} = \bigl(\mathfrak{H}_t\bigr)_{t\in\mathbb{T}}$ of Hilbert spaces. Moreover, by Proposition 3.3.5 we see that two product systems $E^{\overline{\odot}^s}, E'^{\overline{\odot}^s}$ are isomorphic, if and only if the corresponding product systems $\mathfrak{H}^{\overline{\otimes}}, \mathfrak{H}'^{\overline{\otimes}}$ are isomorphic.

The extension of Theorem 11.3.5 to normal CPD-semigroups on von Neumann algebras tells us that we can associate with any normal CP-semigroup on $\mathcal{B}(G)$ a product system of von Neumann $\mathcal{B}(G)$–$\mathcal{B}(G)$–modules and further a product systems of Hilbert spaces. For normal E_0–semigroups we recover Arveson's construction [Arv89a] (where by Example 4.2.13 the tensor product of elements in $\mathfrak{H}_s, \mathfrak{H}_t$ is just the multiplication of the corresponding elements in the centers of $E_s, E_t \subset \mathcal{B}(G)$). If we look at Example 11.1.3 in this case, we recover his result that normal E_0–semigroups on $\mathcal{B}(G)$ are classified by their (Hilbert space) product systems up to outer conjugacy. For general normal CP-semigroups on $\mathcal{B}(G)$ we recover the (Hilbert space) product system constructed in a very different way by Bhat [Bha96]; see Section 14.1 where we discuss the generalization to Hilbert modules.

11.1.5 Example. Let F be a (pre-)Hilbert $\mathcal{B}$–$\mathcal{B}$–module. By Theorem 7.1.3 the time ordered Fock modules $\underline{\mathbb{I}\Gamma}_t(F)$ form a product system of pre-Hilbert modules. We call $\underline{\mathbb{I}\Gamma}^{\odot}(F) = \bigl(\underline{\mathbb{I}\Gamma}_t(F)\bigr)_{t\in\mathbb{T}}$ the product system (of pre-Hilbert modules) *associated* with the time ordered Fock module $\underline{\mathbb{I}\Gamma}(F)$. We use similar notations for $\mathbb{I}\Gamma(F)$ and $\mathbb{I}\Gamma^s(F)$. More generally, we speak of a *time ordered product system* $E^{\odot}$ (of Hilbert modules $E^{\overline{\odot}}$, of von Neumann modules $E^{\overline{\odot}^s}$), if $E^{\odot}$, ($E^{\overline{\odot}}$, $E^{\overline{\odot}^s}$) is isomorphic to $\underline{\mathbb{I}\Gamma}^{\odot}(F)$ (to $\mathbb{I}\Gamma^{\odot}(F)$, to $\mathbb{I}\Gamma^{s\odot}(F)$).

Let $\lambda > 0$. Then $[\mathfrak{I}_t^{\lambda} f](s) = \sqrt{\lambda} f(\lambda s)$ ($s \in [0, \frac{t}{\lambda}]$) defines a two-sided isomorphism $L^2([0,t]) \to L^2([0, \frac{t}{\lambda}])$. Clearly, the family of second quantizations $\mathcal{F}(\mathfrak{I}_t^{\lambda}) \upharpoonright \underline{\mathbb{I}\Gamma}_t(F)$ defines an isomorphism from $\underline{\mathbb{I}\Gamma}^{\odot}(F)$ to the time rescaled product system $\bigl(\underline{\mathbb{I}\Gamma}_{\frac{t}{\lambda}}(F)\bigr)_{t\in\mathbb{T}}$.

11.1.6 Example. With each pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module E we can associate a *discrete* product system $\bigl(E^{\odot n}\bigr)_{n\in\mathbb{N}_0}$. Conversely, any discrete product system $\bigl(E_n\bigr)_{n\in\mathbb{N}_0}$ can be obtained in that way from E_1.

• • •

In Examples 11.1.3 and 11.1.4 we have rediscovered the way how Arveson [Arv89a] constructs product systems of Hilbert spaces from normal CP-semigroups on $\mathcal{B}(G)$ and classifies in this way E_0–semigroups by product systems up to outer conjugacy. At the same time we pointed out that his construction may be understood as a specialization from a more general construction for Hilbert modules, making use of the particularly simple centered structure of von Neumann $\mathcal{B}(G)$–$\mathcal{B}(G)$–modules.

On the other hand, Arveson's classification is *one-to-one* in the sense that any product system arises from an E_0–semigroup in the described way [Arv90b]. This statement is, however, not true in the above general algebraic framework (no conditions on the product system, no conditions on the E_0–semigroup except normality, no conditions on the Hilbert space G). First of all, $\mathbb{T} = \mathbb{R}_+$. In Arveson's set-up G is always infinite-dimensional and separable. E_0–semigroups are, besides being normal, also strongly continuous (see Definition A.5.1). The case of automorphisms (which are always inner for $\mathcal{B}(G)$) is excluded explicitly. Product systems fulfill the following topological and measurability conditions. All $\mathfrak{H}_t$ ($t > 0$) are also infinite-dimensional and separable (this corresponds to the exclusion of automorphism semigroups) and, therefore, isomorphic to a fixed Hilbert space H. Allowing for infinite-dimensional *fibers* $\mathfrak{H}_t$, we can say the *vector bundle* $\mathfrak{H} = \bigl(\mathfrak{H}_t\bigr)_{t\in\mathbb{R}_+\backslash\{0\}}$ is topologically isomorphic to the *trivial bundle* $(0,\infty)\times H$. Of course, the *projection* $p\colon \mathfrak{H} \to \mathbb{R}_+\backslash\{0\}$ (sending (t, h_t) to t so that $p^{-1}(t) = \mathfrak{H}_t$) is measurable. Finally, the *inner product*, considered as function on $\bigl(\mathfrak{H}_t \times \mathfrak{H}_t\bigr)_{t\in\mathbb{R}_+\backslash\{0\}} \subset \mathfrak{H}\times\mathfrak{H}$, is measurable. We will call such a product system in the narrow sense an ***Arveson system***.

The product systems associated with time ordered Fock spaces (isomorphic to symmetric Fock spaces), so-called type I product systems, play a crucial role in the classifcation of Arveson systems. For a long time they were the only explicitly known product systems, and only by indirect proofs Powers [Pow87] showed existence of E_0–semigroups which have other Arveson systems. Only recently, Tsirelson [Tsi00a, Tsi00b] constructed examples of non-type I. The first step in the classification is done with the help of *units* which we discuss in the following section.

Also in the classification of our product systems the time ordered Fock module will play a crucial role. We know from Example 11.1.5 that all members of the associated product system (except $t = 0$) are isomorphic. Hence, requiring this property will not exclude many interesting examples. Of course, all examples of Arveson are still contained. Presently, we hesitate to include this property and the other topological constraints of Arveson into our definition and remain algebraical. We discuss these questions in Chapter 15.

11.2 Units and CPD-semigroups

11.2.1 Definition. A *unit* for a product system $E^\odot = \bigl(E_t\bigr)_{t\in\mathbb{T}}$ is a family $\xi^\odot = \bigl(\xi_t\bigr)_{t\in\mathbb{T}}$ of elements $\xi_t \in E_t$ such that

$$\xi_s \odot \xi_t \;=\; \xi_{s+t} \tag{11.2.1}$$

in the identification (11.1.2) and $\xi_0 = \mathbf{1} \in \mathcal{B} = E_0$. By $\mathfrak{U}(E^\odot)$ we denote the set of all units for $E^\odot$. A unit $\xi^\odot$ is *unital* and *contractive*, if $\langle \xi_t, \xi_t \rangle = \mathbf{1}$ and $\langle \xi_t, \xi_t \rangle \leq \mathbf{1}$, respectively. A unit is *central*, if $\xi_t \in C_{\mathcal{B}}(E_t)$ for all $t \in \mathbb{T}$.

11.2.2 Observation. Obviously, a morphism $w^\odot \colon E^\odot \to F^\odot$ sends units to units. For this the requirement $w_0 = \mathsf{id}_{\mathcal{B}}$ is necessary. For a subset $S \subset \mathfrak{U}(E^\odot)$ of units for $E^\odot$ we denote by $w^\odot S \subset \mathfrak{U}(F^\odot)$ the subset of units for $F^\odot$, consisting of the units $w\xi^\odot = \bigl(w_t \xi_t\bigr)_{t\in\mathbb{T}}$ $(\xi^\odot \in S)$.

11.2.3 Proposition. *The family $\mathfrak{U} = \bigl(\mathfrak{U}_t\bigr)_{t\in\mathbb{T}}$ of kernels $\mathfrak{U}_t$ in $\mathcal{K}_{\mathfrak{U}(E^\odot)}(\mathcal{B})$, defined by setting*

$$\mathfrak{U}_t^{\xi,\xi'}(b) \;=\; \langle \xi_t, b\xi'_t \rangle$$

is a CPD-semigroup. More generally, the restriction $\mathfrak{U} \upharpoonright S$ to any subset $S \subset \mathfrak{U}(E^\odot)$ is a CPD-semigroup.

PROOF. Completely positive definiteness follows from the second half of Theorem 5.2.3 (i.e. Example 1.7.7). The semigroup property follows from

$$\mathfrak{U}_{s+t}^{\xi,\xi'}(b) \;=\; \langle \xi_{s+t}, b\xi'_{s+t} \rangle \;=\; \langle \xi_s \odot \xi_t, b\xi'_s \odot \xi'_t \rangle \;=\; \bigl\langle \xi_t, \langle \xi_s, b\xi'_s \rangle \xi'_t \bigr\rangle \;=\; \mathfrak{U}_t^{\xi,\xi'} \circ \mathfrak{U}_s^{\xi,\xi'}(b)$$

and $\langle \xi_0, b\xi'_0 \rangle = b$. ■

Observe that here and on similar occasions, where it is clear that the superscripts refer to units, we prefer to write the shorter $\mathfrak{U}^{\xi,\xi'}$ instead of the more correct $\mathfrak{U}^{\xi^\odot,\xi'^\odot}$.

In Section 11.3 we will see that any CPD-semigroup, i.e in particular, any CP-semigroup, can be recovered in this way from its *GNS-system*. In other words, any CPD-semigroup is obtained from units of a product system. However, the converse must not be true (see Tsirelson [Tsi00a]). Nevertheless, the units of a product system *generate* a product subsystem, determined uniquely by $\mathfrak{U}$. In the following proposition we explain this even for subsets $S \subset \mathfrak{U}(E^\odot)$. Although both statements are fairly obvious, we give a detailed proof of the first one, because it gives us immediately the idea of how to construct the product system of a CPD-semigroup. See Appendix B.3 for details about the lattices $\mathbb{I}_t$ and $\mathbb{J}_t$.

11.2.4 Proposition. *Let $E^\odot$ be a product system and let $S \subset \mathfrak{U}(E^\odot)$. Then the spaces*

$$E_t^S \;=\; \mathsf{span}\bigl\{b_n\xi_{t_n}^n \odot \ldots \odot b_1\xi_{t_1}^1 b_0 \mid n \in \mathbb{N}, b_i \in \mathcal{B}, \xi^{i\odot} \in S, (t_n, \ldots, t_1) \in \mathbb{J}_t\bigr\} \qquad (11.2.2)$$

*form a product subsystem $E^{S\odot}$ of $E^\odot$, the (unique) subsystem **generated** by the units in S.*

Moreover, if $E'^\odot$ is another product system with a subset of units set-isomorphic to S (and, therefore, identified with S) such that $\mathfrak{U} \restriction S = \mathfrak{U}' \restriction S$, then $E'^{S\odot}$ is isomorphic to $E^{S\odot}$ (where the identification of the subset $S \subset \mathfrak{U}(E^\odot)$ and $S \subset \mathfrak{U}(E'^\odot)$ and extension via (11.2.2) *gives the isomorphism).*

Proof. The restriction of u_{st} to $E_s^S \odot E_t^S$ in the identification (11.1.2) gives

$$\begin{aligned}(b_{n+m}\xi_{r_{n+m}}^{n+m} \odot \ldots \odot b_{n+1}\xi_{r_{n+1}}^{n+1}b_n') \odot (b_n\xi_{r_n}^n \odot \ldots \odot b_1\xi_{r_1}^1 b_0)\\ = b_{n+m}\xi_{r_{n+m}}^{n+m} \odot \ldots \odot b_{n+1}\xi_{r_{n+1}}^{n+1} \odot b_n' b_n\xi_{r_n}^n \odot \ldots \odot b_1\xi_{r_1}^1 b_0\end{aligned}$$

where $(r_{n+m}, \ldots, r_{n+1}) \in \mathbb{J}_s$ and $(r_n, \ldots, r_1) \in \mathbb{J}_t$. Therefore, $E_s^S \odot E_t^S \subset E_{s+t}^S$. To see surjectivity let $\mathfrak{r} = (r_k, \ldots, r_1) \in \mathbb{J}_{s+t}$ and $b_i \in \mathcal{B}$, $\xi^i \in S$ $(i = 0, \ldots, k)$. If $\mathfrak{r}$ *hits* t, i.e. $\mathfrak{r} = \mathfrak{s} \smile \mathfrak{t}$ for some $\mathfrak{s} \in \mathbb{J}_s, \mathfrak{t} \in \mathbb{J}_t$, then clearly

$$b_k\xi_{r_k}^k \odot \ldots \odot b_1\xi_{r_1}^1 b_0 \qquad (11.2.3)$$

is in $E_s^S \odot E_t^S$. If $\mathfrak{r}$ does not hit t, then we may easily achieve this by splitting that $\xi_{r_\ell}^\ell$ with $\sum_{i=1}^{\ell-1} r_i < t < \sum_{i=1}^{\ell} r_i$ into a tensor product of two; cf. Example 4.2.8. More precisely, we write $\xi_{r_\ell}^\ell$ as $\xi_{r_2'}^\ell \odot \xi_{r_1'}^\ell$ such that $r_1' + r_2' = r_\ell$ and $r_1' + \sum_{i=1}^{\ell-1} r_i = t$. Also here we find that (11.2.3) is in $E_s^S \odot E_t^S$. ∎

Like for Arveson systems, the question, whether a product system is generated by its units or even some subset of units in the stated way, is crucial for the classification of product systems. However, for Hilbert spaces the property of a certain subset to be total or not, does not depend on the topology, whereas for Hilbert modules we must distinguish clearly between the several possibilities. Furthermore, we can opt to consider only subsets of units distinguished by additional properties like continuity (which, unlike for Arveson systems, again must be split into different topologies).

11.2.5 Definition. A product system $E^\odot = \bigl(E_t\bigr)_{t\in\mathbb{T}}$ of pre-Hilbert modules is of ***type $\underline{I}$***, if it is ***generated*** by some subset $S \subset \mathfrak{U}(E^\odot)$ of its units, i.e. if $E^\odot = E^{S\odot}$. It is of ***type I***, of ***type $I^{\mathcal{B}}$***, and of ***type I^s***, if $E^\odot$ is the closure of $E^{S\odot}$ in norm, in $\mathcal{B}$–weak, and in strong topology, respectively. We say the set S is ***totalizing*** (in the respective topology).

We add subscripts c, c_0, s, n and m, if S can be chosen such that $\mathfrak{U} \upharpoonright S$ is uniformly continuous, C_0–, strongly continuous, normal and measurable, respectively. We add the subscript C if S can be chosen to consist of central units.

Obviously, type I_c implies type I_{c_0} implies type I_s and type I_m and each of them implies type I (and similarly for types $\underline{\mathrm{I}}$, $\mathrm{I}^{\mathcal{B}}$, and I^s), whereas n is a *local* property of the CPD-semigroup which may or may not hold independently (and which is automatic for von Neumann modules). For each subscript type $\underline{\mathrm{I}}$ implies type I implies type $\mathrm{I}^{\mathcal{B}}$ implies type I^s.

11.2.6 Example. The product system constructed in Example 11.1.3 from an E_0–semigroup ϑ on $\mathcal{B}$ is generated by the unit $\xi_t = \mathbf{1}$. Of course, the same is true for the special case $\mathcal{B} = \mathcal{B}(G)$ considered in Example 11.1.4. Notice, however, that for the product system $\mathcal{B}_{\vartheta_t} = \mathcal{B}(G) \bar{\otimes}^s \mathfrak{H}_t$ the unit $\mathbf{1}$ is non-central (if ϑ is non-trivial), whereas any unit $h^\otimes = \bigl(h_t\bigr)$ for $\mathfrak{H}^\otimes$ gives rise to a central unit $\mathbf{1} \otimes h_t$ (which generates, conversely, the trivial semigroup).

We see that non-trivial product systems of pre-Hilbert modules can be generated by a single unit (this is true, in particular, for the GNS-system of a CP-semigroup), whereas a product system of pre-Hilbert spaces generated by a single unit is the trivial one.

11.2.7 Example. Let ξ_t be a semigroup in $\mathcal{B}$ and consider the CP-semigroup $T = \bigl(T_t\bigr)_{t\in\mathbb{T}}$ with $T_t(b) = \xi_t^* b \xi_t$ on $\mathcal{B}$. From Example 4.1.10 and simple computations similar to Example 11.1.3 we conclude that the trivial product system with unit ξ_t gives us back the semigroup T. Checking, whether the product system is generated by this unit can be quite complicated and may, contrary to the case of the trivial product system of Hilbert spaces where any unit is generating, fail. If, however, ξ_t alone is generating for $\mathcal{B}$ (for instance, if ξ_t invertible for all $t > 0$), then the trivial product system is generated by the unit ξ_t.

11.2.8 Example. Let F be a Hilbert $\mathcal{B}$–$\mathcal{B}$–module and consider the time ordered product system $\mathrm{I\!\Gamma}^\odot(F)$ of Hilbert modules with the set $\mathcal{U}_c(F) = \{\xi^\odot(\beta, \zeta)\colon \beta \in \mathcal{B}, \zeta \in F\}$ of units. By Theorem 7.3.1 $\mathfrak{U} \upharpoonright \mathcal{U}_c(F)$ is a uniformly continuous CPD-semigroup. By Theorem 7.2.2 the exponential unit $\xi^\odot(0, \zeta)$ $(\zeta \in F)$ alone generate $\mathrm{I\!\Gamma}(F)$. Therefore, $\mathrm{I\!\Gamma}(F)$ is type I_c. Similarly, if $\mathcal{B}$ is a von Neumann algebra and F is also a von Neumann $\mathcal{B}$–module, then the product system $\mathrm{I\!\Gamma}^{s\odot}(F)$ is type I_c^s. So far, it need not be type I_{cn}^s. Only if F is a two-sided von Neumann module, then $\mathrm{I\!\Gamma}^{s\odot}(F)$ is a time ordered product system of von Neumann modules and, therefore, type I_{cn}^s. If F is centered (for some topology) then the exponential units to elements in the center of F are already totalizing for that topology. So we may add a subscript C in any of these cases. Theorem 7.3.4 and Observation 7.3.5 (together with Lemma 11.6.6) tell us that for both $\mathrm{I\!\Gamma}^\odot(F)$ and $\mathrm{I\!\Gamma}^{s\odot}(F)$ the set $S = \mathcal{U}_c(F) =$

$\{\xi^{\odot}(\beta,\zeta) \colon \beta \in \mathcal{B}, \zeta \in F\}$ has no proper extension such that the CPD-semigroup associated with this extension is still uniformly continuous. $\mathfrak{U} \upharpoonright \mathcal{U}_c(F)$ is *maximal continuous*.

If a unit has a non-zero component in the 1–particle sector, then it, usually, also has non-zero components in all n–particle sectors for $n > 1$. (For the time ordered Fock space this is, clearly, the case. An exception is, for instance, the Fock module for boolean calculus where $F \odot F = \{0\}$.) This shows that, usually, in $\underline{\mathbb{I}\Gamma}^{\odot}(F)$ there are only the *vacuum units* $\xi^{\odot}(\beta, 0)$. In this case, $\underline{\mathbb{I}\Gamma}^{\odot}(F)$ is not type I, unless $F = \{0\}$.

• • •

A unit in an Arveson system $\mathfrak{H}^{\otimes}$ is a measurable *cross section* $\big(h_t\big)_{t\in\mathbb{T}}$ such that $h_s \otimes h_t = h_{s+t}$. The measurable function $\psi_{h,h'}(t) = \langle h_t, h'_t\rangle$ fulfills the functional equation $\psi_{h,h'}(s+t) = \psi_{h,h'}(s)\psi_{h,h'}(t)$ and is, therefore, of the form $e^{t\ell(h,h')}$ for some constant $\ell(h,h')$ depending only on the units. From positivity of the inner product on each $\mathfrak{H}_t$ it follows that $\ell \colon (h,h') \mapsto \ell(h,h')$ is a *conditionally positive definite function* (in the usual sense) on the set of Arveson units, which in [Arv89a] is called the *covariance function* and plays a crucial role. We see that in our context the covariance function is replaced by the generator of the associated CPD-semigroup.

Arveson classified his product systems into three types. An Arveson system is type I, if it is generated by its units. Arveson shows [Arv89a] that these are precisely the symmetric Fock spaces. We will recover the same statement for product systems of type I^s_{cn} when we show that these are (strong closures of) time ordered Fock modules (Theorem 13.3.2). This is also our motivation for Definition 11.2.5. Type II Arveson systems are such which have at least one unit but are not type I, and type III Arveson systems are those whithout unit. For type II and III the analogue definitions for modules are not so clear. A solution is suggested by the observation in Example 11.2.6 that a unit in the Arveson system comming from an E_0–semigroup on $\mathcal{B}(G)$ corresponds to a central unit in the corresponding product system of $\mathcal{B}(G)$–modules. We came back to this point in Chapter 15. Existence of a unit for the distinction between type II and type III might depend on Arveson's measurability requirement. Surprisingly, by a result of Liebscher [Lie00a] for (algebraic) product systems of Hilbert spaces this is not true. Each such system with a unit can be equipped with a measurable structure such that the unit is measurable.

For $\mathcal{B} = \mathbb{C}$ measurability of units implies already continuity of inner products of them, and all reasonable types of continuity coincide. In our context the situation is not so pleasant. Therefore, we do not speak so much about measurability and require continuity directly. In the sequel, we stick mainly on uniform continuity, because this gives us back Arveson's classification of type I as Fock modules (spaces). We want to emphasize, however, that this does not mean that there do not exist other interesting units in type I_c product systems; see Example 7.3.7. Like weak dilations which we consider as a tool to study general dilations, we consider the *continuous* units of a product system as a tool to study the product system. The units of physically interesting CP-semigroups will be only strongly continuous. But they exist also in type I product systems in abundance. Any dilation of a CP-semigroup with unbounded generator constructed on a symmetric Fock space tensored with some initial space may serve as an example.

11.3 CPD-semigroups and product systems

In this Section we construct for each CPD-semigroup $\mathfrak{T}$ on S a product system $E^{\odot}$ with a totalizing set of units such that $\mathfrak{T}$ is recovered as in Proposition 11.2.3 by matrix elements with these units. The construction is a direct generalization from CP-semigroups to CPD-semigroups of the construction in Bhat and Skeide [BS00], and it contains the case of CP-semigroups as the special case where S consists of one element.

The idea can be looked up from the proof of Proposition 11.2.4 together with Example 4.2.8 and its generalization to completely positive definite kernels in Observation 5.4.3. Indeed, the two-sided submodule of E_t^S in Proposition 11.2.4 generated by $\{\xi_t(\xi^{\odot} \in S)\}$ is just the Kolmogorov module $\breve{E}_t$ of the kernel $\mathfrak{U}_t \upharpoonright S \in \mathcal{K}_S(\mathcal{B})$. Splitting ξ_t into $\xi_{t-s} \odot \xi_s$ (for all $\xi^{\odot} \in S$), as done in that proof, means to embed $\breve{E}_t$ into the bigger space $\breve{E}_{t-s} \odot \breve{E}_s$. By definition we obtain all of E_t^S, if we continue this procedure by splitting the interval $[0, t]$ into more and more disjoint subintervals. In other words, E_t^S is the inductive limit over tensor products of an increasing number of Kolmogorov modules $\breve{E}_{t_i}$ (t_i summing up to t) of $\mathfrak{U}_{t_i} \upharpoonright S$.

For a general CPD-semigroup $\mathfrak{T}$ on some set S we proceed precisely in the same way, with the only exception that now the spaces E_t^S do not yet exist. We must construct them. So let $(\breve{E}_t, \breve{\xi}_t)$ denote the Kolmogorov decomposition for $\mathfrak{T}_t$, where $\breve{\xi}_t \colon \sigma \mapsto \breve{\xi}_t^\sigma$ is the canonical embedding $S \to \breve{E}_t$. (Observe that $\breve{E}_0 = \mathcal{B}$ and $\breve{\xi}_0^\sigma = \mathbf{1}$ for all $\sigma \in S$.) See Appendix B.3 for the lattice $\mathbb{J}_t$. Let $\mathfrak{t} = (t_n, \ldots, t_1) \in \mathbb{J}_t$. We define

$$\breve{E}_{\mathfrak{t}} = \breve{E}_{t_n} \odot \ldots \odot \breve{E}_{t_1} \quad \text{and} \quad \breve{E}_{()} = \breve{E}_0.$$

In particular, we have $\breve{E}_{(t)} = \breve{E}_t$. By obvious generalization of Example 4.2.8

$$\breve{\xi}_t^\sigma \ \longmapsto\ \breve{\xi}_{\mathfrak{t}}^\sigma \ :=\ \breve{\xi}_{t_n}^\sigma \odot \ldots \odot \breve{\xi}_{t_1}^\sigma$$

defines an isometric two-sided homomorphism $\beta_{\mathfrak{t}(t)} \colon \breve{E}_t \to \breve{E}_{\mathfrak{t}}$.

Now suppose that $\mathfrak{t} = (t_n, \ldots, t_1) = \mathfrak{s}_m \smile \ldots \smile \mathfrak{s}_1 \geq \mathfrak{s} = (s_m, \ldots, s_1)$ with $|\mathfrak{s}_j| = s_j$. By

$$\beta_{\mathfrak{t}\mathfrak{s}} = \beta_{\mathfrak{s}_m(s_m)} \odot \ldots \odot \beta_{\mathfrak{s}_1(s_1)}$$

we define an isometric two-sided homomorphism $\beta_{\mathfrak{t}\mathfrak{s}}\colon \breve{E}_{\mathfrak{s}} \to \breve{E}_{\mathfrak{t}}$. Obviously, $\beta_{\mathfrak{t}\mathfrak{r}}\beta_{\mathfrak{r}\mathfrak{s}} = \beta_{\mathfrak{t}\mathfrak{s}}$ for all $\mathfrak{t} \geq \mathfrak{r} \geq \mathfrak{s}$. See Appendix A.10 for details about inductive limits. By Proposition A.10.10 we obtain the following result.

11.3.1 Proposition. *The family $\big(\breve{E}_{\mathfrak{t}}\big)_{\mathfrak{t}\in\mathbb{J}_t}$ together with $\big(\beta_{\mathfrak{t}\mathfrak{s}}\big)_{\mathfrak{s}\leq\mathfrak{t}}$ forms an inductive system of pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules. Hence, also the inductive limit $E_t = \liminds_{\mathfrak{t}\in\mathbb{J}_t} \breve{E}_{\mathfrak{t}}$ is a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module and the canonical mappings $i_{\mathfrak{t}}\colon \breve{E}_{\mathfrak{t}} \to E_t$ are isometric two-sided homomorphisms.*

In order to distinguish this inductive limit, where the involved isometries preserve left multiplication, from a different one in Section 11.4, where this is not the case, we refer to it as the *two-sided inductive limit*. This is a change of nomenclature compared with [BS00], where this limit was refered to as the *first inductive limit*.

Before we show that the E_t form a product system, we observe that the elements $\breve{\xi}^\sigma_t$ survive the inductive limit.

11.3.2 Proposition. *Let $\xi^\sigma_t = i_{(t)}\breve{\xi}^\sigma_t$ for all $\sigma \in S$. Then $i_{\mathfrak{t}}\breve{\xi}^\sigma_{\mathfrak{t}} = \xi^\sigma_t$ for all $\mathfrak{t} \in \mathbb{J}_t$. Moreover,*

$$\langle \xi^\sigma_t, b\xi^{\sigma'}_t \rangle \;=\; \mathfrak{T}^{\sigma,\sigma'}_t(b). \tag{11.3.1}$$

Proof. Let $\mathfrak{s},\mathfrak{t} \in \mathbb{J}_t$ and choose $\mathfrak{r}$, such that $\mathfrak{r} \geq \mathfrak{s}$ and $\mathfrak{r} \geq \mathfrak{t}$. Then $i_{\mathfrak{s}}\breve{\xi}^\sigma_{\mathfrak{s}} = i_{\mathfrak{r}}\beta_{\mathfrak{r}\mathfrak{s}}\breve{\xi}^\sigma_{\mathfrak{s}} = i_{\mathfrak{r}}\breve{\xi}^\sigma_{\mathfrak{r}} = i_{\mathfrak{r}}\beta_{\mathfrak{r}\mathfrak{t}}\breve{\xi}^\sigma_{\mathfrak{t}} = i_{\mathfrak{t}}\breve{\xi}^\sigma_{\mathfrak{t}}$.

Moreover, $\langle \xi^\sigma_t, b\xi^{\sigma'}_t \rangle = \langle i_{(t)}\breve{\xi}^\sigma_t, bi_{(t)}\breve{\xi}^{\sigma'}_t \rangle = \langle i_{(t)}\breve{\xi}^\sigma_t, i_{(t)}b\breve{\xi}^{\sigma'}_t \rangle = \langle \breve{\xi}^\sigma_t, b\breve{\xi}^{\sigma'}_t \rangle = \mathfrak{T}^{\sigma,\sigma'}_t(b)$. ∎

11.3.3 Corollary. *$(\xi^\sigma_t)^* i_{\mathfrak{t}} = \breve{\xi}^{\sigma*}_{\mathfrak{t}}$ for all $\mathfrak{t} \in \mathbb{J}_t$. Therefore, $\breve{\xi}^{\sigma*}_{\mathfrak{t}}\beta_{\mathfrak{t}\mathfrak{s}} = \breve{\xi}^{\sigma*}_{\mathfrak{s}}$ for all $\mathfrak{s} \leq \mathfrak{t}$.*

11.3.4 Remark. Clearly, $E_0 = \breve{E}_0 = \mathcal{B}$ and $\xi^\sigma_0 = \breve{\xi}^\sigma_0 = \mathbf{1}$ such that $E_t = E_0 \odot E_t = \xi_0 \odot E_t$ in the identification according to Definition 4.2.1.

11.3.5 Theorem. *The family $E^\odot = \big(E_t\big)_{t\in\mathbb{T}}$ (with E_t as in Proposition 11.3.1) forms a product system. Each of the families $\xi^{\sigma\odot} = \big(\xi^\sigma_t\big)_{t\in\mathbb{T}}$ (with ξ^σ_t as in Proposition 11.3.2) forms a unit and the set $\mathfrak{U}(S) = \{\xi^{\sigma\odot}\ (\sigma \in S)\}$ of units is totalizing for $E^\odot$.*

Proof. Let $s,t \in \mathbb{T}$ and choose $\mathfrak{s} \in \mathbb{J}_s$ and $\mathfrak{t} \in \mathbb{J}_t$. Then the proof that the E_t form a product system is almost done by observing that

$$\breve{E}_{\mathfrak{s}} \odot \breve{E}_{\mathfrak{t}} = \breve{E}_{\mathfrak{s}\smile\mathfrak{t}}. \tag{11.3.2}$$

From this, intuitively, the mapping $u_{st}\colon i_{\mathfrak{s}}x_{\mathfrak{s}} \odot i_{\mathfrak{t}}y_{\mathfrak{t}} \mapsto i_{\mathfrak{s}\smile\mathfrak{t}}(x_{\mathfrak{s}} \odot y_{\mathfrak{t}})$ should define a surjective isometry. Surjectivity is clear, because (as in the proof of Proposition 11.2.4) elements of the form $i_{\mathfrak{s}\smile\mathfrak{t}}(x_{\mathfrak{s}} \odot y_{\mathfrak{t}})$ are total in $\breve{E}_{s+t}$. To see isometry we observe that $i_{\mathfrak{s}}x_{\mathfrak{s}} = i_{\hat{\mathfrak{s}}}\beta_{\hat{\mathfrak{s}}\mathfrak{s}}x_{\mathfrak{s}}$ and $i_{\mathfrak{t}}y_{\mathfrak{t}} = i_{\hat{\mathfrak{t}}}\beta_{\hat{\mathfrak{t}}\mathfrak{t}}y_{\mathfrak{t}}$ for $\hat{\mathfrak{t}} \geq \mathfrak{t}$ and $\hat{\mathfrak{s}} \geq \mathfrak{s}$. Similarly, $i_{\mathfrak{s}\smile\mathfrak{t}}(x_{\mathfrak{s}} \odot y_{\mathfrak{t}}) = i_{\hat{\mathfrak{s}}\smile\hat{\mathfrak{t}}}(\beta_{\hat{\mathfrak{s}}\mathfrak{s}}x_{\mathfrak{s}} \odot \beta_{\hat{\mathfrak{t}}\mathfrak{t}}y_{\mathfrak{t}})$. Therefore, for checking the equation

$$\langle i_{\mathfrak{s}}x_{\mathfrak{s}} \odot i_{\mathfrak{t}}y_{\mathfrak{t}}, i_{\mathfrak{s}'}x'_{\mathfrak{s}'} \odot i_{\mathfrak{t}'}y'_{\mathfrak{t}'}\rangle = \langle i_{\mathfrak{s}\smile\mathfrak{t}}(x_{\mathfrak{s}} \odot y_{\mathfrak{t}}), i_{\mathfrak{s}'\smile\mathfrak{t}'}(x'_{\mathfrak{s}'} \odot y'_{\mathfrak{t}'})\rangle$$

we may assume that $\mathfrak{t}' = \mathfrak{t}$ and $\mathfrak{s}' = \mathfrak{s}$. (This is also a key observation in showing that $E_s \odot E_t = \liminf_{(\mathfrak{s},\mathfrak{t})\in\mathbb{J}_s\times\mathbb{J}_t} \breve{E}_{\mathfrak{s}} \odot \breve{E}_{\mathfrak{t}}$.) Now isometry is clear, because both $i_{\mathfrak{s}} \odot i_{\mathfrak{t}}\colon \breve{E}_{\mathfrak{s}} \odot \breve{E}_{\mathfrak{t}} \to E_s \odot E_t$ and $i_{\mathfrak{s}\smile\mathfrak{t}}\colon \breve{E}_{\mathfrak{s}\smile\mathfrak{t}} = \breve{E}_{\mathfrak{s}} \odot \breve{E}_{\mathfrak{t}} \to \breve{E}_{s+t}$ are (two-sided) isometries. The associativity condition follows directly from associativity of (11.3.2).

The fact that the ξ_t^σ form a unit is obvious from Proposition 11.3.2 and Observation 5.4.3. The set $\mathfrak{U}(S)$ of units is generating, because E_t is generated by vectors of the form $i_{\mathfrak{t}}(b_n\breve{\xi}^n_{t_n} \odot \ldots \odot b_1\breve{\xi}^1_{t_1}b_0)$ $(b_i \in \mathcal{B}, \xi^{i\odot} \in \mathfrak{U}(S))$. ■

11.3.6 Remark. We, actually, have shown, using the identifications (11.1.2) and (11.3.2), that $i_{\mathfrak{s}} \odot i_{\mathfrak{t}} = i_{\mathfrak{s}\smile\mathfrak{t}}$.

11.3.7 Definition. We refer to $E^\odot$ as the ***GNS-system*** of $\mathfrak{T}$. Proposition 11.2.4 tells us that the pair $(E^\odot, \mathfrak{U}(S))$ is determined up to isomorphism by the requirement that $\mathfrak{U}(S)$ be a totalizing set of units fufilling (11.3.1). We refer to $E^{\bar{\odot}}$ as the GNS-system of ***Hilbert*** modules. If $\mathcal{B}$ is a von Neumann algebra and $\mathfrak{T}$ a normal CPD-semigroup, then by Propositions 4.1.13, 4.2.24, and A.10.10 all $\overline{E}^s_t$ are von Neumann modules. We refer to $E^{\bar{\odot}^s}$ as the GNS-system of ***von Neumann*** modules.

11.4 Unital units, dilations and flows

By Corollary 4.2.6 a unit vector $\xi \in E$ gives rise to an isometric embedding $\xi\odot\mathsf{id}\colon F \to E\odot F$ with adjoint $\xi^*\odot\mathsf{id}$. Hence, we may utilize a *unital unit* $\xi^\odot$ for a product system $E^\odot$ to embed E_s into E_t for $t \geq s$ and, finally, end up with a *second inductive limit* (in the nomenclature of [BS00]). However, since the embeddings no longer preserve left multiplication, we do not have a unique left multiplication on the inductive limit $E = \liminf_{t\to\infty} E_t$. We, therefore, refer to it as the ***one-sided inductive limit***. It is, however, possible to define on E for each time t a different left multiplication, which turns out to be more or less the left multiplication from E_t (see Proposition 11.4.9). This family of left multiplications turns out to be a weak Markov flow for the (unital) CP-semigroup $T_t(b) = \langle\xi_t, b\xi_t\rangle$ associated with the unit. Also the identification by (11.1.2) has a counter part obtained by sending, formally, s to ∞. The embedding of $\mathcal{B}^a(E_s)$ into $\mathcal{B}^a(E_{s+t})$, formally, becomes an embedding $\mathcal{B}^a(E_{\text{“}\infty\text{”}})$ into $\mathcal{B}^a(E_{\text{“}\infty+t\text{”}})$, i.e. an endomorphism of $\mathcal{B}^a(E)$. This endomorphism depends, however, on t.

The family formed by all these endomorphisms turns out to be an E_0–semigroup dilating T.

Let $t, s \in \mathbb{T}$ with $t \geq s$. We define the isometry

$$\gamma_{ts} \;=\; \xi_{t-s} \odot \mathsf{id}\colon E_s \;\longrightarrow\; E_{t-s} \odot E_s = E_t.$$

Let $t \geq r \geq s$. Since $\xi^{\odot}$ is a unit, we have

$$\gamma_{ts} = \xi_{t-s} \odot \mathsf{id} = \xi_{t-r} \odot \xi_{r-s} \odot \mathsf{id} = \gamma_{tr}\gamma_{rs}.$$

By Proposition A.10.10 that leads to the following result.

11.4.1 Proposition. *The family $\bigl(E_t\bigr)_{t\in\mathbb{T}}$ together with $\bigl(\gamma_{ts}\bigr)_{s\leq t}$ forms an inductive system of right pre-Hilbert $\mathcal{B}$–modules. Hence, also the inductive limit $E = \limind_{t\to\infty} E_t$ is a right pre-Hilbert $\mathcal{B}$–module. Moreover, the canonical mappings $k_t\colon E_t \to E$ are isometries.*

E contains a distinguished unit-vector.

11.4.2 Proposition. *Let $\xi = k_0\xi_0$. Then $k_t\xi_t = \xi$ for all $t \in \mathbb{T}$. Moreover, $\langle \xi, \xi \rangle = \mathbf{1}$.*

Proof. Precisely, as in Proposition 11.3.2. ■

By Example 4.4.10 we have a representation of $\mathcal{B}$ and a conditional expectation.

11.4.3 Corollary. *By $j_0(b) = \xi b \xi^*$ we define a faithful representation of $\mathcal{B}$ by operators in $\mathcal{B}^a(E)$. Moreover, $\varphi\colon a \mapsto j_0(\mathbf{1})aj_0(\mathbf{1})$ defines a conditional expectation $\mathcal{B}^a(E) \to j_0(\mathcal{B})$.*

11.4.4 Theorem. *For all $t \in \mathbb{T}$ we have*

$$E \odot E_t = E, \tag{11.4.1}$$

extending (11.1.2) *in the natural way. Moreover,*

$$E \odot (E_s \odot E_t) \;=\; (E \odot E_s) \odot E_t. \tag{11.4.2}$$

Proof. The mapping $u_t\colon k_s x_s \odot y_t \mapsto k_{s+t}(x_s \odot y_t)$ defines a surjective isometry. We see that this is an isometry precisely as in the proof of Theorem 11.3.5. To see surjectivity recall that any element in E can be written as $k_r x_r$ for suitable $r \in \mathbb{T}$ and $x_r \in E_r$. If $r \geq t$ then consider x_r as an element of $E_{r-t} \odot E_t$ and apply the prescription to see that $k_r x_r$ is in the range of u_t. If $r < t$, then apply the prescription to $\xi_0 \odot \gamma_{tr} x_r \in E_0 \odot E_t$. Of course,

$$u_{s+t}(\mathsf{id} \odot u_{st}) \;=\; u_t(u_s \odot \mathsf{id}) \tag{11.4.3}$$

which, after the identifications (11.4.1) and (11.1.2), implies (11.4.2). ■

11.4.5 Corollary. *The family $\vartheta = \big(\vartheta_t\big)_{t\in\mathbb{T}}$ of endomorphisms $\vartheta_t\colon \mathcal{B}^a(E) \to \mathcal{B}^a(E \odot E_t) = \mathcal{B}^a(E)$ defined by setting*

$$\vartheta_t(a) = a \odot \mathsf{id}_{E_t} \tag{11.4.4}$$

is a strict E_0–semigroup.

PROOF. The semigroup property follows directly from $E \odot E_{s+t} = E \odot (E_s \odot E_t) = (E \odot E_s) \odot E_t$. Strictness of each ϑ_t trivially follows from the observation that vectors of the form $x \odot x_t$ $(x \in E, x_t \in E_t)$ span E. ■

11.4.6 Remark. Making use of the identification (11.4.1), the proof of Theorem 11.4.4, actually, shows that, $k_s \odot \mathsf{id} = k_{s+t}$. Putting $s = 0$ and making use of Remark 11.3.4, we find

$$k_t = (k_0 \odot \mathsf{id})(\xi_0 \odot \mathsf{id}) = \xi \odot \mathsf{id}\,.$$

In particular, $\xi = \xi \odot \xi_t$.

11.4.7 Corollary. *k_t is an element of $\mathcal{B}^a(E_t, E)$. The adjoint mapping is*

$$k_t^* \;=\; \xi^* \odot \mathsf{id}\colon E = E \odot E_t \;\longrightarrow\; E_t.$$

Therefore, $k_t^* k_t = \mathsf{id}_{E_t}$ and $k_t k_t^*$ is a projection onto the range of k_t.

11.4.8 Theorem. *Define the family $j = \big(j_t\big)_{t\in\mathbb{T}}$ of representations, by setting $j_t = \vartheta_t \circ j_0$. Then (E, j, ξ) is a weak Markov flow of the CP-semigroup T on E and (E, ϑ, ξ) is a weak dilation on E.*

PROOF. The statements are clear, if we show the Markov property $p_s j_t(b) p_s = j_s(T_{t-s}(b))$ for $s \le t$ (with $p_t = j_t(\mathbf{1})$). By definition of j and the semigroup property of ϑ it is enough to restrict to $s = 0$. We have

$$\langle \xi, j_t(b)\xi \rangle \;=\; \langle \xi \odot \xi_t, (j_0(b) \odot \mathsf{id})(\xi \odot \xi_t) \rangle \;=\; \langle \xi_t, b\xi_t \rangle \;=\; T_t(b)$$

by Corollary 11.4.3. Hence, $p_0 j_t(b) p_0 = \xi T_t(b) \xi^* = j_0(T_t(b))$. ■

It seems interesting to see clearly that j_t is nothing but the left multiplication from E_t. The following obvious proposition completely settles this problem.

11.4.9 Proposition. *We have $k_t b k_t^* = \vartheta_t\big(\xi b \xi^*\big) = j_t(b)$. In particular, $p_t = k_t k_t^*$.*

11.4.10 Remark. Let $(k_t^*)_s = k_t^* k_s$ be the family associated with k_t^* by Proposition A.10.3. One may check that

$$(k_t^*)_s \;=\; \begin{cases} \gamma_{ts} & \text{for } s \leq t \\ \gamma_{st}^* & \text{for } s \geq t. \end{cases}$$

Hence, the action of k_t^* on an element $k_s x_s \in E$ coming from E_s can be interpreted as *lifting* this element to E_t via $\xi_{t-s} \odot \mathsf{id}$, if s is too small, and *truncating* it to E_t via $(\xi_{s-t})^* \odot \mathsf{id}$, if s is too big.

The construction in this section was done in [BS00], in particular, to find the GNS-dilation of a unital CP-semigroup T.

11.4.11 Definition. By the *GNS-dilation* of a unital CP-semigroup T, we mean the weak dilation obtained by constructing the one-sided inductive limit over the GNS-system of T (considered as CPD-semigroup) for the single generating (unital) unit in this system giving back T. Sometimes, we write E^{min} for the dilation module.

We discuss this and other results related to the special case of CP-semigroups in Chapter 12. There we also comment on the relations to earlier work. We explain in how far it is possible to construct the one-sided inductive limit, when the unit is non-unital. In Section 14.1 we will see that we can obtain any (strict) dilation on a Hilbert module E as in (11.4.4) for a suitable product system $E^\odot$. We only loose the interpretation of E as an inductive limit (i.e. the subspaces $k_t E_t$ do not necessarily increase to E).

We close with some continuity results on ϑ. First of all, it is clear that we may complete both the product system $E^\odot$ and the one-sided inductive limit E. If $\mathcal{B}$ is a von Neumann algebra and the left action of $\mathcal{B}$ on $\overline{E}_t^s$ is normal (for instance, if $E^\odot$ is the GNS-system of a normal CPD-semigroup), then we may pass to the strong closure of $E^\odot$ and E. All ϑ_t extend to normal endomorphisms of the von Neumann algebra $\mathcal{B}^a(\overline{E}^s)$. (This follows from Proposition 4.2.24 and the trivial observation that $\overline{E}^s$ is a von Neumann $\mathcal{B}^a(\overline{E}^s)$–$\mathcal{B}$–module.)

11.4.12 Theorem. *1. Suppose $\overline{E}^{\overline{\odot}}$ is a type I_{c_0} system (for instance, the GNS-system of a CPD–C_0–semigroup). Then ϑ is strictly continuous (by uniform boundedness of ϑ this means $t \mapsto \vartheta_t(a)x$ is continuous for all $a \in \mathcal{B}^a(\overline{E})$ and $x \in \overline{E}$).*

2. Suppose $\overline{E}^{s\overline{\odot}^s}$ is a type I_s^s system of von Neumann $\mathcal{B}$–$\mathcal{B}$–modules (for instance, the GNS-system of a normal CPD-semigroup) and, therefore, a I_{sn}^s system. Then ϑ is strongly continuous (by uniform boundedness of ϑ this means $t \mapsto \vartheta_t(a)x \odot g$ is continuous for all $a \in \mathcal{B}^a(\overline{E})$, $x \in \overline{E}$, and g in the representation space G of $\mathcal{B}$).

PROOF. The *right shift* $\mathsf{s}^r_t\colon x \mapsto x \odot \xi_t$ is, clearly, bounded by $\|\xi_t\|$ and extends to $\overline{E}$ and $\overline{E}^s$. The family $x \odot \xi_t$ depends continuously on t. (On the dense subset E this follows from the fact that inner products of elements of the form (11.2.3) depend jointly continuously on all time arguments. By boundedness of s^r_t this extends to the whole of $\overline{E}$.) Now we easily see that for each $a \in \mathcal{B}^a(\overline{E})$ and for each $x \in \overline{E}$

$$\begin{aligned} ax - \vartheta_t(a)x &= ax - ax \odot \xi_t + ax \odot \xi_t - \vartheta_t(a)x = (ax) - (ax) \odot \xi_t + \vartheta_t(a)(x \odot \xi_t - x) \\ &= (\mathsf{id} - \mathsf{s}^r_t)(ax) + \vartheta_t(a)(\mathsf{s}^r_t - \mathsf{id})(x) \end{aligned}$$

is small for t sufficiently small. Replacing a by $\vartheta_s(a)$ we obtain continuity at all times s.

The second case follows precisely in the same manner, but starting from the mapping $t \mapsto x \odot \xi_t \odot g$ instead of $t \mapsto x \odot \xi_t$. ■

11.4.13 Remark. Since $\vartheta_t \circ j_0(\mathbf{1}) = j_t(\mathbf{1})$ is an increasing family of projections, ϑ is in general not a C_0–semigroup.

11.4.14 Remark. Of course, s^r_t is not an element of $\mathcal{B}^r(E)$, therefore, certainly neither adjointable, nor isometric (unless T is trivial). In particular, passing to the Stinespring construction (Example 4.1.9), s^r_t will never be implemented by an operator in $\mathcal{B}(\overline{H})$, i.e. the operator $x \odot g \mapsto x \odot \xi_t \odot g$ on H is, in general, ill-defined. It follows that the j_t (interpreted as mapping $\mathcal{B} \to \mathcal{B}(\overline{H})$) do not form a stationary process in the sense of [Bel85]. In the Hilbert space picture obtained by Stinespring construction, in general, there is no time shift like s^r_t, acting directly on the Hilbert space.

11.5 Central units and white noise

We ask, under which circumstances the dilation constructed in Section 11.4 is a white noise. Obviously, it is necessary and sufficient that the unit $\xi^\odot$ is also central. In this case all the embeddings γ_{ts} are two-sided and there is a single (non-degenerate!) left multiplication on E such that also the k_t are two-sided. In this section we show that this white noise of endomorphisms may be extended to a unitarily implemented white noise of automorphisms on the algebra of operators on a bigger pre-Hilbert module.

We start by constructing a *reverse* inductive limit. Thanks to centrality of $\xi^\odot$ we may define the two-sided isometries $\overrightarrow{\gamma}_{ts}\colon E_s \to E_t$ $(t \geq s)$ by setting

$$\overrightarrow{\gamma}_{ts} x_s \;=\; x_s \odot \xi_{t-s}.$$

Similarly as in Section 11.4 the E_t with $\overrightarrow{\gamma}_{ts}$ form an inductive system. The inductive limit $\overrightarrow{E}$ is a two-sided pre-Hilbert module with a central unit vector $\overrightarrow{\xi}$, fulfilling

$$E_t \odot \overrightarrow{E} \;=\; \overrightarrow{E} \quad \text{and} \quad \xi_t \odot \overrightarrow{\xi} \;=\; \overrightarrow{\xi}.$$

The canonical embedding $E_t \to \overrightarrow{E}$ is given by $\overrightarrow{k}_t = \mathsf{id} \odot \overrightarrow{\xi}$ and has the adjoint $\overrightarrow{k}_t^* = \mathsf{id} \odot \overrightarrow{\xi}^*$.

Let $x \in E, x_t \in E_t, \overrightarrow{x} \in \overrightarrow{E}$ and set $y = x \odot x_t \in E \odot E_t = E$ and $\overrightarrow{y} = x_t \odot \overrightarrow{x} \in E_t \odot \overrightarrow{E} = \overrightarrow{E}$. There are two possibilities to interprete $x \odot x_t \odot \overrightarrow{x}$ as an element of $\overleftrightarrow{E} = E \odot \overrightarrow{E}$, namely, $y \odot \overrightarrow{x}$ or $x \odot \overrightarrow{y}$. The mapping

$$\overleftrightarrow{u_t}(x \odot \overrightarrow{y}) \;=\; y \odot \overrightarrow{x}$$

defines a two-sided unitary in $\mathcal{B}^a(\overleftrightarrow{E})$. Clearly, the $\overleftrightarrow{u_t}$ form a semigroup in $\mathcal{B}^a(\overleftrightarrow{E})$ which extends to a unitary group.

11.5.1 Proposition. *The automorphism group $\alpha = \big(\alpha_t\big)_{t \in \widehat{\mathbb{T}}}$ on $\mathcal{B}^a(\overleftrightarrow{E})$, defined by setting $\alpha_t(a) = \overleftrightarrow{u_t} a \overleftrightarrow{u_t}^*$, with the conditional expectation $\varphi(a) = \langle \overleftrightarrow{\xi}, a \overleftrightarrow{\xi} \rangle$ onto the subalgebra $\mathcal{B} \ni \mathsf{id}_{\overleftrightarrow{E}}$ of $\mathcal{B}^a(\overleftrightarrow{E})$ is a white noise.*

The restriction of α to a semigroup on $\mathcal{B}^a(E) \cong \mathcal{B}^a(E) \odot \mathsf{id} \subset \mathcal{B}^a(\overleftrightarrow{E})$ is ϑ.

11.5.2 Example. The one-sided inductive limit over the product system $\mathbb{I}\Gamma^\odot(F)$ of time ordered Fock modules for the vacuum unit $\omega^\odot$ is just $\mathbb{I}\Gamma(F)$. Of course, $\overleftrightarrow{\mathbb{I}\Gamma(F)} = \breve{\mathbb{I}\Gamma}(F)$ and $\alpha = \mathcal{S}$ is the time shift group.

Notice that it is far from being clear, whether the one-sided inductive limit for one of the unital units $\xi^\odot\big(-\frac{\langle\zeta,\zeta\rangle}{2}, \zeta\big)$ is isomorphic to $\mathbb{I}\Gamma(F)$. (Even if this is so, then the non-coincidence of the left multiplications shows us that the identification of $\mathbb{I}\Gamma_t(F)$ in $\mathbb{I}\Gamma(F)$ cannot be the canonical one. See also Section 14.1.) For $\mathcal{B} = \mathcal{B}(G)$ (i.e. symmetric Fock space with initial space G) Bhat [Bha01] shows with the help of a quantum stochastic calculus (Hudson and Parthasarathy [HP84, Par92]) that this is the case. Our calculus on the full Fock module [Ske00d] in Part IV does not help to answer this question in general, because the time ordered Fock module (contained in the full Fock module) is not left invariant.

11.5.3 Observation. Denote by $\mathfrak{i}$ the canonical left multiplication of E and suppose $\mathfrak{u}$ is a unitary right cocycle for ϑ such that $(E, \vartheta^{\mathfrak{u}}, \mathfrak{i}, \xi)$ is a (unital) dilation for a (unital) CP-semigroup T. Then $(\overleftrightarrow{E}, \alpha^{\mathfrak{u}\odot\mathsf{id}}, \mathfrak{i} \odot \mathsf{id}, \overleftrightarrow{\xi})$ is a unital dilation to an automorphism group.

11.6 Morphisms, units and cocycles

Let $E^\odot$ be a product system and E the one-sided inductive limit comming from some unital unit $\xi^\odot$ for $E^\odot$ with the E_0–semigroup ϑ on $\mathcal{B}^a(E)$ as constructed in Section 11.4. In this section we we investigate two types of cocycles for ϑ, one giving a one-to-one correspondence with endomorphisms of $E^\odot$ and the other giving a one-to-one correspondence with units for

$E^\odot$. In both cases the uniqueness of the correspondence is due to a special notion of adaptedness. Then we find criteria for strong continuity of cocycles for type I product systems and relations to the associated CPD-semigroups.

Let $w^\odot = \bigl(w_t\bigr)_{t\in\mathbb{T}}$ be an endomorphism of $E^\odot$. Then, clearly, setting $\mathfrak{w}_t = \mathsf{id}\odot w_t$ we define a local cocycle $\mathfrak{w} = \bigl(\mathfrak{w}_t\bigr)_{t\in\mathbb{T}}$ for ϑ. Conversely, if $\mathfrak{w} = \bigl(\mathfrak{w}_t\bigr)_{t\in\mathbb{T}}$ is a local cocycle, then by Theorem 4.2.18 there are unique elements $w_t = (\xi^*\odot\mathsf{id})\mathfrak{w}_t(\xi\odot\mathsf{id}) = k_t^*\mathfrak{w}_t k_t \in \mathcal{B}^{a,bil}(E_t)$ such that $\mathfrak{w}_t = \mathsf{id}\odot w_t$. We find

$$\begin{aligned} w_{s+t} &= k_{s+t}^*\mathfrak{w}_{s+t}k_{s+t} = k_{s+t}^*\vartheta_t(\mathfrak{w}_s)\mathfrak{w}_t k_{s+t} \\ &= (\xi^*\odot\mathsf{id}_{E_s}\odot\mathsf{id}_{E_t})(\mathsf{id}\odot w_s\odot\mathsf{id}_{E_t})(\mathsf{id}\odot\mathsf{id}_{E_s}\odot w_t)(\xi\odot\mathsf{id}_{E_s}\odot\mathsf{id}_{E_t}) = w_s\odot w_t. \end{aligned}$$

We summarize.

11.6.1 Theorem. *The formula $\mathfrak{w}_t = \mathsf{id}\odot w_t$ establishes a one-to-one correspondence between local cocycles $\mathfrak{w}$ for ϑ and endomorphisms $w^\odot$ of $E^\odot$.*

11.6.2 Observation. The E_0–semigroup ϑ, or better the space $\mathcal{B}^a(E)$ where it acts, depends highly on the choice of a (unital) unit. (However, if two inductive limits coincide for two unital units $\xi^\odot, \xi'^\odot$, then the corresponding E_0–semigroups are outer conjugate; see Theorem 14.1.5.) On the contrary, the set of endomorphisms is an intrinsic property of $E^\odot$ not depending on the choice a unit. Therefore, we prefer very much to study product systems by properties of their endomorphisms, instead of cocycles with respect to a fixed E_0–semigroup.

Clearly, locality is a kind of adaptedness, as it asserts that a certain operator $\mathfrak{w}_t \in \mathcal{B}^a(E)$ is of the form $\mathsf{id}\odot w_t$ for some $w_t \in \mathcal{B}^{a,bil}(E_t)$. This unital identification of $\mathcal{B}^{a,bil}(E_t)$ with the subset $\mathsf{id}\odot\mathcal{B}^{a,bil}(E_t)$ in $\mathcal{B}^a(E)$ is restricted to bilinear operators. If we want to embed all of $\mathcal{B}^a(E_t)$ into $\mathcal{B}^a(E)$, then we must content ourselves (unless there is additional structure like centeredness) with a non-unital embedding like $a \mapsto k_t a k_t^*$.

11.6.3 Definition. A *weakly adapted* cocycle is a weak cocycle $\mathfrak{w} = \bigl(\mathfrak{w}_t\bigr)_{t\in\mathbb{T}}$ for ϑ such that $p_t\mathfrak{w}_t p_t = \mathfrak{w}_t$ for all $t \in \mathbb{T}$ and $\mathfrak{w}_0 = p_0$. In other words, there exist unique elements $w_t = k_t^*\mathfrak{w}_t k_t \in \mathcal{B}^a(E_t)$ such that $\mathfrak{w}_t = k_t w_t k_t^*$.

Notice that the members $\mathfrak{w}_t$ of a weakly adapted right cocycle $\mathfrak{w}$ are necessarily of the form $\mathfrak{w}_t = (k_t\zeta_t)(k_t\xi_t)^* = k_t\zeta_t\xi^*$ where ζ_t are the unique elements $k_t^*\mathfrak{w}_t\xi \in E_t$. Indeed, by the cocycle property we have $\mathfrak{w}_t = \vartheta_0(\mathfrak{w}_t)\mathfrak{w}_0 = \mathfrak{w}_t p_0$. By adaptedness we have $\mathfrak{w}_t = p_t\mathfrak{w}_t$. Hence, $\mathfrak{w}_t = k_t k_t^*\mathfrak{w}_t\xi\xi^* = k_t\zeta_t\xi^*$.

11.6.4 Theorem. *The formula $\zeta_t = k_t^* \mathfrak{w}_t \xi$ establishes a one-to-one correspondence between weakly adapted right cocycles $\mathfrak{w}$ for ϑ and units $\zeta^\odot = \big(\zeta_t\big)$ for $E^\odot$.*

Proof. Let $\mathfrak{w}$ be an adapted right cocycle. Then

$$\zeta_{s+t} = k_{s+t}^* \mathfrak{w}_{s+t}\xi = k_{s+t}^* \vartheta_t(\mathfrak{w}_s) p_t \mathfrak{w}_t \xi = k_{s+t}^*(\mathfrak{w}_s \xi \odot k_t^* \mathfrak{w}_t \xi) = k_s^* \mathfrak{w}_s \xi \odot k_t^* \mathfrak{w}_t \xi = \zeta_s \odot \zeta_t.$$

Since also $\zeta_0 = k_0 p_0 \xi = \xi_0 = \mathbf{1}$, it follows that $\zeta^\odot$ is a unit.

Conversely, let $\zeta^\odot$ be a unit and set $\mathfrak{w}_t = k_t \zeta_t \xi^*$. Then by Corollary 11.4.7 and Proposition 11.4.9

$$\vartheta_t(\mathfrak{w}_s)\mathfrak{w}_t \xi = (\mathfrak{w}_s \odot \mathsf{id})(\xi \odot \zeta_t) = k_{s+t}(\zeta_s \odot \zeta_t) = \mathfrak{w}_{s+t}\xi.$$

Moreover, $\mathfrak{w}_{s+t}$ is 0 on the orthogonal complement $(\mathbf{1} - \xi\xi^*)E$ of ξ. In other words, $\mathfrak{w}_{s+t} = \vartheta_t(\mathfrak{w}_s)\mathfrak{w}_t$ so that the $\mathfrak{w}$ is a weakly adapted right cocycle for ϑ. Finally, $\mathfrak{w}_0 = \zeta_0 \xi_0^* = \xi_0 \xi_0^* = p_0$. ■

11.6.5 Remark. We mention a small error in [BS00] where we did not specify the value of a cocycle at $t = 0$, which is, of course, indispensable, if we want that cocycles map units to units (cf. Observation 11.2.2). As the value assigned to a cocycle at 0 depends on the type of cocycle (local or weakly adapted) some attention should be paid.

Cocycles may be continuous or not. In Theorem 7.3.4 we have computed all units for $\mathbb{I}\Gamma^\odot(F)$ which are continuous in $\mathbb{I}\Gamma(F)$. In Example 11.5.2 we explained that $\mathbb{I}\Gamma(F)$ is the one-sided inductive limit over $\mathbb{I}\Gamma^\odot(F)$ for the vacuum unit. Now we investigate how such continuity properties can be expressed intrinsically, without reference to the inductive limit.

We say a unit $\xi^\odot$ is ***continuous***, if the associated CP-semigroup $T_t^\xi(b) = \langle \xi_t, b\xi_t \rangle$ is uniformly continuous. In general, we say a set S of units is ***continuous***, if the CPD-semigroup $\mathfrak{U} \upharpoonright S$ is uniformly continuous.

11.6.6 Lemma. *Let $\xi^\odot$ be a unital continuous unit for $E^\odot$, and denote by E the one-sided inductive limit for $\xi^\odot$. Let $\zeta^\odot$ be another unit. Then the following conditions are equivalent.*

1. *The function $t \mapsto \xi \odot \zeta_t \in E$ is continuous.*

2. *The semigroups $\mathfrak{U}^{\zeta,\xi}$ and T^ζ are uniformly continuous.*

3. *The functions $t \mapsto \langle \zeta_t, \xi_t \rangle$ and $t \mapsto \langle \zeta_t, \zeta_t \rangle$ are continuous.*

Moreover, if $\zeta^\odot, \zeta'^\odot$ are two units both fulfilling one of the three conditions above, then also the function $t \mapsto \langle \zeta_t, \zeta'_t \rangle$ is continuous, hence, also the semigroup $\mathfrak{U}^{\zeta,\zeta'}$ is uniformly continuous.

PROOF. The crucial step in the proof is the observation that the norm of mappings on $\mathcal{B}$ of the form $b \mapsto \langle x, by\rangle$ (for x, y in some pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module) can be estimated by $\|x\| \, \|y\|$.

1⇒2. We have

$$\xi \odot \zeta_{t+\varepsilon} - \xi \odot \zeta_t \;=\; \xi \odot \zeta_\varepsilon \odot \zeta_t - \xi \odot \xi_\varepsilon \odot \zeta_t \;=\; \xi \odot (\zeta_\varepsilon - \xi_\varepsilon) \odot \zeta_t \tag{11.6.1}$$

so that $t \mapsto \xi \odot \zeta_t$ is continuous, if and only if $\|\zeta_t - \xi_t\| \to 0$ for $t \to 0$. Thus, 1 implies

$$\|\mathfrak{U}_t^{\zeta,\xi} - \mathsf{id}\| \;\leq\; \|\mathfrak{U}_t^{\zeta,\xi} - T_t^\xi\| + \|T_t^\xi - \mathsf{id}\| \;\to\; 0,$$

because the norm of $\mathfrak{U}_t^{\zeta,\xi} - T_t^\xi \colon b \mapsto \langle \zeta_t - \xi_t, b\xi_t\rangle$ is smaller than $\|\zeta_t - \xi_t\| \, \|\xi_t\| \to 0$, and

$$\|T_t^\zeta - \mathsf{id}\| \;\leq\; \|T_t^\zeta - \mathfrak{U}_t^{\zeta,\xi}\| + \|\mathfrak{U}_t^{\zeta,\xi} - \mathsf{id}\| \;\to\; 0,$$

because the norm of $T_t^\zeta - \mathfrak{U}_t^{\zeta,\xi} \colon b \mapsto \langle \zeta_t, b(\zeta_t - \xi_t)\rangle$ is smaller than $\|\zeta_t\| \, \|\zeta_t - \xi_t\| \to 0$ and by the preceding estimate.

2⇒3 is trivial, so let us come to 3⇒1. We have

$$\|\zeta_t - \xi_t\|^2 \;\leq\; \|\langle \zeta_t, \zeta_t\rangle - \mathbf{1}\| + \|\langle \zeta_t, \xi_t\rangle - \mathbf{1}\| + \|\langle \xi_t, \zeta_t\rangle - \mathbf{1}\| + \|\langle \xi_t, \xi_t\rangle - \mathbf{1}\|$$

which tends to 0 for $t \to 0$ so that (11.6.1) implies continuity of $\xi \odot \zeta_t$.

Now let $\zeta^\odot, \zeta'^\odot$ be two units fulfilling 3. Then

$$\|\langle \zeta_t, \zeta'_t\rangle - \mathbf{1}\| \;\leq\; \|\langle \zeta_t, \zeta'_t - \xi_t\rangle\| + \|\langle \zeta_t - \xi_t, \xi_t\rangle\| + \|\langle \xi_t, \xi_t\rangle - \mathbf{1}\| \;\to\; 0$$

for $t \to 0$ so that $t \mapsto \langle \zeta_t, \zeta'_t\rangle$ is continuous. As before, this implies that $\mathfrak{U}^{\zeta,\zeta'}$ is uniformly continuous. ■

The following theorem is simple corollary of Theorem 11.3.5 and Lemma 11.6.6. Taking into account also the extensions following Corollary 13.3.3 which assert that a continuous unit is contained in a time ordered poduct systems of von Neumann $\mathcal{B}^{**}$–$\mathcal{B}^{**}$–modules, and the fact discussed in the proof of Lemma 13.2.6 that units in such product systems may be normalized within that system, one may show that we can drop the assumption in brackets.

11.6.7 Theorem. *For a CPD-semigroup $\mathfrak{T}$ on a set S containing an element σ such that $\mathfrak{T}^{\sigma,\sigma}$ is uniformly continuous (and that $\mathfrak{T}_t^{\sigma,\sigma}(\mathbf{1}) = \mathbf{1}$ for all $t \in \mathbb{R}_+$) the following statements are equivalent.*

1. *$\mathfrak{T}$ is uniformly continuous.*

2. *The functions $t \mapsto \mathfrak{T}_t^{\sigma,\sigma'}(\mathbf{1})$ are continuous for all $\sigma, \sigma' \in S$.*

3. *The functions $t \mapsto \mathfrak{T}_t^{\sigma,\sigma'}(\mathbf{1})$ and $t \mapsto \mathfrak{T}_t^{\sigma',\sigma'}(\mathbf{1})$ are continuous for all $\sigma' \in S$.*

The main idea in the proof of Lemma 11.6.6 is that a certain (completely bounded) mapping can be written as $b \mapsto \langle x, by\rangle$ for some vectors in some *GNS-space*. Theorem 11.6.7 is an intrinsic result about CPD-semigroups obtained, roughly speaking, by rephrasing all statements from Lemma 11.6.6 involving units in terms of the associated CPD-semigroup. It seems difficult to show Theorem 11.6.7 directly without reference to the GNS-system of the CPD-semigroup.

Another consequence of Lemma 11.6.6 concerns continuity properties of local cocycles.

11.6.8 Corollary. *Let $E^\odot$ be generated by a subset $S \subset \mathcal{U}(E^\odot)$ of units such that $\mathfrak{U} \restriction S$ is a uniformly continuous CPD-semigroup. Let $\xi^\odot \in S$ be a unital unit, and denote by E the one-sided inductive limit for $\xi^\odot$. Then for a morphism $w^\odot$ and the associated local cocycle $\mathfrak{w} = \big(\mathsf{id} \odot w_t\big)_{t\in\mathbb{T}}$ the following equivalent conditions*

1. *The CPD-semigroup $\mathfrak{U} \restriction (S \cup w^\odot S)$ (see Observation 11.2.2) is uniformly continuous. (In particular, if S is maximal continuous, then $w^\odot$ leaves S invariant.)*

2. *For some $\xi'^\odot \in S$ all functions $t \mapsto \langle \xi'_t, \zeta_t\rangle$, $t \mapsto \langle \zeta_t, \zeta_t\rangle$ $(\zeta^\odot \in w^\odot S)$ are continuous.*

both imply that $\mathfrak{w}$ is strongly continuous.

Proof. By simple applications of Lemma 11.6.6, 1 and 2 are equivalent, and for the remaining implication it is sufficient to choose $\xi'^\odot = \xi^\odot$. So assume that all functions $t \mapsto \langle \zeta_t, \zeta_t\rangle$, $t \mapsto \langle \xi_t, \zeta_t\rangle$ $(\zeta^\odot \in S \cup w^\odot S)$ are continuous. Then

$$\|w_t\zeta_t - \zeta_t\| \;=\; \|\xi \odot w_t\zeta_t - \xi \odot \zeta_t\| \;\leq\; \|\xi \odot w_t\zeta_t - \xi\| + \|\xi \odot \zeta_t - \xi\| \;\to\; 0 \tag{11.6.2}$$

for $t \to 0$. Applying $\mathfrak{w}_{s+\varepsilon} - \mathfrak{w}_s = \mathsf{id} \odot (w_\varepsilon - \mathsf{id}_{E_\varepsilon}) \odot w_s$ to a vector of the form $\xi \odot x_t$ where $x_t \in E_t$ is as in (11.2.3), we conclude from (11.6.2) (choosing $\varepsilon > 0$ so small that $w_\varepsilon - \mathsf{id}_{E_\varepsilon}$ comes to act on a single unit only) that the function $s \mapsto \mathfrak{w}_s(\xi \odot x_t)$ is continuous. Since the vectors $\xi \odot x_t$ span E, $\mathfrak{w}$ is strongly continuous. ■

11.6.9 Observation. If $\mathfrak{w}$ is bounded locally uniformly (for instance, if $w^\odot$ is contractive) or, equivalently, if the extension of $\mathfrak{w}$ to $\overline{E}$ is also strongly continuous, then also the reverse implication holds. (We see by the same routine arguments that the inner product $\langle \xi_t, w_t\zeta_t\rangle = \langle \xi \odot \xi_t, \xi \odot w_t\zeta_t\rangle = \langle \xi, \mathfrak{w}_t(\xi \odot \zeta_t)\rangle$ depends continuously on t and, similarly, also $\langle w_t\zeta_t, w_t\zeta_t\rangle$.)

11.6.10 Definition. A morphism is *continuous*, if it sends some totalizing continuous subset of units to a continuous subset of units.

Chapter 12

The case of CP-semigroups

In this chapter we interrupt the analysis of general product systems and restrict to the GNS-system of a CP-semigroup. We present some related results mainly from Bhat and Skeide [BS00]. Although this set-up was the starting point of product systems, the chapter is independent and may be skipped.

12.1 The GNS-dilation

Let $(E = E^{min}, \vartheta, \xi)$ be the GNS-dilation from Definition 11.4.11 with the weak Markov flow (E, j, ξ). We show that E earns the superscript *min*.

12.1.1 Proposition. *ξ is cyclic for the algebra $\mathcal{A}_\infty = \mathsf{alg}\, j_{\mathbb{T}}(\mathcal{B})$, i.e. $E = \mathsf{span}(\mathcal{A}_\infty \xi)$.*

Proof. It is enough to show that for each $t \in \mathbb{T}$, $\mathfrak{t} = (t_n, \ldots, t_1) \in \mathbb{J}_t$, $b_n, \ldots, b_0 \in \mathcal{B}$, and $(s_n, \ldots, s_1) = \mathfrak{o}(\mathfrak{t}) \in \mathbb{I}_t$ (cf. Proposition B.3.2)

$$j_{s_n}(b_n) \ldots j_{s_1}(b_1) j_0(b_0)\xi = \xi \odot b_n \xi_{t_n} \odot \ldots \odot b_1 \xi_{t_1} b_0, \tag{12.1.1}$$

because by Remark 11.4.6, $\xi \odot b_n \xi_{t_n} \odot \ldots \odot b_1 \xi_{t_1} b_0 = k_t(b_n \xi_{t_n} \odot \ldots \odot b_1 \xi_{t_1} b_0)$, and E is spanned by these vectors. First, observe that

$$j_t(b)\xi \;=\; \vartheta_t \circ j_0(b)\xi \;=\; (\xi b \xi^* \odot \mathsf{id})(\xi \odot \xi_t) \;=\; \xi \odot b\xi_t. \tag{12.1.2}$$

Now we proceed by induction on n. Extending (12.1.1) to the empty tuple (i.e. $t = 0$), the statement is true for $n = 0$. Let us assume that (12.1.1) holds for n and choose $t_{n+1} > 0$

and $b_{n+1} \in \mathcal{B}$. Then by (12.1.2) and Remark 11.4.6

$$\begin{aligned} j_{t_{n+1}+t}(b_{n+1})(\xi \odot b_n \xi_{t_n} \odot \ldots \odot b_1 \xi_{t_1} b_0) \\ &= (j_{t_{n+1}}(b_{n+1}) \odot \mathsf{id}_{E_t})(\xi \odot b_n \xi_{t_n} \odot \ldots \odot b_1 \xi_{t_1} b_0) \\ &= (j_{t_{n+1}}(b_{n+1})\xi) \odot b_n \xi_{t_n} \odot \ldots \odot b_1 \xi_{t_1} b_0 \\ &= \xi \odot b_{n+1} \xi_{t_{n+1}} \odot b_n \xi_{t_n} \odot \ldots \odot b_1 \xi_{t_1} b_0. \blacksquare \end{aligned}$$

In Section 12.4 we will see that a weak dilation is determined up to unitary isomorphism by this cyclicity condition.

The e_0–semigroup $\vartheta \upharpoonright \mathcal{A}_\infty$ is (up to completion) the e_0–dilation constructed in Bhat [Bha99]. More precisely, if $\mathcal{B}$ is represented faithfully on a Hilbert space G, then the Stinespring construction (Example 4.1.9) gives rise to a (pre-)Hilbert space $H = E \odot G$ and a faithful representation ρ of $\mathcal{A}_\infty$ by operators on H. Lifting ϑ to $\rho(\mathcal{A}_\infty)$, we obtain the e_0–semigroup from [Bha99]. Observe that the e_0–semigroup on $\mathcal{A}_\infty$ was defined in [Bha99] by setting $\vartheta_s \circ j_t(b) = j_{s+t}(b)$ and that it was quite hard to show that this mapping extends as a contractive homomorphisms to all of $\mathcal{A}_\infty$. Identifying $\vartheta_t(a)$ as $a \odot \mathsf{id}_{E_t}$, well-definedness and contractivity even on the bigger algebra $\mathcal{B}^a(E)$ become trivialities in the approach by Hilbert modules.

For the weak Markov flow j alone (i.e. without showing that $j_s(b) \mapsto j_{s+t}(b)$ extends to an endomorhism of $\mathcal{A}_\infty$) the construction of the space H was done by Bhat and Parthasarathy in [BP94, BP95] (as in usual proofs of the Stinespring construction) by a Kolmogorov decomposition for positive definite kernel constructed from j. A similar construction was done for flows indexed by more general index sets by Belavkin [Bel85]. However, in contrast with [BP94] where positivity of the kernel can be proved, in [Bel85] it must be assumed that the kernel in question be positive definite.

The extension of Bhat's e_0–semigroup to an E_0–semigroup of strict endomorphisms of $\mathcal{B}^a(E)$ is from Bhat and Skeide [BS00]. Of course, also $\mathcal{B}^a(E)$ has a faithful image in $\mathcal{B}(H)$. However, it seems not possible to find this subalgebra easily without reference to the module description.

The module description also allows us to show that, if T is a normal and strongly continuous CP-semigroup on a von Neumann algebra $\mathcal{B} \subset \mathcal{B}(G)$ (i.e. T is continuous in the strong topology of $\mathcal{B}$), then ϑ is normal and strongly continuous, too (see Definition 11.3.7 and Theorem 11.4.12). This answers a question raised in [Bha99] in the affirmative sense. In the case when $\mathcal{B} = \mathcal{B}(G)$, whence $\mathcal{B}^a(\overline{E}^s) = \mathcal{B}(\overline{H})$ by Example 3.1.2, we recover the result from Bhat [Bha01] that a normal strongly continuous CP-semigroup on $\mathcal{B}(G)$ allows for a weak dilation on some $\mathcal{B}(\overline{H})$ which is determined uniquely by the requirement that H is generated by the flow j from its subspace $G \cong j_0(\mathbf{1})H$.

12.2 Central flows

Let (E, ϑ, ξ) be a weak dilation (not necessarily the GNS-dilation) with weak Markov flow (E, j, ξ) constructed from a product system $E^{\odot}$ and a unital unit $\xi^{\odot}$ as in Theorem 11.4.8 with associated unital CP-semigroup T. The notion of weak Markov flow is essentially non-commutative. The reason for this is that by definition $j_t(\mathbf{1})$ is a projection (at least in non-trivial examples) which "levels out" whatever happened before "in the future of t". As a consequence, $j_t(b)$ and $j_s(b)$ have no chance to commute in general. Indeed, for $s < t$ we find

$$j_t(b)j_s(b)x \odot x_{t-s} \odot x_s = \xi \odot b\xi_{t-s} \odot \langle \xi \odot \xi_{t-s}b^*, x \odot x_{t-s}\rangle x_s, \tag{12.2.1a}$$

whereas

$$j_s(b)j_t(b)x \odot x_{t-s} \odot x_s = \xi \odot \xi_{t-s}b \odot \langle \xi \odot b^*\xi_{t-s}, x \odot x_{t-s}\rangle x_s. \tag{12.2.1b}$$

Since b and ξ_{t-s} do not commute, unless T is the trivial semigroup, the elements of E decribed by Equations (12.2.1a,12.2.1b) are different.

If we restrict ourselves to the center of $\mathcal{B}$, then the weak Markov flow j can be modified as shown in Bhat [Bha93] to a commutative flow k called the *central flow*. If the initial algebra $\mathcal{B}$ is commutative to begin with, then the flow k can be interpreted as the classical Markov process obtained by the *Daniell-Kolmogorov construction*. Central flows play a crucial role in Attal and Parthasarathy [AP96]. In this section we recover k as a process of operators on E. This example, almost a triviality now, illustrates once again the power of the module approach. (The central flow k appears only in this section and should not be confused with the canonical mappings $k_t \colon E_t \to E$.)

Recall from Example 1.6.8 that for $b \in C_{\mathcal{B}}(\mathcal{B})$ the operator $b^r \colon x \mapsto xb$ of right multiplication by b is a well-defined element of $\mathcal{B}^a(E)$ (even of $\mathcal{B}^{a,bil}(E)$, if E is two-sided). Writing $E = E \odot \mathcal{B}$, a simple application of Theorem 4.2.18 tells us that $C_{\mathcal{B}^a(E)}(\mathcal{B}^a(E)) = (\mathcal{B}^a(E) \odot \mathsf{id})' = \mathsf{id} \odot \mathcal{B}^{a,bil}(\mathcal{B}) = \mathsf{id} \odot C_{\mathcal{B}}(\mathcal{B}) \cong C_{\mathcal{B}}(\mathcal{B})$. In other words, $k_0 \colon b \mapsto b^r$ defines an isomorphism from the center of $\mathcal{B}$ onto the center of $\mathcal{B}^a(E)$. We define $k_t = \vartheta_t \circ k_0$.

12.2.1 Theorem. *The process $k = \big(k_t\big)_{t \in \mathbb{T}}$ is commutative (i.e. $\big[k_{\mathbb{T}}(C_{\mathcal{B}}(\mathcal{B})), k_{\mathbb{T}}(C_{\mathcal{B}}(\mathcal{B}))\big] = \{0\}$) and $\langle \xi, k_t(b)\xi \rangle = \langle \xi, j_t(b)\xi \rangle = T_t(b)$ for all $t \in \mathbb{T}, b \in C_{\mathcal{B}}(\mathcal{B})$. In particular, if $T_{\mathbb{T}}(C_{\mathcal{B}}(\mathcal{B})) \subset C_{\mathcal{B}}(\mathcal{B})$, then k is a classical Markov process.*

Proof. Clearly, $k_0(C_{\mathcal{B}}(\mathcal{B}))$ commutes with $k_t(C_{\mathcal{B}}(\mathcal{B})) \subset \mathcal{B}^a(E)$. The remaining statements follow by time shift. ■

The explicit action of k_t is

$$k_t(b)x \odot x_t = xb \odot x_t. \tag{12.2.2}$$

Let us have a closer look at the difference between j and k. Both $j_t(b)$ and $k_t(b)$ let act the algebra element b at time t. This can be seen explicitly by the observation that the actions of $j_t(b)$ and $k_t(b)$ restricted to the submodule $\xi \odot E_t$ coincide. In other words, both $j_t(b)$ and $k_t(b)$ can be thought of as the left multiplication of E_t, first, lifted to $\xi \odot E_t \subset E$ and, then, extended to the whole of E. It is this extension which makes the difference. $j_t(b)$ is extended just by 0 to the orthogonal complement of $\xi \odot E_t$ in E. Correspondingly, $j_t(\mathbf{1})$ projects down the future of t to the presence. Whereas $k_t(b)$ inserts b at time t without changing the future part x of $x \odot x_t$. Therefore, all k_t are unital homomorphisms.

A look at Equation (12.2.2) reminds us of the *ampliation* $\mathsf{id} \odot l_b$ of the operator of left multiplication $l_b \colon x_t \mapsto bx_t$ on E_t by b to the tensor product $E \odot E_t$. Once again, we emphasize that in contrast to $a \odot \mathsf{id}$, a mapping $\mathsf{id} \odot a$ on a tensor product of pre-Hilbert modules, in general, only exists, if a is $\mathcal{B}$–$\mathcal{B}$–linear.

12.3 Contractive units

In this section we study the construction of the GNS-dilation via the procedures in Sections 11.3 and 11.4 in the case, when $\mathcal{B}$ still is unital, however, T may be non-unital. We still assume that all T_t are contractive and, of course, that $T_0 = \mathsf{id}$.

Let us remark that considering contractive CP-semigroups T is a restriction. There is no problem, if T is exponentially bounded, i.e. there is a $c \in \mathbb{R}$ such that the CP-semigroup $e^{ct}T_t$ consists of contractions. (A result from the theory of strongly continuous semigroups in the the operators on some Banach space B asserts that for such semigroups we may find $c \in \mathbb{R}$ and $C \neq 0$ such that all $Ce^{ct}T_t$ are contractions, but these do not form a semigroup, unless $C = 1$.) The following example we owe to V. Liebscher (private communication).

12.3.1 Example. Consider the semigroup

$$a_t = \begin{pmatrix} \mathfrak{s}_t & 0 \\ \mathbb{I}_{[0,t]}\mathfrak{s}_t & \mathfrak{s}_t \end{pmatrix}$$

on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R}_+)$. One easily proves that $\left(a_t\right)_{t\in\mathbb{R}_+}$ is a semigroup and strongly continuous. On the other hand, $\|a_t\| = \sqrt{2}$ for $t > 0$. Consequently, T with $T_t(b) = a_t^* b a_t$ is a strongly continuous CP-semigroup with $\|T\|$ close to 2 for small $t > 0$.

Let us return to contractive CP-semigroups. There are two essentially different ways to proceed. The first way as done in [Bha99] uses only the possibly non-unital CP-semigroup T. Although we may construct the pair $(E^\odot, \xi^\odot)$ in the two-sided inductive limit of Section 11.3, the one-sided inductive limit of Section 11.4 breaks down, and the inner product must be defined *a priori*. The second way to proceed as in Bhat and Skeide [BS00] uses the unitization $\widetilde{T}$ on $\widetilde{\mathcal{B}}$ as in Theorem 4.1.11(3).

Here we mainly follow the second approach. In other words, we do the constructions of Sections 11.3 and 11.4 for the unital CP-semigroup $\widetilde{T}$. As a result we obtain a pre-Hilbert $\widetilde{\mathcal{B}}$–module $\widetilde{E}$, a cyclic vector $\widetilde{\xi}$, a weak Markov flow $\widetilde{j}$ acting on $\widetilde{E}$, and an E_0–semigroup $\widetilde{\vartheta}$ on $\mathcal{B}^a(\widetilde{E})$. The restriction of $\widetilde{\vartheta}$ to the submodule E which is generated by $\widetilde{\xi}$ and $\widetilde{\vartheta}_{\mathbb{T}} \circ \widetilde{j}_0(\mathcal{B})$ is *cum grano salis* a dilation of T. We will see that the (linear) codimension of E in $\widetilde{E}$ is 1.

Recall that $\widetilde{\mathcal{B}} = \mathcal{B} \oplus \mathbb{C}\widetilde{\mathbf{1}}$, and that ($\mathcal{B}$ is unital)

$$\mathcal{B} \oplus \mathbb{C} \longrightarrow \widetilde{\mathcal{B}}, \quad (b, \mu) \longmapsto (b - \mu\mathbf{1}) \oplus \mu\widetilde{\mathbf{1}}$$

is an isomorphism of C^*–algebras, where $\mathcal{B} \oplus \mathbb{C}$ is the usual C^*–algebraic direct sum. In [BP94] the unitization has been introduced in the picture $\mathcal{B} \oplus \mathbb{C}$. In the sequel, we will switch between the pictures $\widetilde{\mathcal{B}}$ and $\mathcal{B} \oplus \mathbb{C}$ according to our needs.

We start by reducing the GNS-construction $(\widetilde{\breve{E}}_t, \widetilde{\breve{\xi}}_t)$ for $\widetilde{T}_t$ to the GNS-construction $(\breve{E}_t, \breve{\xi}_t)$ for T_t. By Example 1.6.5 we may consider $\breve{E}_t$ also as a pre-Hilbert $\widetilde{\mathcal{B}}$–$\widetilde{\mathcal{B}}$–module. Since T_t is not necessarily unital, $\breve{\xi}_t$ is not necessarily a unit vector. However, $\langle \breve{\xi}_t, \breve{\xi}_t \rangle \leq \mathbf{1}$ as T_t is contractive. Denote by $\widehat{\breve{\xi}}_t$ the positive square root of $\widetilde{\mathbf{1}} - \langle \breve{\xi}_t, \breve{\xi}_t \rangle$ in $\widetilde{\mathcal{B}}$. Denote by $\widehat{E}_t = \widehat{\breve{\xi}}_t \widetilde{\mathcal{B}}$ the right ideal in $\widetilde{\mathcal{B}}$ generated by $\widehat{\breve{\xi}}_t$ considered as a right pre-Hilbert $\widetilde{\mathcal{B}}$–module (see Example 1.1.5). By defining the left multiplication $b\widehat{\breve{\xi}}_t = 0$ for $b \in \mathcal{B}$ and $\widetilde{\mathbf{1}}\widehat{\breve{\xi}}_t = \widehat{\breve{\xi}}_t$, we turn $\widehat{E}_t$ into a pre-Hilbert $\widetilde{\mathcal{B}}$–$\widetilde{\mathcal{B}}$–module. We set $\widetilde{\breve{E}}_t = \breve{E}_t \oplus \widehat{E}_t$ and $\widetilde{\breve{\xi}}_t = \breve{\xi}_t \oplus \widehat{\breve{\xi}}_t$. One easily checks that $(\widetilde{\breve{E}}_t, \widetilde{\breve{\xi}}_t)$ is the GNS-construction for $\widetilde{T}_t$.

12.3.2 Observation. Among many other simple relations connecting $\breve{\xi}_t$, $\widetilde{\breve{\xi}}_t$, and $\widehat{\breve{\xi}}_t$ with the central projections $\mathbf{1}$, and $\widetilde{\mathbf{1}} - \mathbf{1}$ like e.g. $\mathbf{1}\widetilde{\breve{\xi}}_t = \breve{\xi}_t$, $(\widetilde{\mathbf{1}} - \mathbf{1})\widetilde{\breve{\xi}}_t = \widehat{\breve{\xi}}_t$, or $\widetilde{\breve{\xi}}_t(\widetilde{\mathbf{1}} - \mathbf{1}) = (\widetilde{\mathbf{1}} - \mathbf{1})\widetilde{\breve{\xi}}_t(\widetilde{\mathbf{1}} - \mathbf{1})$, the relation

$$\widehat{\breve{\xi}}_t \mathbf{1} \;=\; (\widetilde{\mathbf{1}} - \mathbf{1})\widetilde{\breve{\xi}}_t \mathbf{1} \;=\; \widetilde{\mathbf{1}}\widetilde{\breve{\xi}}_t \mathbf{1} - \mathbf{1}\widetilde{\breve{\xi}}_t \mathbf{1} \;=\; \widetilde{\breve{\xi}}_t \mathbf{1} - \mathbf{1}\widetilde{\breve{\xi}}_t$$

is particularly crucial for the proof of Theorem 12.3.5.

We denote $\breve{\Omega}_t = \widetilde{\breve{\xi}}_t(\widetilde{\mathbf{1}} - \mathbf{1})$ and $\widetilde{b} = (b, \mu)$ in the picture $\mathcal{B} \oplus \mathbb{C}$. The following proposition is verified easily by looking at the definition of $\widehat{\breve{\xi}}_t$ and by the rules in Observation 12.3.2.

12.3.3 Proposition. *$\breve{\Omega}_t$ may be identified with the element $\widetilde{\mathbf{1}} - \mathbf{1}$ in the right ideal $\widehat{E}_t$ in $\widetilde{\mathcal{B}}$. We have*

$$\widetilde{b}\widetilde{\breve{\xi}}_t \widetilde{b}'(\widetilde{\mathbf{1}} - \mathbf{1}) = \breve{\Omega}_t \mu\mu' = (\widetilde{\mathbf{1}} - \mathbf{1})\breve{\Omega}_t \mu\mu'.$$

In particular, $x_t \mapsto x_t(\widetilde{\mathbf{1}} - \mathbf{1})$ is a projection onto $\mathbb{C}\breve{\Omega}_t$. The orthogonal complement of $\breve{\Omega}_t$ is a right ideal in $\mathcal{B}$ and may, therefore, be considered as a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module.

Doing the constructions of Sections 11.3 and 11.4 for $\widetilde{T}$, we refer to $\widetilde{E}_{\mathfrak{t}}$, $\widetilde{E}_t = \limind_{\mathfrak{t}\in\mathbb{J}_t} \widetilde{E}_{\mathfrak{t}}$, and $\widetilde{E} = \limind_t \widetilde{E}_t$. Also other ingredients of these constructions are indicated by the *dweedle*. Letters without *dweedle* like $(E^{\odot}, \xi^{\odot})$ refer to analogue quantities coming from T_t.

By sending $b_n\xi_{t_n} \odot \ldots \odot b_1\xi_{t_1}b_0$ to $b_n\xi_{t_n} \odot \ldots \odot b_1\xi_{t_1}b_0$ $(\mathfrak{t} = (t_n, \ldots, t_1) \in \mathbb{J}_t; b_n, \ldots, b_0 \in \mathcal{B})$ we establish a $\mathcal{B}$–$\mathcal{B}$–linear isometric embedding $E_t \to \widetilde{E}_t$. In this identification we conclude from

$$\mathbf{1}\widetilde{b}_n\widetilde{\xi}_{t_n} \odot \ldots \odot \widetilde{b}_1\widetilde{\xi}_{t_1}\widetilde{b}_0 = b_n\xi_{t_n} \odot \ldots \odot b_1\xi_{t_1}b_0$$

that $\mathbf{1}\widetilde{E}_t = E_t$. We remark that here and in the remainder of this section it does not matter, whether we consider the tensor products as tensor products over $\mathcal{B}$ or over $\widetilde{\mathcal{B}}$. By definition of the tensor product the inner products coincide, so that the resulting pre-Hilbert modules are isometrically isomorphic. As long as the inner product takes values in $\mathcal{B}$ we are free to consider them as $\mathcal{B}$–modules or as $\widetilde{\mathcal{B}}$–modules.

12.3.4 Proposition. *Let $t \in \mathbb{T}$ and set $\Omega_t = \widetilde{\xi}_t(\widetilde{\mathbf{1}} - \mathbf{1}) \in \widetilde{E}_t$. Then $\Omega_{t_n} \odot \ldots \odot \Omega_{t_1} = \Omega_t$ for all $\mathfrak{t} \in \mathbb{J}_t$. Moreover, the Ω_t form a unit for $\widetilde{E}_t$.*

Set $\widehat{\xi}_t = (\widetilde{\mathbf{1}} - \mathbf{1})\widetilde{\xi}_t \in \widetilde{E}_t$. Then $\widetilde{i}_{(t)}\widehat{\widetilde{\xi}}_t = \widehat{\xi}_t$ for all $t \in \mathbb{T}$.

Set $\Omega = \widetilde{\xi}(\widetilde{\mathbf{1}} - \mathbf{1}) \in \widetilde{E}$. Then $\widetilde{k}_t\Omega_t = \Omega$ for all $t \in \mathbb{T}$.

Proof. From Observation 12.3.2 we find

$$\Omega_t = \widetilde{\xi}_t(\widetilde{\mathbf{1}} - \mathbf{1}) = \widetilde{\xi}_{t_n} \odot \ldots \odot \widetilde{\xi}_{t_1}(\widetilde{\mathbf{1}} - \mathbf{1}) = \Omega_{t_n} \odot \ldots \odot \Omega_{t_1}$$

from which all assertions of the first part follow. The second and third part are proved in an analogue manner. ■

Clearly, we have $\widetilde{E}(\widetilde{\mathbf{1}} - \mathbf{1}) = \mathbb{C}\Omega$. Denote by $E = \widetilde{E}\mathbf{1}$ the orthogonal complement of this submodule and denote by $\xi = \widetilde{\xi}\mathbf{1}$ the component of $\widetilde{\xi}$ in E. We may consider E as a pre-Hilbert $\mathcal{B}$–module.

12.3.5 Theorem. *The operators in $\widetilde{j}_{\mathbb{T}}(\mathcal{B})$ leave invariant E, i.e. $\widetilde{j}_t(b)$ and the projection $\mathbf{1}^r$ onto E commute for all $t \in \mathbb{T}$ and $b \in \mathcal{B}$. For the restrictions $j_t(b) = \widetilde{j}_t(b) \upharpoonright E$ the following holds.*

1. *E is generated by $j_{\mathbb{T}}(\mathcal{B})$ and ξ.*

2. *The j_t fulfill the Markov property* (10.4.3) *and, of course, j_0 is faithful.*

3. *The restriction of $\widetilde{\vartheta}$ to $\mathcal{B}^a(E)$ defines an E_0–semigroup ϑ on $\mathcal{B}^a(E)$, which fulfills $\vartheta_t \circ j_s = j_{s+t}$. Clearly, ϑ leaves invariant $\mathcal{A}_\infty = \operatorname{span} j_{\mathbb{T}}(\mathcal{B})$.*

Proof. Observe that $\widetilde{j}_t(\mathbf{1})\widetilde{E} = \widetilde{\xi} \odot E_t$. By $\widehat{E} \subset E$ we denote the linear span of all these spaces. Clearly, $\widehat{E}$ is a pre-Hilbert $\mathcal{B}$–module. Moreover, all $\widetilde{j}_t(b)$ leave invariant $\widehat{E}$. We will show that $\widehat{E} = E$, which implies that also E is left invariant by $\widetilde{j}_t(b)$.

$\widetilde{E}$ is spanned by the subspaces $\widetilde{\xi} \odot \widetilde{E}_t$, so that E is spanned by the subspaces $\widetilde{\xi} \odot \widetilde{E}_t\mathbf{1}$. The space $\widetilde{E}_t\mathbf{1}$ is spanned by elements of the form $x_t = \widetilde{b}_n\widetilde{\xi}_{t_n} \odot \ldots \odot \widetilde{b}_1\widetilde{\xi}_{t_1}b_0$. For each $1 \leq k \leq n$ we may assume that either $\widetilde{b}_k = b_k \in \mathcal{B}$ or $\widetilde{b}_k = \mu_k(\widetilde{\mathbf{1}} - \mathbf{1})$. If $\widetilde{b}_k = \mu_k(\widetilde{\mathbf{1}} - \mathbf{1})$ for some k, then we may assume that $\widetilde{b}_\ell = \mu_\ell(\widetilde{\mathbf{1}} - \mathbf{1})$ for all $\ell \geq k$. (Otherwise, the expression is 0.) We have to distinguish two cases. Firstly, all $\widetilde{b}_k$ are in $\mathcal{B}$. Then x_t is in E_t so that $\widetilde{\xi} \odot x_t \in \widehat{E}$. Secondly, there is a unique smallest number $1 \leq k \leq n$, such that $\widetilde{b}_\ell = \mu_\ell(\widetilde{\mathbf{1}} - \mathbf{1})$ for all $\ell \geq k$. Then it is easy to see that

$$x_t = \Omega_{s_3} \odot \widehat{\xi}_{s_2} \odot x_{s_1}, \quad \text{i.e.} \quad \widetilde{\xi} \odot x_t = \widetilde{\xi} \odot \widehat{\xi}_{s_2} \odot x_{s_1}$$

where $s_1 + s_2 + s_3 = t$ and $x_{s_1} \in E_{s_1}$. By Observation 12.3.2, we obtain

$$\begin{aligned} \widetilde{\xi} \odot \widehat{\xi}_{s_2} \odot x_{s_1} &= \widetilde{\xi} \odot (\widetilde{\xi}^{s_2}\mathbf{1} - \mathbf{1}\widetilde{\xi}^{s_2}) \odot x_{s_1} = \widetilde{\xi} \odot (\widetilde{\xi}^{s_2}\mathbf{1} - \mathbf{1}\xi^{s_2}) \odot x_{s_1} \\ &= (\widetilde{j}_{s_1}(\mathbf{1}) - \widetilde{j}_{s_2+s_1}(\mathbf{1}))\widetilde{\xi} \odot x_{s_1}, \end{aligned}$$

so that also in this case $\widetilde{\xi} \odot x_t \in \widehat{E}$. Therefore, $E \subset \widehat{E}$.

1. It remains to show that $\widetilde{\xi} \odot x_t$ for $x_t \in E_t$ can be expressed by applying a suitable collection of operators $j_t(b)$ to ξ and building linear combinations. But this follows inductively by the observation that $j_t(b)(\widetilde{\xi} \odot x_s) = \widetilde{\xi} \odot b\xi_{t-s} \odot x_s$ for $t > s$.
2. This assertion follows by applying $\mathbf{1}^r$ to the Markov property of $\widetilde{j}$.
3. Clear. ■

12.3.6 Remark. Considering $\mathcal{B}$ as a C^*–subalgebra of $\mathcal{B}(G)$ for some Hilbert space G and doing the Stinespring construction for E as described in Example 4.1.9, we obtain the results from [Bha99]. It is quite easy to see that the inner products of elements in E_t (that is for fixed t) coincide, when tensorized with elements in the *initial space* G, with the inner products given in [Bha99]. We owe the reader to compute the inner products of elements in $\widetilde{k}_tE_t \subset E$ and $\widetilde{k}_sE_s \subset E$ for $t \neq s$. Let $x_t \in E_t$ and $y_s \in E_s$ and assume without loss of generality that $s < t$. We find

$$\langle \widetilde{\xi} \odot x_t, \widetilde{\xi} \odot y_s \rangle = \langle \widetilde{\xi} \odot x_t, \widetilde{\xi} \odot \widetilde{\xi}^{t-s} \odot y_s \rangle = \langle x_t, \widetilde{\xi}^{t-s} \odot y_s \rangle = \langle x_t, \xi^{t-s} \odot y_s \rangle.$$

(In the last step we made use of $\mathbf{1}x_t = x_t$ and $\mathbf{1}\widetilde{\xi}^{t-s} = \xi^{t-s}$.) This shows in full correspondence with [Bha99] that an element in E_s has to be lifted to E_t by "inserting a $\mathbf{1}$ at time t", before we can compare it with an element in E_t. This lifting is done by tensorizing ξ_{t-s}. As this operation is no longer an isometry, the second inductive limit breaks down in the non-unital case. Cf. also Remark A.10.8.

12.3.7 Example. Now we study in detail the most simple non-trivial example. We start with the non-unital CP-semigroup $T_t\colon z \mapsto e^{-t}z$ on $\mathbb{C}$. Here the product system $E_t = \mathbb{C}$ consists of one-dimensional Hilbert spaces and the unit consists of the vectors $\xi^t = e^{-\frac{t}{2}} \in E_t$.

For the unitization we find it more convenient to consider $\mathbb{C}^2$ rather than $\widetilde{\mathbb{C}}$. The mappings $\widetilde{T}_t\colon \mathbb{C}^2 \to \mathbb{C}^2$ are given by $\widetilde{T}_t\binom{a}{b} = b\binom{1}{U} + (a-b)\binom{e^{-t}}{0}$. The first component corresponds to the original copy of $\mathbb{C}$, whereas the second component corresponds to $\mathbb{C}(\widetilde{\mathbf{1}} - \mathbf{1})$.

We continue by writing down $\widetilde{E}$ and $\widetilde{E}_t$, showing afterwards that these spaces are the right ones. (To be precise we are dealing rather with their completions, but, this difference is not too important.) We define the Hilbert $\mathbb{C}^2$–module $\widetilde{E}$ and its inner product by

$$\widetilde{E} = L^2(\mathbb{R}_-) \oplus \mathbb{C}\breve{\Omega} \quad \text{and} \quad \left\langle \binom{f}{\mu}, \binom{g}{\nu} \right\rangle = \binom{\langle f,g\rangle}{\overline{\mu}\nu}.$$

The inner product already determines completely the right multiplication by elements of $\mathbb{C}^2$ to be the obvious one.

Let us define $e_t \in L^2(\mathbb{R}_-)$ by setting $e_t(t) = \chi_{[t,\infty)}(t)e^{-\frac{t}{2}}$. Observe that $\langle e_t, e_t\rangle = e^{-t}$. We define the Hilbert $\mathbb{C}^2$–submodule $\widetilde{E}_t$ of $\widetilde{E}$ by $\widetilde{E}_t = L^2(0,t) \oplus \mathbb{C}e_t \oplus \mathbb{C}\breve{\Omega}$. (Observe that, indeed, $\langle L^2(0,t), e_t\rangle = \{0\}$.) We turn $\widetilde{E}_t$ into a Hilbert $\mathbb{C}^2$–$\mathbb{C}^2$–module by defining the left multiplication

$$\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} g \\ \beta \\ \nu \end{pmatrix} = b \begin{pmatrix} g \\ 0 \\ \nu \end{pmatrix} + a \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix}.$$

We define the homomorphism $\widetilde{j}_t\colon \mathbb{C}^2 \to \mathcal{B}^a(\widetilde{E})$ by, first, projecting down to the submodule $\widetilde{E}_t$, and then, applying the left multiplication of $\mathbb{C}^2$ on $\widetilde{E}_t \subset \widetilde{E}$. Clearly, the $\widetilde{j}_t$ form a weak Markov flow of $\widetilde{T}$.

Observe that also $\langle L^2(0,t), \mathfrak{s}_t L^2(\mathbb{R}_-)\rangle = \{0\}$. One easily checks that the mappings

$$\begin{pmatrix} f \\ \mu \end{pmatrix} \odot \begin{pmatrix} g \\ \beta \\ \nu \end{pmatrix} \longmapsto \begin{pmatrix} \mu g + e^{-\frac{t}{2}}\beta \mathfrak{s}_t f \\ \mu\nu \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f \\ \alpha \\ \mu \end{pmatrix} \odot \begin{pmatrix} g \\ \beta \\ \nu \end{pmatrix} \longmapsto \begin{pmatrix} \mu g + e^{-\frac{t}{2}}\beta \mathfrak{s}_t f \\ \alpha\beta \\ \mu\nu \end{pmatrix} \tag{12.3.1}$$

define isomorphisms $\widetilde{E} \odot \widetilde{E}_t \to \widetilde{E}$ and $\widetilde{E}_s \odot \widetilde{E}_t \to \widetilde{E}_{s+t}$, respectively. Remarkably enough, no completion is necessary here.

It remains to show that $\widetilde{E}$ (and, similarly, also $\widetilde{E}_t$) is generated by $\widetilde{\xi} = \binom{e_0}{1}$ and $\widetilde{j}_{\mathbb{T}}(\mathbb{C}^2)$. But this is simple, as we have $\widetilde{j}_0\binom{0}{1}\widetilde{\xi} = \breve{\Omega}$ and $\big(\widetilde{j}_t\binom{1}{0} - \widetilde{j}_s\binom{1}{0}\big)\widetilde{\xi} = \binom{\chi_{[s,t]}e_0}{0}$ for $s < t$. Therefore, we obtain all functions which consist piecewise of arcs of the form $e^{-\frac{t}{2}}$. Clearly, these functions form a dense subspace of $L^2(\mathbb{R}_-)$. Until now we, tacitly, have assumed to speak about Hilbert modules. It is, however, clear that the arcwise exponentials form an algebraically invariant subset.

In this example we see in an extreme case that the product system of a non-conservative CP-semigroup T may be *blown up* considerably, when changing to its unitization $\widetilde{T}$. Notice that the original one-dimensional product system of T is present in the middle component $\left(\begin{smallmatrix}0\\ \alpha\\ 0\end{smallmatrix}\right) \odot \left(\begin{smallmatrix}0\\ \beta\\ 0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0\\ \alpha\beta\\ 0\end{smallmatrix}\right)$ in the factorization by (12.3.1). (Recall that $\left\|\left(\begin{smallmatrix}0\\ \beta\\ 0\end{smallmatrix}\right)\right\|^2 = |\beta|^2\, e^{-t}$ depends on t.) Responsible for the blow up is the part $\widehat{\xi}_t\mathbf{1}$ of $\widehat{\xi}_t$ which lies in $\mathcal{B}$. If T was already unital, then this part is 0.

12.4 Weak Markov quasiflows

In this section study the structure of weak Markov quasiflows in an algebraic fashion, i.e. to begin with we do not refer to a representation module. Other structures like a family of conditional expectations $\varphi_t = p_t \bullet p_t$ can be reconstructed.

If we want to encode properties of a weak Markov flow, which are of an *essentially spatial* nature, then we have to require that the GNS-representation of the conditional expectation φ_0 is suitably faithful. This leads to the notion of an *essential* weak Markov flow. Among all such flows we are able to single out two universal objects, a *maximal* weak Markov flow, which is realized by $(\mathcal{B}^a(E), j)$ as in Theorem 11.4.8, and a *minimal* weak Markov flow being just the restriction $(\mathcal{A}_\infty = \mathsf{alg}\, j_{\mathbb{T}}(\mathcal{B}), j)$ of $(\mathcal{B}^a(E), j)$.

12.4.1 Definition. Let $(\mathcal{A}, j)$ be a weak Markov quasiflow for a unital CP-semigroup T. Let I be a subset of $\mathbb{T}$. We set $\mathcal{A}_I = \mathsf{alg}\, j_I(\mathcal{B})$. In particular, we set $\mathcal{A}_{t]} = \mathcal{A}_{[0,t]}$, $\mathcal{A}_{[t} = \mathcal{A}_{[t,\infty)}$, and $\mathcal{A}_\infty = \mathcal{A}_{[0,\infty)} = \mathsf{alg}\, j_{\mathbb{T}}(\mathcal{B})$.

A *morphism* from a weak Markov quasiflow $(\mathcal{A}, j)$ of T to a weak Markov quasiflow $(\mathcal{C}, k)$ of T is a contractive $*$–algebra homomorphism $\alpha\colon \mathcal{A} \to \mathcal{C}$ fulfilling

$$\alpha \circ j_t = k_t \quad \text{for all} \quad t \in \mathbb{T}.$$

α is an *isomorphism*, if it is also an isomorphism between pre–C^*–algebras (i.e. α is isometric onto). The class consisting of weak Markov quasiflows and morphisms among them forms a category.

The p_t form an approximate unit for $\mathcal{A}_\infty$. This shows, in particular, that $\mathcal{A}_\infty$ is non-unital, unless $p_t = \mathbf{1}$ for some $t \in \mathbb{T}$. In a faithful non-degenerate representation of $\mathcal{A}_\infty$ the p_t converge to $\mathbf{1}$ strongly.

Of course, the main goal in constructing a weak Markov quasiflow is to recover T_t in terms of j_t. This is done by $p_0 j_t(b) p_0 = j_0(T_t(b))$ and, naturally, leads to the requirement that j_0 should be injective. Nevertheless, as the following remark shows, there are interesting examples of weak Markov quasiflows where j_0 is not injective.

12.4.2 Remark. If j is a weak Markov quasiflow, then also the time shifted family j^s with $j^s_t = j_{s+t}$ for some fixed $s \in \mathbb{T}$ is a weak Markov quasiflow. The j_t are, in general, far from being injective. Of course, a trivial example is $j_t = 0$ for all t.

Now we are going to construct a univeral mapping $\mathfrak{T}$ very similar to the correlation kernels introduced in [AFL82]. We will see that $\mathfrak{T}$ and j_0 determine $(\mathcal{A}_\infty, j)$ completely. Moreover, $(\mathcal{A}_\infty, j)$ always admits a faithful representation on a suitable pre-Hilbert $j_0(\mathcal{B})$–module E^j (closely related to $E = E^{min}$ as constructed in Theorem 11.4.8). This quasiflow is determined by j_0 up to unitary equivalence.

12.4.3 Lemma. *Denote by* $\mathbb{B} = \bigcup_{n \in \mathbb{N}_0} (\mathbb{T} \times \mathcal{B})^n$ *the set of all finite tuples* $\big((t_1, b_1), \ldots, (t_n, b_n)\big)$ $(n \in \mathbb{N})$ *of pairs in* $\mathbb{T} \times \mathcal{B}$*. Let* V *be a vector space and* $\mathfrak{T} \colon \mathbb{B} \to V$ *a mapping, fulfilling*

$$\mathfrak{T}\big((t_1, b_1), \ldots, (s, a), (t, b), (s, c), \ldots, (t_n, b_n)\big) \\ = \mathfrak{T}\big((t_1, b_1), \ldots, (s, aT_{t-s}(b)c), \ldots, (t_n, b_n)\big), \quad (12.4.1)$$

whenever $s \le t; a, b, c \in \mathcal{B}$*, and*

$$\mathfrak{T}\big((t_1, b_1), \ldots, (t_k, \mathbf{1}), \ldots, (t_n, b_n)\big) = \mathfrak{T}\big((t_1, b_1), \ldots, \widehat{(t_k, \mathbf{1})}, \ldots, (t_n, b_n)\big), \qquad (12.4.2)$$

whenever $t_{k-1} \le t_k$ $(1 < k)$*, or* $t_{k+1} \le t_k$ $(k < n)$*, or* $k = 1$*, or* $k = n$*.*

Then $\mathfrak{T}$ *is determined uniquely by the values* $\mathfrak{T}\big((0, b)\big)$ $(b \in \mathcal{B})$*. Moreover, the range of* $\mathfrak{T}$ *is contained in* $\operatorname{span} \mathfrak{T}\big((0, \mathcal{B})\big)$*.*

Proof. In a tuple $\big((t_1, b_1), \ldots, (t_n, b_n)\big) \in \mathbb{B}$ go to the position with maximal time t_m. By (12.4.1) we may reduce the length of this tuple by 2, possibly, after having inserted by (12.4.2) a $\mathbf{1}$ at a suitable time in the neighbourhood of (t_m, b_m). This procedure may be continued until the length is 1. If this is achieved, then we insert $(0, \mathbf{1})$ on both sides and, making again use of (12.4.1), we arrive at a tuple of the form $\big((0, b)\big)$. ∎

12.4.4 Corollary. *Let* $(\mathcal{A}, j)$ *be a weak Markov quasiflow of a unital CP-semigroup* T*. Then the mapping* $\mathfrak{T}_j$*, defined by setting*

$$\mathfrak{T}_j\big((t_1, b_1), \ldots, (t_n, b_n)\big) = p_0 j_{t_1}(b_1) \ldots j_{t_n}(b_n) p_0,$$

is the unique mapping $\mathfrak{T}_j \colon \mathbb{B} \to j_0(\mathcal{B})$*, fullfilling* (12.4.1)*,* (12.4.2)*, and*

$$\mathfrak{T}_j\big((0, b)\big) = j_0(b). \qquad (12.4.3)$$

12.4.5 Corollary. *The mapping* $\varphi_0 \colon a \mapsto p_0 a p_0$ *defines a conditional expectation* $\mathcal{A}_\infty \to \mathcal{A}_0$*.*

12.4.6 Proposition. *For all* $t \in \mathbb{T}$ *the mapping* $\varphi_t \colon a \mapsto p_t a p_t$ *defines a conditional expectation* $\mathcal{A}_\infty \to \mathcal{A}_t$*.*

Proof. Consider the time shifted weak Markov quasiflow j^t as in Remark 12.4.2. Since $j_0^t = j_t$, it follows by Corollary 12.4.5 that $p_t \bullet p_t$ defines a conditional expectation $\mathcal{A}_{[t} \to j_t(\mathcal{B})$.

Now consider a tuple in $\mathbb{B}$ and split it into subtuples which consist either totally of elements at times $\le t$, or totally at times $> t$. At the ends of these tuples we may insert p_t, so that the elements at times $> t$ are framed by p_t. By the first part of the proof the product over such a subtuple (including the surrounding p_t's) is an element of $j_t(\mathcal{B})$. The remaining assertions follow by the fact that p_t is a unit for $\mathcal{A}_{t]}$. ■

12.4.7 Theorem. *There exists a unique mapping $\mathfrak{T}\colon \mathbb{B} \longrightarrow \mathcal{B}$, fullfilling* (12.4.1), (12.4.2), *and $\mathfrak{T}\big((0,b)\big) = b$ for all $b \in \mathcal{B}$. We call $\mathfrak{T}$ the correlation kernel of T.*

Proof. Suppose that j is a weak Markov flow. Then the mapping $j_0^{-1} \circ \mathfrak{T}_j$ has the desired properties. Existence of a weak Markov flow has been settled in Theorem 11.4.8. ■

12.4.8 Corollary. *Let j be a weak Markov quasiflow. Then $\mathfrak{T}_j = j_0 \circ \mathfrak{T}$.*

12.4.9 Remark. The module E from Theorem 11.4.8 may be considered as the Kolmogorov decomposition of the positive definite kernel $\mathfrak{k}\colon \mathbb{B} \times \mathbb{B} \to \mathcal{B}$, defined by setting

$$\mathfrak{k}\Big(\big((t_n, b_n), \ldots, (t_1, b_1)\big)\,,\,\big((s_m, c_m), \ldots, (s_1, c_1)\big)\Big) \\ = \mathfrak{T}\big((t_1, b_1^*), \ldots, (t_n, b_n^*), (s_m, c_m), \ldots, (s_1, c_1)\big).$$

More generally, if $(\mathcal{A}, j)$ is a weak Markov quasiflow, then the *GNS-module* E^j associated with j (see Definition 12.4.10 below) is the Kolmogorov decomposition for the positive definite kernel $j_0 \circ \mathfrak{k}$.

This interpretation throws a bridge to the reconstruction theorem in [AFL82] and the original construction of the minimal weak Markov quasiflow in [BP94], where $\mathfrak{k}$ is a usual $\mathbb{C}$–valued kernel on $\mathbb{B} \times G$ (where G denotes a Hilbert space on which $\mathcal{B}$ is represented). Cf. also [Acc78] and [Bel85].

12.4.10 Definition. Let $(\mathcal{A}, j)$ be a weak Markov quasiflow. Then by (E^j, ξ^j) we denote the GNS-representation of $\varphi_0\colon \mathcal{A}_\infty \to \mathcal{A}_0$. We call E^j the *GNS-module* associated with $(\mathcal{A}, j)$. Denote by $\alpha^j\colon \mathcal{A}_\infty \to \mathcal{B}^a(E^j)$ the canonical homomorphism. Obviously, $\alpha^j\colon (\mathcal{A}_\infty, j) \to (\alpha^j(\mathcal{A}_\infty), \alpha^j \circ j)$ is a morphism of weak Markov quasiflows. We call $(\alpha^j(\mathcal{A}_\infty), \alpha^j \circ j)$ the *minimal* weak Markov quasiflow associated with $(\mathcal{A}, j)$ and we call $(\mathcal{B}^a(E^j), \alpha^j \circ j)$ the *maximal* weak Markov quasiflow associated with $(\mathcal{A}, j)$.

If j_0 is faithful, the we leave out 'quasi'.

It is natural to ask under which conditions the representation of $\mathcal{A}_\infty$ on E^j is faithful or, more generally, extends to a faithful (isometric) representation of $\mathcal{A}$ on E^j. In other words, we ask under which conditions the pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module E^j is essential. The following definition and proposition settle this problem.

12.4.11 Definition. A weak Markov quasiflow $(\mathcal{A}, j)$ is called *essential*, if the ideal I_0 in $\mathcal{A}_\infty$ generated by p_0 is an ideal also in $\mathcal{A}$, and if I_0 is *essential* in the C^*–completion of $\mathcal{A}$ (i.e. for all $a \in \overline{\mathcal{A}}$ we have that $aI_0 = \{0\}$ implies $a = 0$).

12.4.12 Proposition. *E^j is an essential pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module, if and only if $(\mathcal{A}, j)$ is essential. In this case the mapping $a \mapsto p_0 a p_0$ maps also $\mathcal{A}$ into $\mathcal{A}_0$ and, therefore, defines a conditional expectation $\varphi\colon \mathcal{A} \to \mathcal{A}_0$.*

Proof. We have $\mathsf{span}(\mathcal{A}\mathcal{A}_\infty p_0) = \mathsf{span}(\mathcal{A}(\mathcal{A}_\infty p_0)p_0) \subset \mathsf{span}(\mathcal{A}I_0 p_0) = \mathsf{span}(I_0 p_0) = \mathcal{A}_\infty p_0$. Therefore, $\mathcal{A}_\infty p_0$ is a left ideal in $\mathcal{A}$ so that φ, indeed, takes values in $\mathcal{A}_0$. By construction φ is bounded, hence, extends to $\overline{\mathcal{A}}$. (Observe that $\mathcal{A}_0$ is the range of a C^*–algebra homomorphism and, therefore, complete.) Now our statement follows as in the proof of Proposition 4.4.8. ■

If j is essential, then we identify $\mathcal{A}$ as a pre–C^*–subalgebra of $\mathcal{B}^a(E^j)$. In this case, we write $(\mathcal{A}_\infty, j)$ and $(\mathcal{B}^a(E^j), j)$ for the minimal and the maximal weak Markov quasiflow associated with $(\mathcal{A}, j)$, respectively. An essential weak Markov quasiflow $(\mathcal{A}, j)$ lies in between the minimal and the maximal essential weak Markov quasiflow associated with it, in the sense that $\mathcal{A}_\infty \subset \mathcal{A} \subset \mathcal{B}^a(E^j)$.

Proposition 12.4.12 does not mean that φ_0 is faithful. By Proposition 4.4.7, φ_0 is faithful, if and only if $p_0 = p_t = \mathbf{1}$.

The C^*–algebraic condition in Definition 12.4.11 seems to be out of place in our pre–C^*–algebraic framework for the algebra $\mathcal{A}$. In fact, we need it only in order to know that the GNS-representation of $\mathcal{A}$ is isometric. This is necessary, if we want that the E_0–semigroup ϑ in Theorem 11.4.8 extends to the completion of $\mathcal{B}^a(E)$. Example 4.4.9 shows that the C^*–algebraic version is, indeed, indispensable.

Notice that there exist interesting non-essential weak Markov quasiflows. For instance, the Markov flow in Theorem 11.4.8 comming from a product system $E^\odot$ which is not the GNS-system of T is, usually, non-essential.

12.4.13 Definition. For a (unital) C^*–algebra $\mathcal{B}$ we introduce the *homomorphism category* $\mathfrak{h}(\mathcal{B})$. The objects of $\mathfrak{h}(\mathcal{B})$ are pairs $(\mathcal{A}, j)$ consisting of a C^*–algebra $\mathcal{A}$ and a surjective homomorphism $j\colon \mathcal{B} \to \mathcal{A}$. A morphism $i\colon (\mathcal{A}, j) \to (\mathcal{C}, k)$ in $\mathfrak{h}(\mathcal{B})$ is a homomorphism

$\mathcal{A} \to \mathcal{C}$, also denoted by i, such that $i \circ j = k$. Clearly, such a morphism exists, if and only if $\mathsf{ker}(j) \subset \mathsf{ker}(k)$. If there exists a morphism, then it is unique.

In the sequel, by (E, j, ξ) we always mean the minimal weak Markov flow comming from the GNS-dilation. Also the notions related to $\mathcal{A}_I$ and φ_t refer to this minimal weak Markov flow. $(\mathcal{C}, k)$ stands for an essential weak Markov quasiflow. $\mathcal{C}_I$ and related notions are defined similar to $\mathcal{A}_I$. (The flow k is not to be confused with the canonical mappings k_t in Section 11.4.)

12.4.14 Lemma. *Let $(\mathcal{C}, k)$ be an essential weak Markov quasiflow of T. Furthermore, denote by $(\breve{E}^k, \mathbf{1}^k)$ the GNS-construction of $k_0 \colon \mathcal{B} \to \mathcal{C}_0 = k_0(\mathcal{B})$. Then $E^k = E \odot \breve{E}^k$ and $\xi^k = \xi \odot \mathbf{1}^k$. Moreover, in this identification we have*

$$k_t(b) \;=\; j_t(b) \odot \mathsf{id}\,. \tag{12.4.4}$$

Proof. Clearly, $\breve{E}^k = k_0(\mathcal{B})$, when considered as a Hilbert $\mathcal{B}$–$k_0(\mathcal{B})$–module via $bk_0(b') := k_0(bb')$ and $\mathbf{1}^k = k_0(\mathbf{1})$. It follows that $E \odot \breve{E}^k$ is just E equipped with the new $\mathcal{C}_0$–valued inner product $\langle x, x' \rangle_k = k_0(\langle x, x' \rangle)$ divided by the kernel $\mathcal{N}$ of this inner product. $\xi \odot \mathbf{1}^k$ is just $\xi + \mathcal{N}$.

Let $x = j_{t_n}(b_n) \dots j_{t_1}(b_1)\xi$ and $x' = j_{t'_m}(b'_m) \dots j_{t'_1}(b'_1)\xi$ $(t_i, t'_j \in \mathbb{T}; b_i, b'_j \in \mathcal{B})$ be elements in E. Then

$$\langle x, x' \rangle = \mathcal{T}\big((t_1, b_1^*), \dots, (t_n, b_n^*), (t'_m, b'_m) \dots, (t'_1, b'_1)\big).$$

For $y = k_{t_n}(b_n) \dots k_{t_1}(b_1)\xi^k$ and $y' = k_{t'_m}(b'_m) \dots k_{t'_1}(b'_1)\xi^k$ in E^k we find

$$\langle y, y' \rangle = \mathcal{T}_k\big((t_1, b_1^*), \dots, (t_n, b_n^*), (t'_m, b'_m) \dots, (t'_1, b'_1)\big).$$

Therefore, by sending $x \odot \mathbf{1}^k$ to y we define a unitary mapping $u \colon E \odot \breve{E}^k \to E^k$. Essentially the same computations show that the isomorphism $\mathcal{B}^a(E \odot \breve{E}^k) \to \mathcal{B}^a(E^k), a \mapsto uau^{-1}$ respects (12.4.4). ■

12.4.15 Proposition. *Let $(\mathcal{C} = \mathcal{B}^a(E^k), k)$ and $(\mathcal{C}' = \mathcal{B}^a(E^{k'}), k')$ be two maximal weak Markov quasiflows. Then there exists a morphism $\alpha \colon (\mathcal{C}, k) \to (\mathcal{C}', k')$, if and only if there exists a morphism $i \colon (\mathcal{C}_0, k_0) \to (\mathcal{C}'_0, k'_0)$. If i exists, then α is unique. In particular, $(\mathcal{C}, k)$ and $(\mathcal{C}', k')$ are isomorphic weak Markov quasiflows, if and only if $(\mathcal{C}_0, k_0)$ and $(\mathcal{C}'_0, k'_0)$ are isomorphic objects in $\mathfrak{h}(\mathcal{B})$.*

Proof. If i does not exist, then there does not exist a morphism α. So let us assume that i exists. In this case we denote by $(\mathcal{E}^{kk'}, \mathbf{1}^{kk'})$ the GNS-construction of i. One easily checks that $\mathcal{E}^k \odot \mathcal{E}^{kk'} = \mathcal{E}^{k'}$ and $\mathbf{1}^k \odot \mathbf{1}^{kk'} = \mathbf{1}^{k'}$. Thus, $E^{k'} = E^k \odot \mathcal{E}^{kk'}$. It follows that

$\alpha\colon a \mapsto a \odot \mathsf{id}$ defines a contractive homomorphism $\mathcal{B}^a(E^k) \to \mathcal{B}^a(E^{k'})$. Clearly, we have $k'_t(b) = k_t(b) \odot \mathsf{id}$, so that α is a morphism of weak Markov quasiflows.

If i is an isomorphism, then we may construct $\mathcal{E}^{k'k}$ as the GNS-module of i^{-1}. We find $E^{k'} \odot \mathcal{E}^{k'k} = E^k \odot \mathcal{E}^{kk'} \odot \mathcal{E}^{k'k} = E^k$. This enables us to reverse the whole construction, so that α is an isomorphism. The remaining statements are obvious. ∎

12.4.16 Corollary. *Let $(\mathcal{C}, k)$ be an arbitrary weak Markov quasiflow. Then the minimal and maximal weak Markov quasiflows associated with $(\mathcal{C}, k)$ are determined up to isomorphism by the isomorphism class of $(\mathcal{C}_0, k_0)$ in $\mathfrak{h}(\mathcal{B})$.*

12.4.17 Corollary. *Let $(\mathcal{C}_0, k_0)$ be an object in $\mathfrak{h}(\mathcal{B})$. Then there exist a unique minimal and a unique maximal weak Markov quasiflow extending k_0.*

Proof. Construct again the GNS-module $\mathcal{E}^k$ of k_0 and set $E^k = E \odot \mathcal{E}^k$. Then, obviously, (12.4.4) defines a maximal weak Markov quasiflow $(\mathcal{B}^a(E^k), k)$ with a minimal weak Markov quasiflow sitting inside. By the preceding corollary these weak Markov quasiflows are unique. ∎

The following theorem is proved by appropiate applications of the preceding results.

12.4.18 Theorem. *The maximal weak Markov flow $(\mathcal{B}^a(E), j)$ is the unique universal object in the category of maximal weak Markov quasiflows. In other words, if $(\mathcal{C}, k)$ is another maximal weak Markov quasiflow, then there exists a unique morphism $\alpha\colon (\mathcal{B}^a(E), j) \to (\mathcal{C}, k)$.*

The minimal weak Markov flow $(\mathcal{A}_\infty, j)$ is the unique universal object in the category of all essential weak Markov quasiflows. In other words, if $(\mathcal{C}, k)$ is an essential weak Markov quasiflow, then there exists a unique morphism $\alpha\colon (\mathcal{A}_\infty, j) \to (\mathcal{C}, k)$. Moreover, if $(\mathcal{C}, k)$ is minimal, then α is onto.

Let $(\mathcal{C}, k)$ be an essential weak Markov quasiflow. We could ask, whether the E_0–semigroup ϑ on $\mathcal{B}^a(E)$ gives rise to an E_0–semigroup on $\mathcal{B}^a(E^k)$ (or at least to an e_0–semigroup on $\mathcal{C}_\infty$). A necessary and sufficient condition is that the kernels of T_t should contain the kernel of k_0. (In this case, T_t gives rise to a completely positive mapping T^k_t on $k_0(\mathcal{B})$. Denote by $\breve{E}^k_t$ the GNS-module of T^k_t. It is not difficult to see that $E^k \odot \breve{E}^k_t$ carries a faithful representation of the time shifted weak Markov quasiflow k^t, and that the mapping $a \mapsto a \odot \mathsf{id}$ sends the weak Markov quasiflow k on E^k to the weak Markov quasiflow k_t on $E^k \odot \breve{E}^k_t$. From this it follows that the time shift on $\mathcal{C}_\infty$ is contractive.) However, the following example shows that this condition need not be fufilled, even in the case, when $\mathcal{B}$ is commutative, and when T is uniformly continuous.

12.4.19 Example. Let $\mathcal{B} = \mathbb{C}^2$. By setting $T_t\binom{z_1}{z_2} = \frac{z_1+z_2}{2}\binom{1}{U} + e^{-t}\frac{z_1-z_2}{2}\binom{1}{-1}$ we define a unital CP-semigroup T. We define a homomorphism $k\colon \mathbb{C}^2 \to \mathbb{C}$, by setting $k\binom{z_1}{z_2} = z_1$. Then $k\binom{0}{2} = 0$, but $k \circ T_t\binom{0}{2} = 1 - e^{-t} \neq 0$ (for $t \neq 0$).

12.5 Arveson's spectral algebra and the Evans-Lewis dilation

In [Arv90a, Arv89b] Arveson associates with a product system $\mathfrak{H}^{\bar{\otimes}} = \big(\mathfrak{H}_t\big)_{t\in\mathbb{R}_+}$ of Hilbert spaces $\mathfrak{H}_t$ the *spectral C^*–algebra* $C^*(\mathfrak{H}^{\bar{\otimes}})$. The spectral algebra may be understood in the following way. Denote by $L^2(\mathfrak{H}^{\bar{\otimes}})$ the direct integral over the family $\mathfrak{H}_t$ with respect to the Lebesgue measure on $\mathbb{R}_+$. For $v \in L^1(\mathfrak{H}^{\bar{\otimes}})$ (i.e. the 'function' $t \mapsto v(t) \in \mathfrak{H}_t$ is measurable in a suitable sense, and $\int \|v(t)\|\,dt < \infty$) define the 'creation' operator $\widehat{\ell}^*(v) \in \mathcal{B}(L^2(\mathfrak{H}^{\bar{\otimes}}))$, by setting

$$[\widehat{\ell}^*(v)f](t) = \int_0^t v(s) \odot f(t-s)\,ds$$

for all $f \in L^2(\mathfrak{H}^{\bar{\otimes}})$. Then the C^*–algebra generated by $\widehat{\ell}^*(L^1(\mathfrak{H}^{\bar{\otimes}}))\widehat{\ell}(L^1(\mathfrak{H}^{\bar{\otimes}}))$ is the spectral algebra. (Also here we follow the probabilists and physicists convention, and denote the creator by $*$.) It is the goal of this section to propose a generalization to Hilbert modules. This raises natural questions on the structure of the spectral algebras which have been answered for Arveson systems, like nuclearity (Arveson [Arv90a]) or its K-theory (Zacharias [Zac96]). The discrete version of the spectral algebra stems from an earlier (unbublished) version of Bhat and Skeide [BS00]. At the end of this section we give a rough idea what a 'continuous time' version for Hilbert modules might be.

12.5.1 Remark. Considering a product system on the semigroup $\mathbb{N}_0$ equipped with the counting measure instead of $\mathbb{R}_+$, we recover the usual full Fock space $\widehat{\ell}^2(\mathfrak{H}^{\bar{\otimes}}) = \mathcal{F}(\mathfrak{H}_1)$. The creators $\widehat{\ell}^*(v)$ are just the generalized creators (Definition 6.2.3) which play a crucial role in the calculus on the full Fock module (Part IV). Specializing to functions v with $v(1) = v_1 \in \mathfrak{H}_1$ and $v(t) = 0$ for $t \neq 1$, we find the usual creators $\ell^*(v_1)$. Of course, the creator of the vacuum $\widehat{\ell}^*(\Omega)$ is the identity operator so that $\ell^*(v_1)\widehat{\ell}(\Omega) = \ell^*(v_1)$. Therefore, the spectral algebra may be considered as a *continuous time analogue* of the Cuntz algebra [Cun77].

It should be clear what is the analogue of the spectral algebra for a product system $E^{\odot}$ of Hilbert $\mathcal{B}$–$\mathcal{B}$–modules E_t, acting as an algebra of operators on the 'direct integral' $L^2(E^{\odot})$ of the Hilbert modules E_t. However, a direct integral, in particular, a direct integral of Hilbert modules is connected with technical problems. Therefore, we start with a more algebraic version using direct sums rather than direct integrals, which works for arbitrary product systems, and then see how it can be translated to continuous time, where we require a unit fulfilling a certain measurability condition.

So let us equip $\mathbb{T}$ with the counting measure. Denote by $\ell^2(E^\odot)$ the direct sum over all E_t (i.e. the direct integral with respect to the counting measure). We consider the elements of $\ell^2(E^\odot)$ as functions on $\mathbb{T}$. Denote by $\ell^1(E^\odot)$ the set of all 'functions' $v\colon t \mapsto v(t) \in E_t$ fulfilling $\sum\limits_{t\in\mathbb{T}} \|v_t\| < \infty$. Define for each $v \in \ell^1(E^\odot)$ the *generalized creator* $\widehat{\ell}^*(v) \in \mathcal{B}^a(\ell^2(E^\odot))$, by setting

$$[\widehat{\ell}^*(v)f](t) = \sum_{s\colon 0\leq s\leq t} v(s) \odot f(t-s)$$

for all $f \in \ell^2(E^\odot)$. We define the *discrete spectral algebra* $C(E^\odot)$ associated with $E^\odot$ as the pre–C^*–algebra generated by $\widehat{\ell}^*(\ell^1(E^\odot))\widehat{\ell}(\ell^1(E^\odot))$. Unlike the continuous version, our discrete version contains $\widehat{\ell}^*(\ell^1(E^\odot)) = \widehat{\ell}^*(\ell^1(E^\odot))\widehat{\ell}(\omega)$ where $\omega = \mathbf{1} \in \mathcal{B} = E_0 \subset \ell^2(E^\odot)$.

$\ell^2(E^\odot)$ is a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module so that we may identify $\mathcal{B}$ as a unital subalgebra of $\mathcal{B}^a(\ell^2(E^\odot))$. Moreover, $\varphi = \langle \omega, \bullet\omega\rangle$ defines a conditional expectation $\varphi\colon \mathcal{B}^a(\ell^2(E^\odot)) \to \mathcal{B}$.

Our next goal is to point out that the representation module $\ell^2(E^\odot)$ of the spectral algebra plays a crucial role in the Evans-Lewis dilation theorem [EL77]. We do this by rephrasing the original proof in terms of Hilbert modules. First, let us extend $\ell^2(E^\odot)$ to all times t in $\mathbb{R}$ and in $\mathbb{Z}$, respectively, (i.e. the *Grothendieck group* $\widehat{\mathbb{T}}$ of $\mathbb{T}$). We set $E_t = E_0 = \mathcal{B}$ for $t < 0$ and define the pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module $\widehat{\ell^2(E^\odot)} = \bigoplus\limits_{t\in\widehat{\mathbb{T}}} E_t$.

We consider the C^*–algebra $\mathfrak{F}^\infty(\widehat{\mathbb{T}}, \mathcal{B})$ and the $\mathcal{B}$–$\mathfrak{F}^\infty(\widehat{\mathbb{T}}, \mathcal{B})$–module $E^{\widehat{\mathbb{T}}}$ of all bounded functions $\widehat{\mathbb{T}} \to \mathcal{B}$ and $x\colon t \mapsto E_t$ $(t \in \widehat{\mathbb{T}})$, respectively. By setting

$$\langle x, y\rangle(t) = \langle x(t), y(t)\rangle$$

we turn $E^{\widehat{\mathbb{T}}}$ into a pre-Hilbert $\mathcal{B}$–$\mathfrak{F}^\infty(\widehat{\mathbb{T}}, \mathcal{B})$–module.

We set $\widehat{\mathcal{B}} = \bigoplus\limits_{t\in\widehat{\mathbb{T}}} \mathcal{B}$ and consider it as a pre-Hilbert $\mathfrak{F}^\infty(\widehat{\mathbb{T}}, \mathcal{B})$–$\mathcal{B}$–module in an obvious manner. (The completion of $\widehat{\mathcal{B}}$ is just the exterior tensor product $\mathcal{B} \mathbin{\bar{\otimes}} \ell^2(\widetilde{\mathbb{T}}) = \ell^2(\widetilde{\mathbb{T}}, \mathcal{B})$.) The time shift $\mathfrak{s}_t$ $(t \in \widehat{\mathbb{T}})$, defined by sending $\bigl(b_s\bigr)_{s\in\widehat{\mathbb{T}}}$ to $\bigl(b_{s+t}\bigr)_{s\in\widehat{\mathbb{T}}}$ defines a unitary group on $\widehat{\mathcal{B}}$. Of course, for $\widehat{\mathbb{T}} = \mathbb{R}$ this group is not continuous, neither uniformly continuous, nor in a weak topology.

Clearly, we have $E^{\widehat{\mathbb{T}}} \odot \widehat{\mathcal{B}} = \widehat{\ell^2(E^\odot)}$. Suppose $\xi^\odot$ is a unital unit for $E^\odot$. The family $\widehat{\xi}^\odot = \bigl(\widehat{\xi}_t\bigr)_{t\in\widehat{\mathbb{T}}}$ with $\widehat{\xi}_t = \xi_t$ for $t > 0$ and $\widehat{\xi}_t = \mathbf{1}$ otherwise, is a unit vector in $E^{\widehat{\mathbb{T}}}$. (This is not to be confused with the several $\widehat{\xi}$ which appeared in Section 12.3.) Therefore, the mapping $\widehat{\xi} \odot \mathsf{id}\colon \widehat{\mathcal{B}} \to E^{\widehat{\mathbb{T}}} \odot \widehat{\mathcal{B}} = \widehat{\ell^2(E^\odot)}$ is an (adjointable) isometry.

12.5.2 Observation. Let $v\colon E \to F$ be an adjointable isometry between pre-Hilbert modules. Then $u \mapsto vuv^* + (\mathbf{1} - vv^*)$ defines a group homomorphism from the unitary group $\mathcal{U}(\mathcal{B}^a(E))$ into the unitary group $\mathcal{U}(\mathcal{B}^a(F))$.

We obtain the discrete analogue of what [EL77, Theorem 13.1] asserts for a family of completely positive mappings on $\mathcal{B}$ indexed by $\mathbb{R}$ in a pre-Hilbert module version.

12.5.3 Theorem. *Let T be a unital CP-semigroup on a C^*–algebra $\mathcal{B}$ with GNS-system $E^\odot$ and unital generating unit $\xi^\odot$ such that $T^\xi = T$. Set $T_t = \mathsf{id}$ for $t < 0$. Then*

$$u_t = (\widehat{\xi} \odot \mathsf{id})\mathfrak{s}_t(\widehat{\xi}^* \odot \mathsf{id}) + (\mathbf{1} - \widehat{\xi}\widehat{\xi}^* \odot \mathsf{id})$$

defines a unitary group on $\widehat{\ell^2(E^\odot)}$ such that $\langle \widehat{\xi}_0, u_t b u_t^* \widehat{\xi}_0 \rangle = T_t(b)$.

12.5.4 Remark. Let $t \geq 0$. Then u_{-t} maps elements of $\widehat{\ell^2(E^\odot)}$ which are 0 for $s < 0$ to such elements which are 0 for $s < t$. If we apply an element of $\mathcal{B}$, we do not change this. Application of u_t sends back the result to a family which vanishes for $s < 0$. In other words, $u_t \bullet u_{-t}$ restricts to an E_0–semigroup $\widehat{\vartheta}$ on $\mathcal{B}^a(\ell^2(E^\odot))$. This semigroup sends the unit of $\mathcal{B}$ to $\mathbf{1}$ so that also $(\ell^2(E^\odot), \widehat{\vartheta}, \mathfrak{i} = \mathsf{id} \upharpoonright \mathcal{B}, \widehat{\xi}_0)$ is a unital dilation of T.

12.5.5 Remark. Notice that the submodule $\bigoplus\limits_{t \in \widehat{\mathbb{T}}} \breve{E}_t$ (with $\breve{E}_t = \breve{E}_0 = \mathcal{B}$ for $t < 0$) of $\widehat{\ell^2(E^\odot)}$ is the GNS-module of the completely positive mapping

$$\widehat{T}\colon \mathcal{B} \longrightarrow \mathfrak{F}^\infty(\widehat{\mathbb{T}}, \mathcal{B}), b \longmapsto \big(T_t(b)\big)_{t \in \widehat{\mathbb{T}}}.$$

The mapping $\widehat{T}$ (and its Stinespring construction) and Observation 12.5.2 are also the corner points of the original proof in [EL77]. In contrast with [EL77], where the semigroup structure of the family T plays no role, we are able to describe precisely the space which is *generated* by the time shift and $\mathcal{B}$ in terms of product systems. It is just $\widehat{\ell^2(E^\odot)}$ (when $E^\odot$ is the product system of the minimal dilation).

The idea of Observation 12.5.2 is to send a unitary on E to a unitary on $vE \subset F$, and then to extend it by $\mathbf{1}$ on the orthogonal complement $\ker v^*$ of vE. This shows that the dilation in Theorem 12.5.3 has nothing to do with the *ampliation* of an operator on a factor in a tensor product of Hilbert spaces (or modules) as explained in Sections 12.2 and 6.3. It is not compatible with usual notions of filtration.

The discrete spectral algebra we constructed for arbitrary product systems. Without further technical conditions it seems impossible to go beyond the counting measure. For Theorem 12.5.3 we required a unit giving back T. (The construction of the dilation also works for product systems, being bigger than the GNS-system.) If there is a (unital) unit, then we may embed all E_t into the one-sided inductive limit E and, finally, we have the possibility to pose easily meaurability conditions. (Cf. also the discussion of type II systems in Section 15.2.)

Let us restrict to the case $\mathbb{T} = \mathbb{R}_+$ and let us come back to the Lebesgue measure. In the sequel, we speak always about Hilbert modules. We may construct $L^2(\mathbb{R}_+, E)$ (as analogue of $\ell^2(\mathbb{R}_+, E)$) and we want to identify the subspace of $L^2(\mathbb{R}_+, E)$ consisting of 'functions' f with $f(t) \in E_t$ almost surely (which would be the analogue of $\ell^2(E^\odot)$). A 'reasonable' way to identify these elements is the requirement that the projection p_t (lifted to $L^2(\mathbb{R}_+, E)$ does not change $I\!I_{[0,t]} f$ for all $t \in \mathbb{R}_+$. We denote this space by $L^2(E^\odot)$.

The easiest way to show that $L^2(E^\odot)$ contains many elements, is to construct them. For instance, for any partition $\mathfrak{t} \in \mathbb{P}$ we define a projection $p_\mathfrak{t}$ on the right continuous step functions $\mathfrak{S}^r(\mathbb{R}_+, E)$ by setting $p_\mathfrak{t}(x I\!I_{[s,t)}) = p_{t_{i-1}} x I\!I_{[s,t)}$ for $[s,t) \subset [t_{i-1}, t_i)$. The net $\bigl(p_\mathfrak{t}\bigr)$ of projections is increasing over $\mathbb{P}$ and, therefore, converges to a projection p at least on the strong closure $L^{2,s}(\mathbb{R}_+, E)$. We know neither, whether p leaves invariant $L^2(\mathbb{R}_+, E)$, nor, whether its range contains $L^2(E^\odot)$. We expect that these questions have an affirmative answer under the measurability condition that for all $x \in E$ the function $t \mapsto p_t x \in E$ is (norm) measurable. Observe that this is a manifest condition on the unit, because it is equivalent to the condition that the function $s \mapsto (\xi_s \xi_s^* \odot \mathsf{id}_{E_{t-s}}) x_t$ is mesaurable for all $t \in \mathbb{T}, x_t \in E_t$.

The same type of questions has to be answered for the subspace $L^1(E^\odot)$ (the anlaogue of $\ell^1(E^\odot)$ of $L^2(E^\odot)$ being defined in an analogue manner. For step functions v, f the operator

$$\bigl[\widehat{\ell}^*(v) f\bigr](t) \;=\; \int_0^t v(s) \odot f(t-s)\, ds$$

is perfectly well defined and sends step function in $L^1(E^\odot)$ and $L^2(E^\odot)$, respectively, to a continuous function in $L^2(E^\odot)$. Obviously, $\widetilde{\ell}^*(v)$ is bounded on step functions. Moreover, it has an adjoint, namely,

$$\bigl[\widehat{\ell}(v) f\bigr](t) \;=\; \int_0^\infty (v(s)^* \odot \mathsf{id}_{E_t}) f(s+t)\, ds.$$

Thus, we may propose as a preliminary definition of ***spectral algebra*** to consider the C^*–algebra generated by operators $\widehat{\ell}^*(v)\widehat{\ell}^*(v')$ on the closure of the step functions in $L^2(E^\odot)$ to step functions v, v'. We postpone the solution of the mentioned technical problems to future work. This should also contain a 'continuous time' analogue of Theorem 12.5.3 and the correct interpration of the representation space of the Evans-Lewis dilation as (the strong closure of) the representation space of the spectral algebra.

Chapter 13

Type I_c product systems

After the “intermezzo” about semigroups we come back to type I product systems. Our final goal in this chapter is to show that type I_c^s product systems of von Neumann modules are time ordered Fock modules. This is the analogue of Arveson's result that the type I Arveson systems are symmetric Fock spaces [Arv89a]. We follow Barreto, Bhat, Liebscher and Skeide [BBLS00].

In Section 13.1 we show that a type I_c product system is contained in a time ordered product system, if it contains at least one (continuous) central unit. More precisely, we show that existence of a central unit implies that the generator of the associated CPD-semigroup has Christensen-Evans form (Theorem 13.1.2). This enables us to to give an explicit embedding into a time ordered Fock module (Corollary 13.1.3). In Section 13.2 we study the continuous endomorphisms of the time ordered Fock module (Theorem 13.2.1). We find its projection morphisms (Corollary 13.2.5)and provide a necessary and sufficient criterion for that a given set of (continuous) units is (strongly) totalizing (Theorem 13.2.7). The basic idea (used by Bhat [Bha01] for a comparable purpose) is that a product system of von Neumann modules is generated by a set of units, if and only if there is precisely one projection endomorphism (namely, the idenity morphism), leaving the units of this set invariant. In Section 13.3 we utilize the Christensen-Evans Lemma A.6.1 to show that the GNS-system of a uniformly continuous CPD-semigroup has a central unit and, therefore, is contained in a time ordered Fock module by Section 13.1. By Section 13.2 these units generate a whole time ordered subsystem. We point out that the result by Christensen and Evans is equivalent to show existence of a central unit in any type I_{cn}^s system.

In Sections 13.4 and 13.5 we provide suplementary material. We analyze the positivity structure of contractive positive morphisms of general type I^s product systems and of the time ordered Fock module, and we show how the full information about a the GNS-system of a CPD-semigroup can be put into a single CP-semigroup on a bigger algebra.

13.1 Central units in type I$_c$ product systems

In this section we show that type I$_c$ product systems are contained in time ordered Fock modules, if at least one of the continuous units is central. So let $\omega^\odot$ be a central unit and let $\xi^\odot$ be any other unit. Then

$$\mathfrak{U}_t^{\xi,\omega}(b) \;=\; \langle \xi_t, b\omega_t\rangle \;=\; \langle \xi_t, \omega_t\rangle b \;=\; \mathfrak{U}_t^{\xi,\omega}(\mathbf{1})b \tag{13.1.1}$$

and

$$\mathfrak{U}_{s+t}^{\xi,\omega}(\mathbf{1}) \;=\; \mathfrak{U}_t^{\xi,\omega}(\mathfrak{U}_s^{\xi,\omega}(\mathbf{1})) \;=\; \mathfrak{U}_t^{\xi,\omega}(\mathbf{1})\mathfrak{U}_s^{\xi,\omega}(\mathbf{1}).$$

In other words, $\mathfrak{U}^{\xi,\omega}(\mathbf{1})$ is a semigroup *in* $\mathcal{B}$ and determines $\mathfrak{U}^{\xi,\omega}$ by (13.1.1). In particular, $\mathfrak{U}^{\omega,\omega}(\mathbf{1})$ is a semigroup in $C_{\mathcal{B}}(\mathcal{B})$. If $\omega^\odot$ is continuous, then all $\mathfrak{U}_t^{\omega,\omega}(\mathbf{1})$ are invertible. Henceforth, we may assume without loss of generality that $\omega^\odot$ is unital, i.e. $T^\omega = \mathsf{id}$ is the trivial semigroup.

13.1.1 Lemma. *Let $\omega^\odot$ be a central unital unit and let $\xi^\odot$ be another unit for a product system $E^\odot$ such that the CPD-semigroup $\mathfrak{U} \upharpoonright \{\omega^\odot, \xi^\odot\}$ is uniformly continuous. Let β denote the generator of the semigroup $\mathfrak{U}^{\omega,\xi}(\mathbf{1})$ in $\mathcal{B}$, i.e. $\mathfrak{U}_t^{\omega,\xi}(\mathbf{1}) = e^{t\beta}$, and let $\mathcal{L}^\xi$ denote the generator of the CP-semigroup T^ξ on $\mathcal{B}$. Then the mapping*

$$b \;\longmapsto\; \mathcal{L}^\xi(b) - b\beta - \beta^* b \tag{13.1.2}$$

is completely positive, i.e. $\mathcal{L}^\xi$ is a CE-generator.

Proof. We consider the CP-semigroup $\mathfrak{U}^{(2)} = \big(\mathfrak{U}_t^{(2)}\big)_{t\in\mathbb{R}_+}$ on $M_2(\mathcal{B})$ with $\mathfrak{U}_t^{(2)} = \left(\begin{smallmatrix}\mathfrak{U}_t^{\omega,\omega} & \mathfrak{U}_t^{\omega,\xi}\\ \mathfrak{U}_t^{\xi,\omega} & \mathfrak{U}_t^{\xi,\xi}\end{smallmatrix}\right)$ whose generator is

$$\mathfrak{L}^{(2)}\begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix} \;=\; \frac{d}{dt}\Big|_{t=0}\begin{pmatrix}\mathfrak{U}_t^{\omega,\omega}(b_{11}) & \mathfrak{U}_t^{\omega,\xi}(b_{12})\\ \mathfrak{U}_t^{\xi,\omega}(b_{21}) & \mathfrak{U}_t^{\xi,\xi}(b_{22})\end{pmatrix} \;=\; \begin{pmatrix} 0 & b_{12}\beta\\ \beta^* b_{21} & \mathcal{L}^\xi(b_{22})\end{pmatrix}.$$

By Theorem 5.4.7 and Lemma 5.4.6 $\mathfrak{L}^{(2)}$ is conditionally completely positive. Let $A_i = \left(\begin{smallmatrix}0 & 0\\ a_i & a_i\end{smallmatrix}\right)$ and $B_i = \left(\begin{smallmatrix}0 & -b_i\\ 0 & b_i\end{smallmatrix}\right)$. Then $A_iB_i = 0$, i.e. $\sum\limits_i A_iB_i = 0$, so that

$$\begin{aligned} 0 \;\le\; \sum_{i,j} B_i^*\mathfrak{L}^{(2)}(A_i^*A_j)B_j \;&=\; \sum_{i,j} B_i^*\begin{pmatrix} 0 & a_i^*a_j\beta\\ \beta^*a_i^*a_j & \mathcal{L}^\xi(a_i^*a_j)\end{pmatrix} B_j\\ &=\; \sum_{i,j}\begin{pmatrix} 0 & 0\\ 0 & b_i^*(\mathcal{L}^\xi(a_i^*a_j) - a_i^*a_j\beta - \beta^*a_i^*a_j)b_j\end{pmatrix}.\end{aligned}$$

This means that (13.1.2) is completely positive. ■

Now we show how the generator of CPD-semigroups (i.e. many units) in product systems with a central unit boils down to the generator $\mathcal{L}^\xi$ of a CP-semigroup (i.e. a single unit) as in Lemma 13.1.1. Once again, we exploit Examples 1.7.7 and 4.2.12 as basic idea.

13.1.2 Theorem. *Let $E^\odot$ be a product system with a subset $S \subset \mathcal{U}(E^\odot)$ of units and a central (unital) unit $\omega^\odot$ such that $\mathfrak{U} \upharpoonright S \cup \{\omega^\odot\}$ is a uniformly continuous CPD-semigroup. Then the generator $\mathfrak{L}$ of the (uniformly continuous) CPD-semigroup $\mathfrak{T} = \mathfrak{U} \upharpoonright S$ is a CE-generator.*

PROOF. For $\xi^\odot \in S$ denote by $\beta_\xi \in \mathcal{B}$ the generator of the semigroup $\mathfrak{U}^{\omega,\xi}(\mathbf{1})$ in $\mathcal{B}$. We claim as in Lemma 13.1.1 that the kernel $\mathfrak{L}_0$ on S defined by setting

$$\mathfrak{L}_0^{\xi,\xi'}(b) \;=\; \mathfrak{L}^{\xi,\xi'}(b) - b\beta_{\xi'} - \beta_\xi^* b$$

(for $(\xi^\odot, \xi'^\odot) \in S \times S$) is completely positive definite, what shows the theorem. By Lemma 5.2.1(4) it is equivalent to show that the mapping $\mathfrak{L}_0^{(n)}$ on $M_n(\mathcal{B})$ defined by setting

$$\big(\mathfrak{L}_0^{(n)}(B)\big)_{ij} \;=\; \mathfrak{L}^{\xi^i,\xi^j}(b_{ij}) - b_{ij}\beta_{\xi^j} - \beta_{\xi^i}^* b_{ij}$$

is completely positive for all choices of $n \in \mathbb{N}$ and $\xi^{i\odot} \in S$ $(i = 1, \dots, n)$.

First, observe that by Example 4.2.12 $M_n(E^\odot) = \big(M_n(E_t)\big)_{t\in\mathbb{T}}$ is a product system of $M_n(\mathcal{B})$–$M_n(\mathcal{B})$–modules. Clearly, the diagonal matrices $\Xi_t \in M_n(E_t)$ with entries $\xi_t^i\delta_{ij}$ form a unit $\Xi^\odot$ for $M_n(E^\odot)$. Moreover, the unit $\Omega^\odot$ with entries $\delta_{ij}\omega^\odot$ is central and unital. For the units $\Omega^\odot$ and $\Xi^\odot$ the assumptions of Lemma 13.1.1 are fulfilled. The generator $\widehat{\beta}$ of the semigroup $\mathfrak{U}^{\Omega,\Xi}(\mathbf{1})$ is the matrix with entries $\delta_{ij}\beta_{\xi^i}$. Now (13.1.2) gives us back $\mathfrak{L}_0^{(n)}$ which, therefore, is completely positive. ■

13.1.3 Corollary. *The GNS-system $E^\odot$ of $\mathfrak{T}$ is embeddable into a time ordered product system. More precisely, let (F, ζ) be the (completed) Kolmogorov decomposition for the kernel $\mathfrak{L}_0$ with the canonical mapping $\zeta \colon \xi^\odot \mapsto \zeta_\xi$. Then*

$$\xi^\odot \;\longmapsto\; \xi^\odot(\beta_\xi, \zeta_\xi)$$

extends as an isometric morphism $E^\odot \to \mathrm{I\Gamma}^\odot(F)$.

Notice that (in the notations of Theorem 13.1.2) the preceding morphism may be extended to $E^{S_\omega \odot}$ where $S_\omega = S \cup \{\omega^\odot\}$, by sending $\omega^\odot \in \mathcal{U}(E^\odot)$ to $\omega^\odot \in \mathcal{U}_c(F)$.

13.2 Morphisms of the time ordered Fock module

In the preceding section we found that, roughly speaking, type I product systems with a central unit may be embedded into a time ordered Fock module. In this section we want to find criteria to decide, whether this Fock module is generated by such a subsystem. To that goal, we study the endomorphisms of $\mathbb{I}\Gamma^{\odot}(F)$.

After establishing the general form of (possibly unbounded, but adjointable) continuous morphisms, we find very easily characterizations of isometric, coisometric, unitary, positive, and projection morphisms. The generalizations of ideas from Bhat's "cocycle computations" in [Bha01] are straightforward. Contractivity requires slightly more work and, because we do not need it, we postpone it to Section 13.4. Then we use the characterization of projection morphisms to provide a criterion for checking, whether a set of (continuous) units is strongly totalizing for a time ordered product system, or not.

Besides (11.1.3), the crucial property of a morphism is to consist of adjointable mappings. Adjointability, checked on some total subset of vectors, assures well-definedness by Corollary 1.4.3. If $w^{\odot}$ is a morphism except that the w_t are allowed to be unbounded, then we speak of a *possibly unbounded* morphism.

Recall that a continuous morphism $w^{\odot}$ of time ordered Fock modules corresponds to a transformation

$$\xi^{\odot}(\beta,\zeta) \;\longmapsto\; \xi^{\odot}(\gamma_w(\beta,\zeta),\eta_w(\beta,\zeta)) \tag{13.2.1}$$

among sets of continuous units. We want to know which transformations of the parameter space $\mathcal{B}\times F$ of the continuous units define operators w_t by extending (13.2.1) to vectors of the form (11.2.3).

13.2.1 Theorem. *Let F and F' be Hilbert $\mathcal{B}$–$\mathcal{B}$–modules. By setting*

$$w_t\xi_t(\beta,\zeta) \;=\; \xi_t\big(\,\gamma+\beta+\langle\eta,\zeta\rangle\,,\,\eta'+a\zeta\,\big) \tag{13.2.2}$$

we establish a one-to-one correspondence between possibly unbounded continuous morphisms $w^{\odot}=\big(w_t\big)_{t\in\mathbb{R}_+}$ from $\mathbb{I}\Gamma^{\mathfrak{U}_c\odot}(F)$ to $\mathbb{I}\Gamma^{\mathfrak{U}_c\odot}(F')$ and matrices

$$\Gamma \;=\; \begin{pmatrix}\gamma & \eta^*\\ \eta' & a\end{pmatrix} \;\in\; \mathcal{B}^{a,bil}(\mathcal{B}\oplus F,\mathcal{B}\oplus F') \;=\; \begin{pmatrix}C_{\mathcal{B}}(\mathcal{B}) & C_{\mathcal{B}}(F)^*\\ C_{\mathcal{B}}(F') & \mathcal{B}^{a,bil}(F,F')\end{pmatrix}.$$

Moreover, the adjoint of $w^{\odot}$ is given by the adjoint matrix $\Gamma^*=\begin{pmatrix}\gamma^* & \eta'^*\\ \eta & a^*\end{pmatrix}$

Proof. From bilinearity and adjointability of w_t we have

$$\Big\langle\xi_t(\beta,\zeta)\;,\;b\xi_t\big(\gamma_{w^*}(\beta',\zeta'),\eta_{w^*}(\beta',\zeta')\big)\Big\rangle \;=\; \Big\langle\xi_t\big(\gamma_w(\beta,\zeta),\eta_w(\beta,\zeta)\big)\;,\;b\xi_t(\beta',\zeta')\Big\rangle \tag{13.2.3}$$

for all $t\in\mathbb{R}_+$, $\beta,\beta'\in\mathcal{B}$, $\zeta\in F$, $\zeta'\in F'$ or, equivalently, by differentiating at $t=0$ and (7.4.1)

$$\big\langle\zeta,b\eta_{w^*}(\beta',\zeta')\big\rangle+b\gamma_{w^*}(\beta',\zeta')+\beta^*b \;=\; \big\langle\eta_w(\beta,\zeta),b\zeta'\big\rangle+b\beta'+\gamma_w(\beta,\zeta)^*b. \tag{13.2.4}$$

It is easy to check that validity of (13.2.2) implies (13.2.4) and, henceforth, (13.2.3). Therefore, (13.2.2) defines a unique adjointable bilinear operator $\widehat{w}_t$ from the bimodule generated by all $\xi_t(\beta,\zeta)$ ($\beta \in \mathcal{B}, \zeta \in F$) (i.e. the Kolmogorov decomposition of $\mathfrak{U}_t \upharpoonright \mathfrak{U}_c(F)$) into $\mathrm{I\Gamma}_t^{\mathfrak{U}_c}(F')$. It is clear that (as in the proof of Proposition 13.4.2) the $\widehat{w}_t$ define an operator on $\mathrm{I\Gamma}_t^{\mathfrak{U}_c}(F)$ via Proposition A.10.3, that this operator is the extension of (13.2.2) to vectors of the form (11.2.3), and that the operators fulfill (11.1.3). We put $w_0 = \mathsf{id}_{\mathcal{B}}$, and the w_t form a morphism.

It remains to show that (13.2.2) is also a necessary condition on the form of the functions $\gamma_w \colon \mathcal{B} \times F \to \mathcal{B}$ and $\eta_w \colon \mathcal{B} \times F \to F'$. Putting $\zeta = 0, \zeta' = 0$ in (13.2.4), we find

$$b\gamma_{w^*}(\beta',0) + \beta^* b \;=\; b\beta' + \gamma_w(\beta,0)^* b. \tag{13.2.5}$$

Putting also $\beta = \beta' = 0$ and $b = \mathbf{1}$, we find $\gamma_{w^*}(0,0)^* = \gamma_w(0,0)$. We denote this element of $\mathcal{B}$ by γ. Reinserting arbitrary $b \in \mathcal{B}$, we find that $\gamma \in C_{\mathcal{B}}(\mathcal{B})$. Reinserting arbitrary $\beta \in \mathcal{B}$, we find $\gamma_w(\beta,0) = \gamma + \beta$ and, similarly, $\gamma_{w^*}(\beta',0) = \gamma^* + \beta'$.

Putting in 13.2.4 $\zeta = 0$, inserting $\gamma_w(\beta,0)^*$ and subtracting $\beta^* b$, we obtain

$$b\gamma_{w^*}(\beta',\zeta') \;=\; \big\langle \eta_w(\beta,0), b\zeta' \big\rangle + b\beta' + \gamma^* b \;=\; \big\langle \eta_w(\beta,0), b\zeta' \big\rangle + b\gamma_{w^*}(\beta',0)$$

(recall that γ commutes with b), or

$$b\gamma_{w^*}(\beta',\zeta') - b\gamma_{w^*}(\beta',0) \;=\; \big\langle \eta_w(\beta,0), b\zeta' \big\rangle. \tag{13.2.6}$$

We obtain a lot of information. Firstly, the left-hand side and the right-hand side cannot depend on β' and β, respectively. Therefore, $\eta_w(\beta,0) = \eta_w(0,0)$ which we denote by $\eta' \in F'$. Secondly, we put $b = \mathbf{1}$ and multiply again with an arbitrary $b \in \mathcal{B}$ from the right. Together with the original version of (13.2.6) we obtain that $\eta' \in C_{\mathcal{B}}(F')$. Finally, with $b = \mathbf{1}$ we obtain $\gamma_{w^*}(\beta',\zeta') = \gamma^* + \beta' + \langle \eta', \zeta' \rangle$. A similar computation starting from $\zeta' = 0$, yields $\eta_{w^*}(\beta',0) = \eta_{w^*}(0,0) = \eta$ for some $\eta \in C_{\mathcal{B}}(F)$ and $\gamma_w(\beta,\zeta) = \gamma + \beta + \langle \eta, \zeta \rangle$.

Inserting the concrete form of $\gamma_{w^{(*)}}$ into (13.2.4) and subtracting $\gamma^* b + b\beta' + \beta^* b = b\gamma^* + b\beta' + \beta^* b$, we obtain

$$\big\langle \zeta, b\eta_{w^*}(\beta',\zeta') \big\rangle + b\langle \eta', \zeta' \rangle \;=\; \big\langle \eta_w(\beta,\zeta), b\zeta' \big\rangle + \langle \zeta, \eta \rangle b. \tag{13.2.7}$$

Again, we conclude that $\eta_{w^*}(\beta',\zeta') = \eta_{w^*}(0,\zeta')$ and $\eta_w(\beta,\zeta) = \eta_w(0,\zeta)$ cannot depend on β' and β, respectively. Putting $b = \mathbf{1}$, we find $\big\langle \zeta\,,\, \eta_{w^*}(0,\zeta') - \eta \big\rangle = \big\langle \eta_w(0,\zeta) - \eta'\,,\, \zeta' \big\rangle$. It follows that the mapping $a \colon \zeta \mapsto \eta_w(0,\zeta) - \eta'$ has an adjoint, namely, $a^* \colon \zeta' \mapsto \eta_{w^*}(0,\zeta') - \eta$. Since F and F' are complete, a is an element of $\mathcal{B}^a(F,F')$. Inserting a and a^* in (13.2.7), and taking into account that η and η' are central, we find that $a \in \mathcal{B}^{a,bil}(F,F')$, and $\eta_w(\beta,\zeta) = \eta' + a\zeta$ and $\eta_{w^*}(\beta',\zeta') = \eta + a^*\zeta'$ as desired. ■

13.2.2 Corollary. *A (possibly unbounded) continuous endomorphism $w^\odot$ of $\mathbb{\Gamma}^{\mathfrak{U}_c\odot}(F)$ is self-adjoint, if and only if Γ is self-adjoint.*

Of course, the correspondence is not functorial in the sense that $ww'^\odot = \big(w_t w'_t\big)_{t\in\mathbb{R}_+}$ is not given by $\Gamma\Gamma'$. However, we easily check the following.

13.2.3 Corollary. *Let $w^\odot$ be a morphism with matrix Γ. Then*

$$\begin{pmatrix} \mathbf{1} & 0 & 0 \\ \gamma & \mathbf{1} & \eta^* \\ \eta' & 0 & a \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \beta \\ \zeta \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \gamma_w(\beta,\zeta) \\ \zeta_w(\beta,\zeta) \end{pmatrix} \quad \text{and the mapping} \quad w^\odot \longmapsto \widehat{\Gamma} = \begin{pmatrix} \mathbf{1} & 0 & 0 \\ \gamma & \mathbf{1} & \eta^* \\ \eta' & 0 & a \end{pmatrix}$$

is functorial in the sense that $\widehat{\Gamma}'' = \widehat{\Gamma}\widehat{\Gamma}'$ for $w''^\odot = ww'^\odot$.

13.2.4 Corollary. *The continuous morphism $w^\odot$ with the matrix $\Gamma = \left(\begin{smallmatrix} \gamma & \eta^* \\ \eta' & a \end{smallmatrix}\right)$ is isometric, if and only if a is isometric, $\eta = a^*\eta'$ $(\eta' \in C_{\mathcal{B}}(F'))$ and $\gamma = ih - \frac{\langle \eta',\eta'\rangle}{2}$ $(h = h^* \in C_{\mathcal{B}}(\mathcal{B}))$. It is coisometric, if and only if a is coisometric, $\eta' = a\eta$ $(\eta \in C_{\mathcal{B}}(F))$ and $\gamma = ih - \frac{\langle \eta,\eta\rangle}{2}$ $(h = h^* \in C_{\mathcal{B}}(\mathcal{B}))$. It is unitary (i.e. an isomorphism), if and only if a is unitary, $\eta' = a\eta$ $(\eta \in C_{\mathcal{B}}(F))$ and $\gamma = ih - \frac{\langle \eta,\eta\rangle}{2}$ $(h = h^* \in C_{\mathcal{B}}(\mathcal{B}))$ or, equivalenty, if a is unitary, $\eta = a^*\eta'$ $(\eta' \in C_{\mathcal{B}}(F'))$ and $\gamma = ih - \frac{\langle \eta',\eta'\rangle}{2}$ $(h = h^* \in C_{\mathcal{B}}(\mathcal{B}))$.*

The form of these conditions reminds us very much of the form of the corresponding conditions for solutions of quantum stochastic differential equations; see e.g. Section 16.5. By Theorem 11.6.1 and Example 11.5.2 the morphisms correspond to cocycles, too. These cocycles are, however, local which means that the corresponding cocycle perturbation of the time shift does not change any CP-semigroup T^ξ for any $\xi^\odot$.

After the characterizations of isomorphisms we come to projections. Of course, a projection endomorphism must be self-adjoint and so must be its matrix.

13.2.5 Corollary. *A continuous endomorphism $w^\odot$ of $\mathbb{\Gamma}^{\mathfrak{U}_c\odot}(F)$ is a projection morphism, if and only if its matrix Γ has the form*

$$\Gamma = \begin{pmatrix} -\langle \eta,\eta\rangle & \eta^* \\ \eta & p \end{pmatrix}$$

where p is a projection in $\mathcal{B}^{a,bil}(F)$, and $\eta \in (\mathbf{1}-p)C_{\mathcal{B}}(F)$.

Since a continuous morphism of a product system $\mathbb{\Gamma}^\odot(F)$ or $\mathbb{\Gamma}^{s\odot}(F)$ (or between such) sends continuous units to continuous units, it restricts to a morphism of $\mathbb{\Gamma}^{\mathfrak{U}_c\odot}(F)$ (or between such). Therefore, all characterizations extend to the case of Hilbert modules and the case of von Neumann modules.

Now we characterize strongly totalizing sets of continuous units for time ordered product systems of von Neumann modules. The idea is that, if a set of units is not strongly totalizing, then by Observation 11.1.2 there exists a non-trivial projection morphism onto the subsystem generated by these units. In order to apply our methods we need to know that this morphism is continuous.

13.2.6 Lemma. *Let $p^\odot$ be a projection morphism leaving invariant a non-empty continuous subset $S \subset \mathfrak{U}_c(F)$ of units for $\mathrm{I\!\Gamma}^{s\odot}(F)$ (i.e. $p\xi^\odot = \xi^\odot$ for all $\xi^\odot \in S$). Then $p^\odot$ is continuous.*

Proof. By Lemma 7.4.1(2), the $\mathcal{B}$–weak closure (therefore, *a fortiori* the strong closure) of what a single continuous unit $\xi^\odot(\beta, \zeta) \in S$ generates in a time ordered system contains the unital unit $\xi^\odot\big(-\frac{\langle\zeta,\zeta\rangle}{2}, \zeta\big)$. Therefore, we may assume that S contains a unital unit $\xi^\odot$. Now let $\xi'^\odot$ be an arbitrary unit in $\mathfrak{U}_c(F)$. Then the function $t \mapsto \langle \xi_t, p_t\xi'_t\rangle = \langle p_t\xi_t, \xi'_t\rangle = \langle \xi_t, \xi'_t\rangle$ is continuous. Moreover, we have

$$\langle p_t\xi'_t, p_t\xi'_t\rangle - \langle \xi_t, \xi_t\rangle \;=\; \langle \xi'_t - \xi_t, p_t\xi'_t\rangle + \langle \xi_t, p_t(\xi'_t - \xi_t)\rangle \;\to\; 0$$

for $t \to 0$. From this it follows as, for instance, in (11.6.1) that also the function $t \mapsto \langle p_t\xi'_t, p_t\xi'_t\rangle$ is continuous. By Lemma 11.6.6 also the unit $p\xi'^\odot$ is continuous. As $\xi'^\odot$ was arbitrary, $p^\odot$ is continuous. ■

13.2.7 Theorem. *Let F be a von Neumann $\mathcal{B}$–$\mathcal{B}$–module and let $S \subset \mathfrak{U}_c(F)$ be a continuous subset of units for $\mathrm{I\!\Gamma}^{s\odot}(F)$. Then S is strongly totalizing, if and only if the $\mathcal{B}$–$\mathcal{B}$–submodule*

$$F_0 \;=\; \Big\{\sum_{i=1}^n a_i\zeta_i b_i \;\Big|\; n \in \mathbb{N}\;;\; \zeta_i \in S_F\;;\; a_i, b_i \in \mathcal{B}\colon \sum_{i=1}^n a_i b_i = 0\Big\} \tag{13.2.8}$$

of F is strongly dense in F, where $S_F = \big\{\zeta \in F \mid \exists\, \beta \in \mathcal{B}\colon \xi^\odot(\beta,\zeta) \in S\big\}$.

Proof. Denote by $\mathrm{I\!\Gamma}^{S\odot}$ the strong closure of the product subsystem of $\mathrm{I\!\Gamma}^{s\odot}(F)$ generated by the units in S. We define another $\mathcal{B}$–$\mathcal{B}$–submodule

$$F^0 \;=\; \Big\{\sum_{i=1}^n a_i\zeta_i b_i \;\Big|\; n \in \mathbb{N}\;;\; \zeta_i \in S_F\;;\; a_i, b_i \in \mathcal{B}\Big\}$$

of F. We have $F \supset \overline{F^0}^s \supset \overline{F_0}^s$. Denote by p_0 and p^0 in $\mathcal{B}^{a,bil}(F)$ the projections onto $\overline{F_0}^s$ and $\overline{F^0}^s$, respectively. (Since $\overline{F_0}^s$ and $\overline{F^0}^s$ are complementary by Theorem 3.2.11 and Proposition 1.5.9, the projections exist, and since $\overline{F_0}^s$ and $\overline{F^0}^s$ are $\mathcal{B}$–$\mathcal{B}$–submodules, by Observation 1.6.4 the projections are bilinear.) We have to distinguish three cases.

(i) $F \neq \overline{F^0}^s$. In this case $p^0 \neq \mathbf{1}$ and the matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & p^0 \end{smallmatrix}\right)$ defines a non-trivial projection morphism leaving $\mathrm{I\!\Gamma}^{S\odot}$ invariant.

(ii) $F = \overline{F^0}^s \neq \overline{F_0}^s$. Set $q = \mathbf{1} - p_0$. We may rewrite an arbitrary element of F^0 as

$$\sum_{i=1}^n a_i\zeta_i b_i \;=\; \sum_{i=1}^n (a_i\zeta_i - \zeta_i a_i)b_i + \sum_{i=1}^n (\zeta_i a_i - \zeta a_i)b_i + \zeta\sum_{i=1}^n a_i b_i,$$

where $\zeta \in S_F$ is arbitrary. We find $q\sum_{i=1}^n a_i\zeta_i b_i = q\zeta\sum_{i=1}^n a_i b_i$. Putting $a_i = b_i = \mathbf{1}\delta_{ik}$, we see that the element $\eta = q\zeta$ cannot depend on ζ. Varrying $a_k = b$ for $\zeta_k = \zeta$, we see that $b\eta = \eta b$, i.e. $\eta \in C_{\mathcal{B}}(F)$. Finally, $p_0 \neq \mathbf{1}$ and $\eta \neq 0$. Hence, the matrix $\left(\begin{smallmatrix} -\langle\eta,\eta\rangle & \eta^* \\ \eta & p_0 \end{smallmatrix}\right)$ defines a non-trivial projection morphism leaving $\mathrm{I\!\Gamma}^{S\odot}$ invariant.

(iii) $F = \overline{F^0}^s = \overline{F_0}^s$. Consider the projection morphism with matrix $\left(\begin{smallmatrix} -\langle\eta,\eta\rangle & \eta^* \\ \eta & p \end{smallmatrix}\right)$ and suppose that it leaves $\mathrm{I\!\Gamma}^{S\odot}$ invariant. Then $\zeta = \eta + p\zeta$ for all $\zeta \in S_F$. Since η is in the center, an element in F_0 written as in (13.2.8) does not change, if we replace ζ_i with $p\zeta_i$. It follows $pF = p\overline{F_0}^s = \overline{F_0}^s = F$, whence $p = \mathbf{1}$ and $\eta = (\mathbf{1} - p)\eta = 0$. Therefore, the only (continuous) projection morphism leaving $\mathrm{I\!\Gamma}^{S\odot}$ invariant is the identity morphism. ∎

13.2.8 Corollary. *A single unit $\xi^\odot(\beta,\zeta)$ is totalizing for $\mathrm{I\!\Gamma}^{s\odot}(F)$, if and only if $F = \overline{\mathsf{span}}^s\{(b\zeta - \zeta b)b' \colon b, b' \in \mathcal{B}\}$.*

13.2.9 Remark. In the case where $\mathcal{B} = \mathcal{B}(G)$ for some separable Hilbert space G we have $F = \mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$ where $\mathfrak{H} \cong \mathsf{id} \otimes \mathfrak{H} = C_{\mathcal{B}}(F)$ is the center of F and $\zeta = \sum_n b_n \otimes e_n$ for some ONB $\left(e_n\right)_{n\in N}$ (N a subset of $\mathbb{N}$) and $b_i \in \mathcal{B}$ such that $\sum_n b_n^* b_n < \infty$. The condition stated in Bhat [Bha01], which, therefore, should be equivalent to our cyclicity condition in Corollary 13.2.8, asserts that the set $\{\mathbf{1}, b_1, b_2, \ldots\}$ should be linearly independent in a certain sense (stronger than usual linear independence).

13.2.10 Example. It is an easy exercise to check that the inner derivation to the vector ξ in the von Neumann module $\mathcal{B}(G, G \bar{\otimes} G)$ from Example 4.4.13 generates it as a von Neumann module, if and only if G is infinite-dimensional.

13.2.11 Observation. We see explicitly that the property of the set S to be totalizing or not is totally independent of the parameters β of the units $\xi^\odot(\beta,\zeta)$ in S. Of course, we knew this before from the proof of Lemma 13.2.6.

13.2.12 Remark. We may rephrase Step (ii) as $\overline{F^0}^s = \overline{F_0}^s \oplus q\mathcal{B}$ for some central projection in $q \in \mathcal{B}$ such that $q\mathcal{B}$ is the strongly closed ideal in $\mathcal{B}$ generated by $\langle\eta,\eta\rangle$. By the same argument as in Step (iii) we obtain the most important consequence.

13.2.13 Corollary. *The mapping*

$$\xi^\odot(\beta,\zeta) \longmapsto \xi^\odot\bigl(\beta + \tfrac{\langle\eta,\eta\rangle}{2}, \zeta - \eta\bigr)$$

(which by (7.4.1) *is isometric) extends as an isomorphism from the subsystem of* $\mathrm{I\Gamma}^{s\odot}(F)$ *generated by* S *onto* $\mathrm{I\Gamma}^{s\odot}(\overline{F_0}^s)$*. In other words, each strongly closed product subsystem of the time ordered product system* $\mathrm{I\Gamma}^{s\odot}(F)$ *of von Neumann modules generated by a subset* $S \subset \mathfrak{U}_c(F)$ *of continuous units, is a time ordered product system of von Neumann modules over a von Neumann submodule of* F*.*

13.2.14 Remark. If S contains a unit $\xi^\odot(\beta_0,\zeta_0)$ with $\zeta_0 = 0$ (in other words, as for the condition in Theorem 13.2.7 we may forget about β_0, if S contains the vacuum unit $\omega^\odot = \xi^\odot(0,0)$), then $F_0 = F^0$. (Any value of $\sum\limits_{i=1}^n a_i b_i$ may compensated in $\sum\limits_{i=0}^n a_i b_i$ by a suitable choice of a_0, b_0, because $a_0\zeta_0 b_0$ does not contribute to the sum $\sum\limits_{i=0}^n a_i\zeta_i b_i$.) We recover Theorem 7.4.3.

13.3 Type I^s_{cn} product systems

13.3.1 Theorem. *Let* $T = \bigl(T_t\bigr)_{t\in\mathbb{R}_+}$ *be a normal uniformly continuous CP-semigroup on a von Neumann algebra* $\mathcal{B}$*. Let* F*,* $\zeta \in F$*, and* $\beta \in \mathcal{B}$ *be as in Theorem A.6.3 (by [CE79]), i.e.* F *is a von Neumann* $\mathcal{B}$*–*$\mathcal{B}$*–module such that* $F = \overline{\mathsf{span}}^s\{(b\zeta - \zeta b)b' : b, b' \in \mathcal{B}\}$ *and* $T^{(\beta,\zeta)} = T$*. Then the strong closure of the GNS-system of* T *is (up to isomorphism)* $\mathrm{I\Gamma}^{s\odot}(F)$ *and the generating unit is* $\xi^\odot(\beta,\zeta)$*. Here* F *and* $\xi^\odot(\beta,\zeta)$ *are determined up to unitary isomorphism.*

Proof. This is a direct consequence of Theorem A.6.3 and Corollary 13.2.8 of Theorem 13.2.7. ■

Proof of Theorem 5.4.14. By Theorem 13.3.1 the subsystem of the GNS-system generated by a single unit in S has a central (continuous) unit. By Theorem 13.1.2 the generator of $\mathfrak{T}$ is a CE-generator. The uniqueness statement follows as in Corollary 13.2.13 from the construction of the module $\overline{F_0}^s$. ■

13.3.2 Theorem. *Type* I^s_{cn} *product systems are time ordered product systems of von Neumann modules.*

Proof. By Theorem 5.4.14 (and Corollary 13.1.3) a type I^s_{cn} product system is contained in a time ordered product system. By Corollary 13.2.13 it is all of a time ordered product system. ■

13.3.3 Corollary. *The (strong closure of the) GNS-system of a uniformly continuous normal CPD-semigroup is a time ordered product system of von Neumann modules.*

• • •

Extensions. Section 13.1 works for Hilbert modules F (even for pre-Hilbert modules, but honestly speaking, it is not reasonable to do so, because the construction of sufficiently many units in a time ordered Fock modules involves norm limits). Also the analysis of continuous morphisms in Section 13.2 works for Hilbert modules. In the proof of Theorem 13.2.7 we need projections onto submodules in two different places. Firstly, we need the projections onto the submodules $\overline{F_0}^s$ and $\overline{F^0}^s$ of F. Secondly, if S is not strongly totalizing, then there should exists projections onto the members of the subsystem strongly generated by S.

For both it is sufficient that F is a right von Neumann module (the left action of $\mathcal{B}$ need not be normal). Then the projections onto $\overline{F_0}^s$ and $\overline{F^0}^s$, clearly, exist. But, also for the second condition we simply may pass to the strong closure of the members of the product systems. (For this it is sufficient that $\mathcal{B}$ is a von Neumann algebra. By Proposition 3.1.5 left multiplication by $b \in \mathcal{B}$ is strongly continuous as operation on the module. It just may happen that left multiplication is not strongly continuous as mapping $b \mapsto bx$.) This even shows that $\mathrm{I\Gamma}^{\odot}(F)$ and $\mathrm{I\Gamma}^{s\odot}(F)$ have the same continuous morphisms (in particular, projection morphisms), as soon as F is a right von Neumann module (of course, still a Hilbert $\mathcal{B}$–$\mathcal{B}$–module), because any continuous morphism leaves invariant the continuous units and whatever is generated by them in whatever topology.

As Lemma A.6.1 does not need normality, Theorem 13.3.1 remains true for uniformly continuous CP-semigroups (still on a von Neumann algebra). We find Theorem 5.4.14 for uniformly continuous CPD-semigroups. Consequently, Theorem 13.3.2 remains true for type I_c^s product systems of (right) von Neumann modules and Corollary 13.3.3 remains true for uniformly continuous CPD-semigroups on von Neumann algebras.

Finally, all results can be extended in the usual way to the case when $\mathcal{B}$ is a (unital) C^*–algebra, by passing to the bidual $\mathcal{B}^{**}$. We obtain then the weaker statements that the type I_c product systems and GNS-systems of uniformly continuous CPD-semigroups are strongly dense subsystems of product systems of von Neumann modules associated with time ordered Fock modules. Like in the case of the CE-generator, we can no longer guarantee that the inner products of the canonical units $\xi^{\odot}$ and the β_ξ are in $\mathcal{B}$.

• • •

Resumé. Notice that Theorem 13.3.1 is the first and the only time where we use the results by Christensen and Evans [CE79] quoted in Appendix A.6 (in particular, Lemma A.6.1).

In Sections 13.1 and 13.2 we reduced the proof of Theorem 13.3.2 to the problem to show existence of a central unit among the (continuous) units of a type I^s_{cn} product system. In fact, Lemma 13.1.1 together with Corollary 13.2.13 shows that existence of a central unit is equivalent to Lemma A.6.1. With our methods we are also able to conclude back from the form (5.4.3) of a generator to Lemma A.6.1, a result which seems not to be accessible by the methods in [CE79].

If we are able to show existence of a central unit directly, then we will not only provide a new proof of the results by Christensen and Evans [CE79] but also of the result that bounded derivations on von Neumann algebras are inner. We do not yet have concrete results into that direction. We hope, however, that a proof, if possible, could avoid a subdivision of the argument into the three types of von Neumann algebras.

We remark that the methods from Section 13.1 should work to some extent also for unbounded generators. More precisely, if $E^\odot$ is a product system with a central unital unit $\omega^\odot$ such that the semigroups $\mathfrak{U}^{\xi,\omega}$ in $\mathcal{B}$ have a reasonable generator (not in $\mathcal{B}$, but for instance, a closed operator on G, when $\mathcal{B} \subset \mathscr{B}(G)$), then this should be sufficient to split of a (possibly unbounded) completely positive part from the generator. As Example 7.3.7 shows, it is far from being clear what a "GNS-construction" for such unbounded completely positive mappings could look like. Nevertheless, the splitting of the generator alone, so far a postulated property in literature, would constitute a considerable improvement.

13.4 Appendix: Morphisms and order

The goal of this section is to establish the analogue of Theorem 5.3.3 for the (strong closure of the) GNS-system of a (normal) CPD-semigroup $\mathfrak{T}$ in $\mathcal{K}_S(\mathcal{B})$ for some von Neumann algebra $\mathcal{B}$. It is a straightforward generalization of the result for CP-semigroups obtained in Bhat and Skeide [BS00] and asserts that the set of CPD-semigroups *dominated* by $\mathfrak{T}$ is order isomorphic to the set of positive contractive morphisms of its GNS-system. Then we investigate this order structure for the time ordered Fock module with the methods from Section 13.2.

13.4.1 Definition. Let $\mathfrak{T}$ be a CPD-semigroup in $\mathcal{K}_S(\mathcal{B})$. By $\mathcal{D}_{\mathfrak{T}}$ we denote the set of CPD-semigroups $\mathfrak{S}$ in $\mathcal{K}_S(\mathcal{B})$ *dominated* by $\mathfrak{T}$, i.e. $\mathfrak{S}_t \in \mathcal{D}_{\mathfrak{T}_t}$ for all $t \in \mathbb{T}$, which we indicate by $\mathfrak{T} \geq \mathfrak{S}$. If we restrict to normal CPD-semigroups, then we write $\mathcal{K}^n_S(\mathcal{B})$ and $\mathcal{D}^n_{\mathfrak{T}}$, respectively.

Obviously, $\geq$ defines partial order among the CPD-semigroups.

13.4.2 Proposition. *Let $\mathfrak{T} \geq \mathfrak{S}$ be two CPD-semigroups in $\mathcal{K}_S(\mathcal{B})$. Then there exists a unique contractive morphism $v^\odot = \big(v_t\big)_{t\in\mathbb{T}}$ from the GNS-system $E^\odot$ of $\mathfrak{T}$ to the GNS-system*

$F^\odot$ of $\mathfrak{S}$, fulfilling $v_t\xi_t^\sigma = \zeta_t^\sigma$ for all $\sigma \in S$.

Morever, if all v_t have an adjoint, then $w^\odot = \big(v_t^* v_t\big)_{t\in\mathbb{T}}$ is the unique positive, contractive endomorphism of $E^\odot$ fulfilling $\mathfrak{S}_t^{\sigma,\sigma'}(b) = \langle \xi_t^\sigma, w_t b \xi_t^{\sigma'}\rangle$ for all $\sigma,\sigma' \in S$, $t \in \mathbb{T}$ and $b \in \mathcal{B}$.

Proof. This is a combination of the construction in the proof of Lemma 5.3.2 (which asserts that there is a family of contractions $\breve{v}_t$ from the Kolmogorov decomposition $\breve{E}_t$ of $\mathfrak{T}_t$ to the Kolmogorov decomposition $\breve{F}_t$ of $\mathfrak{S}_t$) and arguments like in Section 11.3. More precisely, denoting by $\beta_{\mathfrak{ts}}^{\mathfrak{T}}, i_{\mathfrak{t}}^{\mathfrak{T}}$ and $\beta_{\mathfrak{ts}}^{\mathfrak{S}}, i_{\mathfrak{t}}^{\mathfrak{S}}$ the mediating mappings and the canonical embeddings for the two-sided inductive limit for the CPD-semigroups $\mathfrak{T}$ and $\mathfrak{S}$, respectively, we have to show that the mappings $i_{\mathfrak{t}}^{\mathfrak{S}}\breve{v}_{\mathfrak{t}}\breve{E}_{\mathfrak{t}} \to F_t$, where $v_t = \breve{v}_{t_n} \odot \ldots \odot \breve{v}_{t_1}$ ($\mathfrak{t} \in \mathbb{J}_t$), define a mapping $v_t \colon E_t \to F_t$ via Proposition A.10.3 (obviously, contractive and bilinear). From

$$\breve{v}_{\mathfrak{s}} \odot \breve{v}_{\mathfrak{t}} \;=\; \breve{v}_{\mathfrak{s}\smile\mathfrak{t}} \tag{13.4.1}$$

we conclude $\beta_{\mathfrak{ts}}^{\mathfrak{S}}\breve{v}_{\mathfrak{s}} = \breve{v}_{\mathfrak{t}}\beta_{\mathfrak{ts}}^{\mathfrak{T}}$. Applying $i_{\mathfrak{t}}^{\mathfrak{S}}$ to both sides the statement follows. Again from (13.4.1) (and Remark 11.3.6) we find that $v_s \odot v_t = v_{s+t}$. Clearly, $v^\odot$ is unique, because we know the values on a totalizing set of units. The statements about $w^\odot$ are now obvious. ■

13.4.3 Theorem. *Let $E^{\overline{\odot}^s} = \big(E_t\big)_{t\in\mathbb{T}}$ be a product system of von Neumann $\mathcal{B}$–$\mathcal{B}$–modules E_t, and let $S \subset \mathfrak{U}(E^{\overline{\odot}^s})$ be a subset of units for $E^{\overline{\odot}^s}$. Then the mapping $\mathfrak{O}\colon w^\odot \mapsto \mathfrak{S}_w$ defined by setting*

$$(\mathfrak{S}_w^{\xi,\xi'})_t(b) \;=\; \langle \xi_t, w_t b \xi'_t\rangle$$

for all $t \in \mathbb{T}$, $\xi,\xi' \in S$, $b \in \mathcal{B}$, establishes an order morphism from the set of contractive, positive morphisms of $E^{\overline{\odot}^s}$ (equipped with pointwise order) onto the set $\mathcal{D}_{\mathfrak{T}}^n$ of normal CPD-semigroups $\mathfrak{S}$ dominated by $\mathfrak{T} = \mathfrak{U} \restriction S$. It is an order isomorphism, if and only if $E^{S^{\overline{\odot}^s}} = E^{\overline{\odot}^s}$.

Proof. If $E^{S^{\overline{\odot}^s}} \neq E^{\overline{\odot}^s}$, then $\mathfrak{O}$ is not one-to-one, because the identity morphism $w_t = \mathsf{id}_{E_t}$ and the morhism $p = \big(p_t\big)_{t\in\mathbb{T}}$ of projections p_t onto $\overline{E_t^S}^s$ are different morphisms giving the same CPD-semigroup $\mathfrak{T}$. On the other hand, any morphism $w^\odot$ for $E^{S^{\overline{\odot}^s}}$ extends to a morphism composed of mappings $w_t p_t$ of $E^{\overline{\odot}^s}$ giving the same Schur semigroup $\mathfrak{S}_w$. Therefore, we are done, if we show the statement for $E^{S^{\overline{\odot}^s}} = E^{\overline{\odot}^s}$.

So let us assume that S is totalizing. Then $\mathfrak{O}$ is one-to-one. It is also order preserving, because $w^\odot \geq w'^\odot$ implies

$$(\mathfrak{S}_w^{\xi,\xi'})_t(b) - (\mathfrak{S}_{w'}^{\xi,\xi'})_t(b) \;=\; \langle \xi_t, (w_t - w'_t) b \xi'_t\rangle \;=\; \langle \sqrt{w_t - w'_t}\xi_t, b\sqrt{w_t - w'_t}\xi'_t\rangle \tag{13.4.2}$$

so that $(\mathfrak{S}_w)_t \geq (\mathfrak{S}_{w'})_t$ in $\mathcal{K}_S(\mathcal{B})$. By obvious extension of Proposition 13.4.2 to von Neumann modules, which guarantees existence of v_t^*, we see that $\mathfrak{O}$ is onto. Now let $\mathfrak{T} \geq \mathfrak{S} \geq \mathfrak{S}'$

with morhisms $w^{\odot} = \mathfrak{O}^{-1}(\mathfrak{S})$ and $w'^{\odot} = \mathfrak{O}^{-1}(\mathfrak{S}')$ and construct $v_t \in \mathcal{B}^{a,bil}(\overline{E}^s_t, \overline{F}^s_t)$, $v'_t \in \mathcal{B}^{a,bil}(\overline{E}^s_t, \overline{F'}^s_t)$, and $u_t \in \mathcal{B}^{a,bil}(\overline{F}_t, \overline{F'}^s_t)$, for the pairs $\mathfrak{T} \geq \mathfrak{S}$, $\mathfrak{T} \geq \mathfrak{S}'$, and $\mathfrak{S} \geq \mathfrak{S}'$, respectively, as in Proposition 13.4.2 and extension to the strong closures. Then by uniqueness we have $v'_t = u_t v_t$. It follows $w_t - w'_t = v^*_t(\mathbf{1} - u^*_t u_t)v_t \geq 0$. This shows that also $\mathfrak{O}^{-1}$ respects the order and, therefore, is an order isomorphism. (Observe that for the last conclusion (13.4.2) is not sufficient, because the vectors $b\xi_t b'$ ($\xi^{\odot} \in S; b, b' \in \mathcal{B}$) do not span E_t.) ■

Observe that this result remains true, if we require that the morphisms respect some subset of units like, for instance, the continuous units in the time ordered Fock module. We investigate now the order structure of the set of (possibily unbounded) positive continuous morphisms on $\mathrm{I\Gamma}^{\mathfrak{U}_c\odot}(F)$. We will see that it is mirrored by the positivity structure of the corresponding matrices $\Gamma \in \mathcal{B}^{a,bil}(\mathcal{B} \oplus F)$ where F is an arbitrary Hilbert $\mathcal{B}$–$\mathcal{B}$–module. Recalling that by Lemma 1.5.2 positive contractions are dominated by $\mathbf{1}$, we find a simple criterion for contractive positive morphisms as those whose matrix Γ is dominated (in $\mathcal{B}^{a,bil}(F)$) by the matrix $\Gamma = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right)$ of the identity morphism.

13.4.4 Lemma. *A (possibly unbounded) continuous endomorphism $w^{\odot}$ of $\mathrm{I\Gamma}^{\mathfrak{U}_c\odot}(F)$ with the matrix $\Gamma = \left(\begin{smallmatrix} \gamma & \eta^* \\ \eta & a \end{smallmatrix}\right)$ is positive, if and only if it is self-adjoint and a is positive.*

Proof. $w^{\odot}$ is certainly positive, if it is possible to write it as a square of a self-adjoint morphism with matrix $\widehat{\Delta} = \left(\begin{smallmatrix} \mathbf{1} & 0 & 0 \\ \delta & \mathbf{1} & \chi^* \\ \chi & 0 & c \end{smallmatrix}\right)$ say (δ and c self-adjoint). In other words, we must have

$$\begin{pmatrix} \mathbf{1} & 0 & 0 \\ \gamma & \mathbf{1} & \eta^* \\ \eta & 0 & a \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 & 0 \\ \delta & \mathbf{1} & \chi^* \\ \chi & 0 & c \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 & 0 \\ \delta & \mathbf{1} & \chi^* \\ \chi & 0 & c \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 2\delta + \langle\chi, \chi\rangle & \mathbf{1} & \chi^* + (c\chi)^* \\ \chi + c\chi & 0 & c^2 \end{pmatrix}.$$

This equation can easily be resolved, if $a \geq 0$. We put $c = \sqrt{a}$. Since $c \geq 0$ we have $\mathbf{1} + c \geq \mathbf{1}$ so that $\mathbf{1} + c$ is invertible. We put $\chi = (\mathbf{1} + c)^{-1}\eta$. Finally, we set $\delta = \frac{\gamma - \langle\chi,\chi\rangle}{2}$ $(= \delta^*)$. Then $\widehat{\Delta}$ determines a self-adjoint endomorphism whose square is $w^{\odot}$.

On the other hand, if $w^{\odot}$ is positive, then Γ is self-adjoint and the generator $\mathfrak{L}_w$ of the CPD-semigroup $\mathfrak{S}_w$ is conditionally completely positive definite. For $\mathfrak{L}_w$ we find (rewritten conveniently)

$$\mathfrak{L}_w^{(\beta,\zeta),(\beta',\zeta')}(b) = \langle\zeta, ba\zeta'\rangle + b\big(\langle\eta, \zeta'\rangle + \beta' + \tfrac{\gamma}{2}\big) + \big(\langle\zeta, \eta\rangle + \beta^* + \tfrac{\gamma}{2}\big)b.$$

For each $\zeta \in F$ we choose $\beta \in \mathcal{B}$ such that $\langle\zeta, \eta\rangle + \beta^* + \frac{\gamma}{2} = 0$. Then it follows as in Remark 13.2.14 ($\zeta = 0 \in F$) that the kernel $b \mapsto \langle\zeta, ba\zeta'\rangle$ on F is not only conditionally completely positive definite, but completely positive definite. This implies that $a \geq 0$. ■

13.4.5 Remark. By applying the lemma to the endomorhism with matrix $\widehat{\Delta}$, we see that it is positive, too.

13.4.6 Lemma. *For two self-adjoint possibly unbounded morphisms $w^\odot$ and $v^\odot$ with matrices $\Gamma = \left(\begin{smallmatrix}\gamma & \eta^* \\ \eta & a\end{smallmatrix}\right)$ and $\Delta = \left(\begin{smallmatrix}\delta & \chi^* \\ \chi & c\end{smallmatrix}\right)$, respectively, we have $w^\odot \geq v^\odot$, if and only if $\Gamma \geq \Delta$ in $\mathcal{B}^{a,bil}(\mathcal{B} \oplus F)$.*

Proof. By Theorem 13.4.3 and Lemma 5.4.12 we have $w^\odot \geq v^\odot$, if and only if $\mathfrak{S}_w \geq \mathfrak{S}_v$, if and only if $\mathfrak{L}_w \geq \mathfrak{L}_v$. By Equations (13.2.2) and (13.2.4) we see that in the last infinitesmal form $\mathfrak{L}_w - \mathfrak{L}_v$, only the difference $\Gamma - \Delta$ enters. Furthermore, evaluating the difference of these kernels at concrete elements $\xi^\odot(\beta, \zeta), \xi^\odot(\beta', \zeta')$, the β, β' do not contribute. Therefore, it is sufficient to show the statement in the case when $\Delta = 0$, i.e. $w^\odot$ dominates (or not) the morphism $v^\odot$ which just projects onto the vacuum, and to check completely positive definiteness only against exponential units. We find

$$\sum_{i,j} b_i^*(\mathfrak{L}_w - \mathfrak{L}_v)^{(0,\zeta_i),(0,\zeta_j)}(a_i^* a_j) b_j = \sum_{i,j} b_i^* \Big(\langle \zeta_i, a_i^* a_j a \zeta_j \rangle + \langle \zeta_i, a_i^* a_j \eta \rangle + a_i^* a_j \langle \eta, \zeta_j \rangle + a_i^* a_j \gamma \Big) b_j$$
$$= \sum_{i,j} \langle a_i \zeta_i b_i, a a_j \zeta_j b_j \rangle + \langle a_i \zeta_i b_i, \eta \rangle a_j b_j + (a_i b_i)^* \langle \eta, a_j \zeta_j b_j \rangle + (a_i b_i)^* \gamma a_j b_j \;=\; \langle Z, \Gamma Z \rangle,$$

where $Z = \sum_i (a_i b_i, a_i \zeta_i b_i) \in \mathcal{B} \oplus F$. Elements of the form Z do, in general, not range over all of $\mathcal{B} \oplus F$. However, to check positivity of Γ with $(\zeta, \beta) \in \mathcal{B} \oplus F$ we choose $\zeta_1 = \lambda \zeta$, $\zeta_2 = 0$, $a_1 = a_2 = \mathbf{1}$, and $b_1 = \frac{\mathbf{1}}{\lambda}$, $b_2 = \beta$. Then $Z \to (\beta, \zeta)$ for $\lambda \to \infty$. This means that $\mathfrak{L}_w - \mathfrak{L}_v \geq 0$, if and only if $\Gamma (= \Gamma - \Delta) \geq 0$. ■

13.4.7 Corollary. *The set of contractive positive continuous morphisms of $\mathrm{I\Gamma}^\odot(F)$ is order isomorphic to the set of those self-adjoint matrices $\Gamma \in \mathcal{B}^{a,bil}(\mathcal{B} \oplus F)$ with $a \geq 0$ and $\Gamma \leq \left(\begin{smallmatrix}0 & 0 \\ 0 & \mathbf{1}\end{smallmatrix}\right)$.*

It is possible to characterize these matrices further. We do not need this characterization.

13.5 Appendix: CPD-semigroups on $\mathcal{K}_S(\mathcal{B})$ versus CP-semigroups on $\mathcal{B}(H_S) \otimes \mathcal{B}$

In the proof of Theorem 13.1.2 we utilized the possibility to pass from a product system $E^\odot$ of $\mathcal{B}$–$\mathcal{B}$–modules to a product system $M_n(E^\odot)$ of $M_n(\mathcal{B})$–$M_n(\mathcal{B})$–modules. Given a family $\xi^{i\odot}$ $(i = 1, \ldots, n)$ of units for $E^\odot$ we defined the diagonal unit $\Xi^\odot$ for $M_n(E^\odot)$ with diagonal entries $\xi^{i\odot}$.

We remark that $\Xi^\odot$ is totalizing for $M_n(E^\odot)$, if and only if the set $S = \{\xi^{1\odot}, \ldots, \xi^{n\odot}\}$ is totalizing for $E^\odot$. In this case $T^\Xi(B) = \langle \Xi_t, B\Xi_t \rangle$ is a CP-semigroup on $M_n(\mathcal{B})$ whose GNS-system is $M_n(E^\odot)$. Moreover, T^Ξ is uniformly continuous, if and only if the CPD-semigroup $\mathfrak{U}(E^\odot) \upharpoonright S$ is (and the same holds for normality, if $\mathcal{B}$ is a von Neumann algebra). We may apply Theorem 13.3.1 to T^Ξ and obtain that the GNS-system of $M_n(\mathcal{B})$–$M_n(\mathcal{B})$–modules is isomorphic to a time ordered product system. Taking into account that by Examples 1.7.7 and 4.2.12 a product system of $M_n(\mathcal{B})$–$M_n(\mathcal{B})$–modules is always of the form $M_n(E_t)$ where the E_t form a product system, we obtain that the two descriptions are interchangable. Specifying that, on the one hand, we look at product systems generated by not more than n units and, on the other hand, that we look only at CP-semigroups on $M_n(\mathcal{B})$ and units for $M_n(E^\odot)$ which are diagonal, we obtain that the analogy is complete.

This way to encode the information of a CPD-semigroup into a single CP-semigroup is taken from Accardi and Kozyrev [AK99] which was also our motivation to study completely positive definite kernels and Schur semigroups of such. In [AK99] the authors considered only the case of the product system of symmteric (i.e. time ordered) Fock spaces $\Gamma^\odot(L^2(\mathbb{R}_+)) \cong \mathrm{I\Gamma}^\odot(\mathbb{C})$, where two exponential units, namely, the vacuum plus any other, are totalizing (Corollary 7.4.4). They were lead to look at semigroups on $M_2(\mathcal{B}(G))$. Notice that in our case we have even interesting results with a single totalizing unit. What we explained so far is the generalization to n generating units (in the case of $\mathcal{B} = \mathcal{B}(G)$ already known to the authors of [AK99]).

Now we want to extend the idea to totalizing sets S containing an arbitrary number of units. It is good to keep the intuitive idea of matrices, now of infinite, even possibly uncountable, dimension. Technically, it is better to change the picture from matrices $M_n(E)$ to exterior tensor products $M_n \otimes E$ as explained in Example 4.3.8. Now the unit $\Xi^\odot$ should have infinitely many entries. For that we must be able to control the norm of each entry. Some sort of continuity should be sufficient, but as we want to control also the norm of the generator, we restrict to the uniformly continuous case.

Let S be a set of continuous units for $\mathrm{I\Gamma}^{s\odot}(F)$ and denote by H_S the Hilbert space with ONB $\left(e_\xi\right)_{\xi^\odot \in S}$. We have

$$\begin{aligned} L^2(\mathbb{R}_+, \mathcal{B}(H_S) \bar{\otimes}^s F) &= L^2(\mathbb{R}_+) \bar{\otimes}^s (\mathcal{B}(H_S) \bar{\otimes}^s F) \\ &= \mathcal{B}(H_S) \bar{\otimes}^s (L^2(\mathbb{R}_+) \bar{\otimes}^s F) = \mathcal{B}(H_S) \bar{\otimes}^s L^2(\mathbb{R}_+, F), \end{aligned}$$

where $\mathcal{B}(H_S) \bar{\otimes}^s F$ and, henceforth, $L^2(\mathbb{R}_+, \mathcal{B}(H_S) \bar{\otimes}^s F)$ is a von Neumann $\mathcal{B}(H_S) \bar{\otimes}^s \mathcal{B}$–$\mathcal{B}(H_S) \bar{\otimes}^s \mathcal{B}$–module by Example 4.3.4. Consequently, we find

$$\mathcal{B}(H_S) \bar{\otimes}^s \mathrm{I\Gamma}^{s\odot}(F) = \mathrm{I\Gamma}^{s\odot}(\mathcal{B}(H_S) \bar{\otimes}^s F).$$

A continuous unit $\xi^\odot(B, Z)$ ($B \in \mathcal{B}(H_S) \bar{\otimes}^s \mathcal{B}, Z \in \mathcal{B}(H_S) \bar{\otimes}^s F$) is *diagonal* (in the matrix picture), if and only if B and Z are diagonal. A diagonal unit $\xi^\odot(B,Z)$ is strongly totalizing for $\mathrm{I\Gamma}^{s\odot}(\mathcal{B}(H_S)\,\bar{\otimes}^s\,F)$, if and only if the set $\{\xi^\odot(\beta,\zeta)\}$ running over the digonal entries of $\xi^\odot(B,Z)$ is strongly totalizing for $\mathrm{I\Gamma}^{s\odot}(F)$.

Can we put together the units from S to a single diagonal unit for $\mathrm{I\Gamma}^{s\odot}(\mathcal{B}(H_S)\,\bar{\otimes}^s\,F)$? In order that a family $\big(a_\xi\big)_{\xi\in S}$ of elements in $\mathcal{B}$ (in F) defines (as strong limit) an element in $\mathcal{B}(H_S)\,\bar{\otimes}^s\,\mathcal{B}$ (in $\mathcal{B}(H_S)\,\bar{\otimes}^s\,F$) with diagonal entries, is it necessary and sufficient (cf. the proof of Lemma 2.3.7) that it is uniformly bounded. This will, in general, not be the case. However, as long as we are only interested in whether S is totalizing or not, we may modify S without changing this property. By Observation 13.2.11 we may forget completely about the parameters β_ξ. Moreover, for the condition in Theorem 13.2.7 the length of the ζ_ξ is irrelevant (as long as it is not 0, of course). We summarize.

13.5.1 Theorem. *Let $\mathfrak{T}$ be a normal uniformly continuous CPD-semigroup on S in $\mathcal{K}_S(\mathcal{B})$. Then there exists a normal uniformly continuous CP-semigroup T on $\mathcal{B}(H_S)\,\bar{\otimes}^s\,\mathcal{B}$ such that the GNS-system (of von Neumann modules) of T is $\mathcal{B}(H_S)\,\bar{\otimes}^s\,E^{\overline{\odot}^s}$ where $E^{\overline{\odot}^s}$ is the GNS-system (of von Neumann modules) of $\mathfrak{T}$.*

So far, we considered diagonal units for the time ordered Fock module $\mathrm{I\Gamma}^{s\odot}(\mathcal{B}(H_S)\,\bar{\otimes}^s\,F)$. Of course, $\xi^\odot(B,Z)$ is a unit for any choice of $B\in\mathcal{B}(H_S)\,\bar{\otimes}^s\,\mathcal{B}$ and $Z\in\mathcal{B}(H_S)\,\bar{\otimes}^s\,F$. The off-diagonal entries of such a unit fulfill a lot of recursive relations. In the case of Hilbert spaces ($\mathcal{B}=\mathbb{C}$) and finite sets S ($\mathcal{B}(H_S)=M_n$) we may hope to compute $\xi^\odot(B,Z)$ explicitly. This should have many applications in the theory of special functions, particularly those involving iterated integrals of exponential functions.

Chapter 14

Other constructions of product systems

In this chapter we present two different constructions of product systems. The first construction in Section 14.1 is the generalization to Hilbert modules from Bhat's approach to Arveson systems [Bha96] as described in Skeide [Ske00a]. As an important application we obtain that any (strict) dilation on a Hilbert module is also a weak dilation (Theorem 14.1.8), another result which seems out of reach without product systems. In Section 14.2 we present the construction from Liebscher and Skeide [LS00b] which allows to construct a product system from *markovian systems of transition expectations* as in Section 11.3. With the help of this product system we construct as in Section 11.4 a *generalized weak Markov flow* which gives us back the original transition expectations.

14.1 E_0–Semigroups on $\mathcal{B}^a(E)$

In [Bha96] Bhat discovered another possibility to construct the Arveson system of a (normal, strongly continuous) E_0–semigroup ϑ on $\mathcal{B}(H)$ (H an infinite-dimensional separable Hilbert space). Contrary to Arveson's original approach [Arv89a] via intertwiner spaces of ϑ_t, Bhat's approach generalizes directly to Hilbert modules.

Let (E, ϑ, ξ) be a triple consisting of a Hilbert $\mathcal{B}$–module, a strict E_0–semigroup ϑ on $\mathcal{B}^a(E)$, and a unit vector $\xi \in E$. (Equation (14.1.1) below forces us to consider completions.) To begin with, we do not assume that (E, ϑ, ξ) is a weak dilation of a CP-semigroup on $\mathcal{B}$.

By $j_0(b) = \xi b \xi^*$ we define a faithful representation of $\mathcal{B}$ on E. We define the representations $j_t = \vartheta_t \circ j_0$ and set $p_t = j_t(\mathbf{1})$. On the Hilbert submodule $E_t = p_t E$ of E we define a left multiplication by $bx_t = j_t(b)x_t$, thus, turning E_t into a Hilbert $\mathcal{B}$–$\mathcal{B}$–module. (Clearly, $\mathbf{1}x_t = x_t$ and $E_0 \cong \mathcal{B}$ via $\xi \mapsto \mathbf{1}$.)

14.1.1 Theorem. *The mapping*

$$u_t \colon x \odot x_t \longmapsto \vartheta_t(x\xi^*)x_t$$

extends as an isomorphism $E \bar{\odot} E_t \to E$. Moreover, the restrictions $u_{st} = u_t \upharpoonright (E_s \bar{\odot} E_t)$ are two-sided isomorphisms $E_s \bar{\odot} E_t \to E_{s+t}$, fulfilling (11.1.1) *so that $E^{\bar{\odot}} = \bigl(E_t\bigr)_{t\in\mathbb{T}}$ is a product system. Using the identifications* (11.4.1) *and* (11.1.2), *we again find* (11.4.2) *and $\vartheta_t(a) = a \odot \mathsf{id}_{E_t}$.*

Proof. From

$$\langle x \odot x_t, x' \odot x'_t\rangle \;=\; \bigl\langle x_t, \langle x, x'\rangle x'_t\bigr\rangle \;=\; \bigl\langle x_t, \vartheta_t(\xi\langle x, x'\rangle\xi^*)x'_t\bigr\rangle \;=\; \bigl\langle \vartheta_t(x\xi^*)x_t, \vartheta_t(x'\xi^*)x'_t\bigr\rangle$$

we see that u_t is isometric. Let $u^\lambda = \sum\limits_{k=1}^{n_\lambda} v_k^\lambda w_k^{\lambda *}$ $(k^\lambda, w_k^\lambda \in E)$ be an approximate unit for $\mathcal{K}(E)$ which, therefore, converges strictly to $\mathbf{1} \in \mathcal{B}^a(E)$. We find

$$\begin{aligned} x \;=\; \lim_\lambda \vartheta_t(u^\lambda)x \;=\; &\lim_\lambda \sum_k \vartheta_t(v_k^\lambda w_k^{\lambda *})x \\ &= \lim_\lambda \sum_k \vartheta_t(v_k^\lambda \xi^*)\vartheta_t(\xi w_k^{\lambda *})x \;=\; \lim_\lambda \sum_k v_k^\lambda \odot \vartheta_t(\xi w_k^{\lambda *})x, \end{aligned} \tag{14.1.1}$$

where $\vartheta_t(\xi w_k^{\lambda *})x = p_t\vartheta_t(\xi w_k^{\lambda *})x$ is in E_t. In other words, u_t is surjective, hence, unitary. Clearly, in the identification (11.4.1) we find

$$\vartheta_t(a)(x \odot x_t) \;=\; \vartheta_t(a)\vartheta_t(x\xi^*)x_t \;=\; \vartheta_t(ax\xi^*)x_t \;=\; ax \odot x_t.$$

Suppose $p_s x = x$. Then $p_{s+t}u_t(x \odot x_t) = \vartheta_{s+t}(\xi\xi^*)\vartheta_t(x\xi^*)x_t = \vartheta_t(p_s x\xi^*)x_t = u_t(x \odot x_t)$ so that u_{st} maps into E_{s+t}. Obviously, $j_{s+t}(b)u_t(x \odot x_t) = u_t(j_s(b)x \odot x_t)$ so that u_t is two-sided on $E_s \bar{\odot} E_t$. Suppose $p_{s+t}x = x$ and apply p_{s+t} to (14.1.1). Then a similar computation shows that we may replace v_k^λ with $p_s v_k^\lambda$ without changing the value. Therefore, $x \in u_t(E_s \bar{\odot} E_t)$. In other words, u_t restricts to a two-sided unitary $u_{st} \colon E_s \bar{\odot} E_t \to E_{s+t}$. The associativity conditions (11.1.1) and (11.4.3) follow by similar computations. ■

14.1.2 Proposition. *The product system $E^{\bar{\odot}}$ does not depend on the choice of the unit vector ξ. More precisely, if $\xi' \in E$ is another unit vector, then $w_t x_t = \vartheta_t(\xi'\xi^*)x_t$ defines an isomorphism $w^\odot = \bigl(w_t\bigr)_{t\in\mathbb{T}}$ from the product system $E^{\bar{\odot}}$ to the product system $E'^{\bar{\odot}} = \bigl(E'_t\bigr)_{t\in\mathbb{T}}$ constructed from ξ'.*

Proof. $p'_t\vartheta_t(\xi'\xi^*) = \vartheta_t(\xi'\xi^*)$ so that w_t maps into E'_t, and $\vartheta_t(\xi'\xi^*)^*\vartheta_t(\xi'\xi^*) = p_t$ so that w_t is an isometry. As $\vartheta_t(\xi'\xi^*)\vartheta_t(\xi'\xi^*)^* = p'_t$, it follows that w_t is surjective, hence, unitary. For $b \in \mathcal{B}$ we find

$$w_t j_t(b) \;=\; \vartheta_t(\xi'\xi^*)\vartheta_t(\xi b\xi^*) \;=\; \vartheta_t(\xi' b\xi^*) \;=\; \vartheta_t(\xi' b\xi'^*)\vartheta_t(\xi'\xi^*) \;=\; j'_t(b)w_t$$

so that w_t is two-sided. In the identification (11.1.2) (applied to $E^{\bar{\odot}}$ and $E'^{\bar{\odot}}$) we find

$$\begin{aligned} w_s x_s \odot w_t y_t &= \vartheta_t(w_s x_s \xi'^*) u_t x_t = \vartheta_t(\vartheta_s(\xi'\xi^*) x_s \xi'^*) \vartheta_t(\xi'\xi^*) y_t \\ &= \vartheta_{s+t}(\xi'\xi^*) \vartheta_t(x_s \xi^*) y_t = w_{s+t}(x_s \odot y_t). \end{aligned}$$

In other words, the w_t form a morphism. ■

14.1.3 Example. If $E = G$ is a Hilbert space with a unit vector g, we recover Bhat's construction [Bha96] resulting in a tensor product system $G^{\bar{\otimes}} = (G_t)_{t\in\mathbb{T}}$ of Hilbert spaces. Let us consider the original product system $\mathfrak{H}^{\bar{\otimes}}$ with the product system $E_t = \mathcal{B}(G, G \bar{\otimes} \mathfrak{H}_t) \cong \mathcal{B}(G)_t$ of von Neumann modules as explained in Examples 11.1.3 and 11.1.4. We know that $E_t \bar{\odot} G = G \bar{\otimes} \mathfrak{H}_t$ and that the isomorphism of $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H}_t)$ and $\mathcal{B}(G)_t$ give rise to an isomorphism of $G \bar{\otimes} \mathfrak{H}_t$ and G. Fixing the subspace $g \otimes \mathfrak{H}_t$, we may identify $\mathfrak{H}_t$ with a subspace of G. We claim that this subspace is G_t. Indeed, $G_t = \vartheta_t(gg^*)G$ which corresponds under the isomorphism to $(gg^* \otimes \mathsf{id})(G \bar{\otimes} \mathfrak{H}_t) = g \otimes \mathfrak{H}_t$. Moreover,

$$g_s \otimes g_t = \vartheta_t(g_s g^*) g_t = (g_s g^* \otimes \mathsf{id})(g \otimes h_t) = g_s \otimes h_t = g \otimes h_s \otimes h_t,$$

so that also the product system structure is the same.

A similar comparison of the products system $E^{\bar{\odot}}$ of Hilbert $\mathcal{B}$–$\mathcal{B}$–modules constructed from (E, ϑ, ξ) and the product system $\mathcal{B}^a(E)^{\bar{\odot}}$ of $\mathcal{B}^a(E)$–$\mathcal{B}^a(E)$–modules $\mathcal{B}^a(E)_t$ as defined in Example 11.1.3 seems not to be possible.

14.1.4 Example. Let us consider $(\breve{\Gamma}(F), \mathcal{S}, \omega)$ where $\mathcal{S}$ is the time shift automorphism group restricted to $t \geq 0$. Then $\mathcal{S}$ leaves invariant $\omega\omega^*$ so that for all $t \in \mathbb{R}_+$ we have $p_t \breve{\Gamma}(F) = \mathcal{B}\omega \cong \mathcal{B}$. If we look instead at $(\Gamma(F), \mathcal{S}, \omega)$ where now $\mathcal{S}$ is the time shift endomorphism obtained by restriction to $\mathcal{B}^a(\Gamma(F)) \cong \mathcal{B}^a(\Gamma(F)) \odot \mathsf{id} \subset \mathcal{B}^a(\breve{\Gamma}(F))$ then $p_0 \odot \mathsf{id} = \omega\omega^* \odot \mathsf{id}$ evolves differently. It is just $p_t \odot \mathsf{id}$ where p_t is the projection onto $\Gamma_t(F)$. We find that the product system of the time shift endomorphism semigroup on $\Gamma(F)$ is $\Gamma^{\odot}(F)$. A similar argument also applies to general product systems $E^{\bar{\odot}}$, if we investigate the relation between the endomorphism white noise constructed in Section 11.5 on the one-sided inductive E with the help of a central unit, and the extension to an automorphism white noise on $\overleftrightarrow{E}$.

Let $E = L^2(\mathbb{R}, F)$ and use the notations from Observation 6.3.8. Again the time shift $\mathcal{S}$ on $\mathcal{F}(E)$ leaves invariant $\omega\omega^*$ so that for $(\mathcal{F}(E), \mathcal{S}, \omega)$ we end up with the trivial product system. On the contrary, if we look at the restriction of the time shift to an E_0–semigroup $\mathcal{S}$ on $\mathcal{B}^a(\mathcal{F}(E_{\mathbb{R}_+}))$ (coming from the identification $\mathcal{B}^a(\mathcal{F}(E_{\mathbb{R}_+})) \cong \mathcal{B}^a(\mathcal{F}(E_{\mathbb{R}_+})) \odot \mathsf{id} \subset \mathcal{B}^a(\mathcal{F}(E))$ in the factorization $\mathcal{F}(E) = \mathcal{F}(E_{\mathbb{R}_+}) \bar{\odot} (\mathcal{B}\omega \oplus E_{\mathbb{R}_-} \bar{\odot} \mathcal{F}(E))$ according to Proposition 6.3.1), then $p_0 \odot \mathsf{id} = \omega\omega^* \odot \mathsf{id}$ evolves as $p_t \odot \mathsf{id}$, where p_t is a family of projections on $\mathcal{F}(E_{\mathbb{R}_+})$. A further applications of Proposition 6.3.1 shows that $p_t\mathcal{F}(E_{\mathbb{R}_+}) = \mathcal{B}\omega \oplus E_{[0,t]} \bar{\odot} \mathcal{F}(E_{\mathbb{R}_+})$. By Theorem 14.1.1 these spaces form a product system and one may check the identification

$$\begin{aligned}
&\bigl(\mathcal{B}\omega \oplus E_{[0,s]} \bar{\odot} \mathcal{F}(E_{\mathbb{R}_+})\bigr) \bar{\odot} \bigl(\mathcal{B}\omega \oplus E_{[0,t]} \bar{\odot} \mathcal{F}(E_{\mathbb{R}_+})\bigr) \\
&\quad\cong\ \mathcal{F}(\mathfrak{s}_t)\bigl(\mathcal{B}\omega \oplus E_{[0,s]} \bar{\odot} \mathcal{F}(E_{\mathbb{R}_+})\bigr) \bar{\odot} \bigl(\mathcal{B}\omega \oplus E_{[0,t]} \bar{\odot} \mathcal{F}(E_{\mathbb{R}_+})\bigr) \\
&\quad=\ \bigl(\mathcal{B}\omega \oplus E_{[t,t+s]} \bar{\odot} \mathcal{F}(E_{[t,\infty)})\bigr) \bar{\odot} \bigl(\mathcal{B}\omega \oplus E_{[0,t]} \bar{\odot} \mathcal{F}(E_{\mathbb{R}_+})\bigr) \\
&\quad=\ \mathcal{B}\omega \,\oplus\, E_{[0,t]} \bar{\odot} \mathcal{F}(E_{\mathbb{R}_+}) \,\oplus\, E_{[t,t+s]} \bar{\odot} \mathcal{F}(E_{[t,\infty)}) \bar{\odot} \bigl(\mathcal{B}\omega \oplus E_{[0,t]} \bar{\odot} \mathcal{F}(E_{\mathbb{R}_+})\bigr) \\
&\quad=\ \mathcal{B}\omega \,\oplus\, E_{[0,t]} \bar{\odot} \mathcal{F}(E_{\mathbb{R}_+}) \,\oplus\, E_{[t,t+s]} \bar{\odot} \mathcal{F}(E_{\mathbb{R}_+}) \ =\ \mathcal{B}\omega \,\oplus\, E_{[0,t+s]} \bar{\odot} \mathcal{F}(E_{\mathbb{R}_+}).
\end{aligned}$$

For F being some separable Hilbert, this result is due to Fowler [Fow95]. He also showed that the product system is isomorphic to $\mathrm{I\!\Gamma}(\ell^2)$, independently of the dimension of F. We expect also in the Hilbert module case that the product system is some type I_c system, but we did not yet investigate that point.

In how far E_0–semigroups on $\mathcal{B}^a(E)$ are classified by their product systems? Of course, we expect as answer that they are classified up to outer conjugacy. First, however, we must clarify in which way we have to ask this question. In Arveson's set-up all Hilbert spaces on which he considers E_0–semigroups are isomorphic. It is this hidden assumption which makes the question for cocycle conjugacy possible. Nothing gets lost (up to unitary isomorphism), if we restrict Arveson's set-up to a single infinite-dimensional separable Hilbert space. Now we can ask the above question in a reasonable way.

14.1.5 Theorem. *Let (E,ξ) be a Hilbert $\mathcal{B}$–module E with a unit vector ξ. Furthermore, let ϑ and ϑ' be two strict E_0–semigroups on $\mathcal{B}^a(E)$. Then the product systems $E^{\bar{\odot}}$ and $E'^{\bar{\odot}}$ associated with ϑ and ϑ', respectively, are isomorphic, if and only if ϑ and ϑ' are outer conjugate.*

Proof. Let $\mathfrak{u}$ be a unitary left cocycle for ϑ such that $\vartheta' = \vartheta^{\mathfrak{u}}$. Then $\mathfrak{u}_t p_t = p'_t \mathfrak{u}_t$. Therefore, $\mathfrak{u}_t$ restricts to a unitary $u_t \colon E_t \to E'_t$ (with inverse $u_t^* = \mathfrak{u}_t^* \upharpoonright E'_t$, of course). Moreover, identifying (very carefully) $E \bar{\odot} E_t = E = E \bar{\odot} E'_t$, we find

$$\mathfrak{u}_t(x \odot x_t) \ =\ \mathfrak{u}_t\vartheta_t(x\xi^*)x_t \ =\ \vartheta'_t(x\xi^*)\mathfrak{u}_t x_t \ =\ x \odot u_t x_t.$$

It follows that $(a \odot \mathsf{id}_{E'_t})\mathfrak{u}_t = \mathfrak{u}_t(a \odot \mathsf{id}_{E_t})$ for all $a \in \mathcal{B}^a(E)$. Specializing to $a = j_0(b)$ so that $a \odot \mathsf{id}_{E_t} = j_t(b)$ and $a \odot \mathsf{id}_{E'_t} = j'_t(b)$, we see that u_t is the (unique) element in $\mathcal{B}^{a,bil}(E_t, E'_t)$ such that $\mathfrak{u}_t = \mathsf{id} \odot u_t$. From

$$\mathsf{id} \odot u_{s+t} \ =\ \mathfrak{u}_{s+t} \ =\ \mathfrak{u}_t\vartheta_t(\mathfrak{u}_s) \ =\ (\mathsf{id} \odot \mathsf{id}_{E'_s} \odot u_t)(\mathsf{id} \odot u_s \odot \mathsf{id}_{E_t}) \ =\ \mathsf{id} \odot u_s \odot u_t$$

we see that $u^{\odot} = \bigl(u_t\bigr)_{t\in\mathbb{T}}$ is a morphism. (Pay again attention to the identifications, e.g. like $\mathsf{id}_E = \mathsf{id}_E \odot \mathsf{id}_{E'_s}$ when writing $\mathfrak{u}_t = \mathsf{id}_E \odot u_t$.)

Conversely, suppose $u^{\odot}$ is an isomorphism from $E^{\bar{\odot}}$ to $E'^{\bar{\odot}}$. Then $\mathfrak{u}_t = \mathsf{id} \odot u_t \colon E = E \bar{\odot} E_t \to E \bar{\odot} E'_t = E$ defines a unitary on E. We find

$$\mathfrak{u}_t \vartheta_t(a)\mathfrak{u}_t^* \;=\; (\mathsf{id} \odot u_t)(a \odot \mathsf{id}_{E_t})(\mathsf{id} \odot u_t^*) \;=\; (a \odot \mathsf{id})(\mathsf{id} \odot u_t u_t^*) \;=\; a \odot \mathsf{id}_{E'_t} \;=\; \vartheta'(a)$$

and as above

$$\mathfrak{u}_{s+t} \;=\; \mathsf{id} \odot u_s \odot u_t \;=\; (\mathsf{id} \odot \mathsf{id}_{E'_s} \odot u_t)(\mathsf{id} \odot u_s \odot \mathsf{id}_{E_t}) \;=\; \mathfrak{u}_t \vartheta_t(\mathfrak{u}_s)$$

In other words, $\mathfrak{u} = \big(\mathfrak{u}_t\big)_{t \in \mathbb{T}}$ is a unitary left cocycle and $\vartheta' = \vartheta^{\mathfrak{u}}$. ■

Now we want to know under which circumstances (E, ϑ, ξ) is a weak dilation, or even a white noise.

14.1.6 Proposition. *For the triple (E, ϑ, ξ) the following conditions are equivalent.*

1. *The family p_t of projections is increasing, i.e. $p_t \geq p_0$ for all $t \in \mathbb{T}$.*

2. *The mappings $T_t(b) = \langle \xi, j_t(b)\xi \rangle$ define a unital CP-semigroup T, i.e. (E, ϑ, ξ) is a weak dilation.*

3. *$T_t(\mathbf{1}) = \mathbf{1}$ for all $t \in \mathbb{T}$.*

Under any of these conditions the elements $\xi_t = \xi \in E_t$ form a unital unit $\xi^{\odot}$ such that $T_t(b) = \langle \xi_t, b\xi_t \rangle$, and the j_t form a weak Markov flow of T on E. The one-sided inductive limit for $\xi^{\odot}$ coincides with the submodule $E_\infty = \lim\limits_{t \to \infty} p_t E$ of E.

Proof. $1 \Rightarrow 2$. If p_t is increasing, then $p_t\xi = p_t p_0 \xi = p_0 \xi = \xi$ so that $T_t(\mathbf{1}) = \langle \xi, p_t \xi \rangle = \mathbf{1}$ and

$$\begin{aligned} T_t \circ T_s(b) \;&=\; \Big\langle \xi, j_t\big(\langle \xi, j_s(b)\xi \rangle\big)\xi \Big\rangle \;=\; \Big\langle \xi, \vartheta_t\big(\xi \langle \xi, \vartheta_s(\xi b \xi^*)\xi \rangle \xi^*\big)\xi \Big\rangle \\ &=\; \big\langle \xi, p_t \vartheta_t \circ \vartheta_s(\xi b \xi^*) p_t \xi \big\rangle \;=\; \big\langle \xi, \vartheta_{s+t}(\xi b \xi^*)\xi \big\rangle \;=\; T_{s+t}(b). \end{aligned}$$

$2 \Rightarrow 3$ is clear. For $3 \Rightarrow 1$ assume that T_t is unital. We find $p_0 = \xi\xi^* = \xi T_t(\mathbf{1})\xi^* = \xi\xi^* \vartheta_t(\xi\xi^*)\xi\xi^* = p_0 p_t p_0$, hence, by Proposition A.7.2(4) $p_t \geq p_0$ for all $t \in \mathbb{T}$.

If p_t is increasing then $p_t \xi = \xi$ so that ξ is, indeed, contained in all E_t. Obviously, $\xi_s \odot \xi_t = \vartheta_t(\xi_s \xi^*)\xi_t = \vartheta_t(\xi\xi^*)\xi = \xi$ so that $\xi^{\odot}$ is a unital unit. However, the identification of ξ as an element $\xi_t \in E_t$ changes the left multiplication, namely, $b\xi_t = j_t(b)\xi$, i.e. $T_t(b) = \langle \xi_t, b\xi_t \rangle$. The Markov property follows as in the proof of Theorem 11.4.8. As above, we have $\xi_s \odot x_t = p_t x_t = x_t$. In other words, $\gamma_{(s+t)t}$ is the canonical embedding of the subspace E_t into E_{s+t}. This identifies E_∞ as the inductive limit for $\xi^{\odot}$. ■

14.1.7 Proposition. *On E_∞ there exists a (unital!) left multiplication of $\mathcal{B}$ such that all E_t are embedded into E_∞ as two-sided submodules, if and only if the unit $\xi^\odot$ is central, i.e. if $(E_\infty, \vartheta, \mathfrak{i}, \xi)$ with $\mathfrak{i}$ being the canonical left multiplication of $\mathcal{B}$ on E is a (unital) white noise.*

PROOF. Existence of a left multiplication on E_∞ implies (in particular) that $b\xi = j_0(b)\xi = \xi b$. The converse direction follows from Section 11.5. ■

This shows once again the importance of existence of a central unit. Without central unit we may not even hope to understand a dilation as a cocycle perturbation of a white noise.

14.1.8 Theorem. *Let (E, ϑ, ξ) be a triple consisting of a Hilbert $\mathcal{B}$–module E, a strict E_0–semigroup ϑ on $\mathcal{B}^a(E)$, and a unit vector $\xi \in E$, and let $\mathfrak{i}\colon \mathcal{B} \to \mathcal{B}^a(E)$ be a representation of $\mathcal{B}$ on E. If $T_t(b) = \langle \xi, \vartheta_t \circ \mathfrak{i}(b)\xi \rangle$ defines a unital CP-semigroup T on $\mathcal{B}$, i.e. if $(E, \vartheta, \mathfrak{i}, \xi)$ is a dilation, then (E, ϑ, ξ) is a weak dilation of T. In other words, any dilation on a Hilbert module has sitting inside also a weak dilation.*

PROOF. We have $\langle \xi, \mathfrak{i}(b)\xi \rangle = b$ so that

$$T_t(b) \;=\; \langle \xi, \vartheta_t \circ \mathfrak{i}(b)\xi \rangle \;=\; \langle \xi \odot \xi_t, (\mathfrak{i}(b) \odot \mathsf{id}_{E_t})(\xi \odot \xi_t) \rangle \;=\; \big\langle \xi_t, \langle \xi, \mathfrak{i}(b)\xi \rangle \xi_t \big\rangle \;=\; \langle \xi, j_t(b)\xi \rangle. \;\blacksquare$$

The set-up of this section is probably most similar to that of Arveson in that we study E_0–semigroups on the algebra of operators on a Hilbert module instead of a Hilbert space. (Normality is replaced by strictness. Passing to von Neumann modules we may again weaken to normality.) The only ingredient what we require is existence of a unit vector, in Hilbert spaces a triviality, here a restriction even in von Neumann modules.

14.1.9 Example. For a projection $p \in \mathcal{B}$ let $E = p\mathcal{B}$ be some right ideal in a von Neumann algebra $\mathcal{B}$. Then by Example 1.1.5 E is a von Neumann $\mathcal{B}$–module. Let $q \in \mathcal{B}$ be the central projection generating the strong closure $q\mathcal{B}$ of $\mathcal{B}_E = \mathsf{span}(\mathcal{B}p\mathcal{B})$. Already in this simple case, the question for a possible unit vector $pb \in E$ has different answers, depending on the choice of $\mathcal{B}$ and p.

Let $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2$. Then E consists of all matrices $B = \left(\begin{smallmatrix} b & b' \\ 0 & 0 \end{smallmatrix}\right)$ $(b, b' \in \mathbb{C})$. Consequently, $\langle B, B \rangle = B^*B = \left(\begin{smallmatrix} \bar{b}b & \bar{b}b' \\ \overline{b'}b & \overline{b'}b' \end{smallmatrix}\right)$. If this is $\mathbf{1}$ then $\overline{b'}b = 0$ from which $b' = 0$ or $b = 0$ so that $\overline{b'}b' = 0$ or $\bar{b}b = 0$. Hence, $\langle B, B \rangle \neq \mathbf{1}$.

Conversely, by definition in a ***purely infinite*** unital C^*–algebra $\mathcal{B}$ for any $a \geq 0$ (in particular, for the projection p) there exists $b \in \mathcal{B}$ such that $b^*ab = \mathbf{1}$. Instead of exploiting this systematically, we give an example. Consider the elements $b = \ell^*\binom{1}{0}$ and $b' = \ell^*\binom{0}{1}$

in $\mathcal{B} = \mathcal{B}(\mathcal{F}(\mathbb{C}^2))$. (Observe that the von Neumann algebra generated by b, b' is $\mathcal{B}$. The C^*–algebra generated by b, b' is the *Cuntz algebra* $\mathcal{O}_2$ [Cun77].) Now the matrix $B \in M_2(\mathcal{B})$ defined as before is a unit vector in $pM_2(\mathcal{B})$ (where p acts in the obvious way).

To answer the interesting question, whether each product system comes from an E_0–semigroup (answered in the affirmative sense by Arveson [Arv90b] for seprarable Hilbert spaces), we certainly have to find first the right technical conditions (measurability in Arveson's set-up). We reserve these and related problems for future work.

We close with a discussion in how far the set-up of Kümmerer [Küm85] (being the basis for Hellmich, Köstler and Kümmerer[HKK98, Hel01, Kös00]) fits into ours. We do not require all assumptions from [Küm85], and phrase only those which we need when we need them. In the first place, we do not consider only von Neumann algebras with faithful invariant normal states, but unital pre–C^*–algebras with a faithful conditional expectation. Invariance of this conditional expectation takes place, if and only if we are concerned with a white noise.

Let $\mathcal{A}$ be a unital pre–C^*–algebra with a C^*–subalgebra $\mathcal{B}$, containing the unit of $\mathcal{A}$, and a faithful essential conditional expectation $\varphi\colon \mathcal{A} \to \mathcal{B}$ (cf. Definition 4.4.1). Denote by (E, ξ) the GNS-construction for φ. Since φ is contractive, so is the pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module E. Since φ is essential, we may identify $\mathcal{A}$ isometrically as a pre–C^*–subalgebra of $\mathcal{B}^a(E)$. Since φ is faithful, the mapping $a \mapsto a\xi$ is a bijection $\mathcal{A} \to E$. Recall that E is also a $\mathcal{B}$–$\mathcal{B}$–module, and that $\xi \in C_{\mathcal{B}}(E)$.

An automorphism α of $\mathcal{A}$ gives rise to a bijective mapping $u\colon a\xi \mapsto \alpha(a)\xi$ on E. If α leaves *invariant* φ (i.e. $\varphi \circ \alpha = \varphi$), then $\langle ua\xi, ua'\xi\rangle = \varphi(\alpha(a^*a')) = \varphi(a^*a') = \langle a\xi, a'\xi\rangle$ so that u is a unitary. Moreover,

$$uau^*a'\xi \;=\; ua\alpha^{-1}(a')\xi \;=\; \alpha(a\alpha^{-1}(a'))\xi \;=\; \alpha(a)a'\xi.$$

In other words, $\alpha = u \bullet u^*$ extends to an inner automorphism of $\mathcal{B}^a(E)$ and further of $\mathcal{B}^a(\overline{E})$. Now suppose that $(\mathcal{A}, \alpha, \mathsf{id}, \varphi)$ is a (unital) white noise with an automorphism group α. It follows that there exists a unitary group u in $\mathcal{B}^a(E)$ such that $(E, u \bullet u^*, \mathsf{id}, \xi)$ is a (unital) white noise with a semigroup of inner automorphisms. In particular, any white noise *over* $\mathcal{B}$ in the sense of [Küm85] may be first identified as a white noise on a subalgebra of $\mathcal{B}^a(E)$ an then extended to a white noise on the Hilbert module $\overline{E}$. It is routine to show that also (after passing to the von Neumann module $\overline{E}^s$) technical conditions like normality and strong continuity of $u \bullet u^*$ are preserved. Of course, the extension $\langle \xi, \bullet \xi\rangle$ to $\mathcal{B}^a(\overline{E}) \supset \mathcal{B}$ of the conditional expectation φ is no longer faithful (but, still essential).

Observe that u leaves invariant ξ so that the associated product system is the trivial one (cf. Example 14.1.4). This situation does also not change, if we pass from the white noise α to a cocycle perturbation $\alpha^{\mathfrak{u}}$ (dilating some non-trivial CP-semigroup T), because by Theorem 14.1.5 the product system is an invariant for outer conjugacy.

Suppose the automorphism white noise $(\mathcal{A}, \alpha, \mathsf{id}, \varphi)$ comes along with a *filtration* which is *covariant* for the time shift automorphism group α. In particular, $\mathcal{A}$ has a *future subalgebra* $\mathcal{A}_+$ which is left invariant by all α_t for $t \geq 0$. Thus, by restricting α to $\mathcal{A}_+$ and $t \geq 0$ as in Example 14.1.4, we obtain an E_0–semigroup ϑ on $\mathcal{A}_+$ (which under a simple minimality condition on the filtration is non-trivial, if α is) and $(\mathcal{A}_+, \vartheta, \mathsf{i}, \varphi \upharpoonright \mathcal{A}_+)$ is a unital endomorphism white noise.

The GNS-module of $\varphi \upharpoonright \mathcal{A}_+$ is the submodule $E_+ = \mathcal{A}_+\xi$ of E. We may ask whether it is possible to extend the E_0–semigroup ϑ on $\mathcal{A}_+$ to an E_0–semigroup on $\mathcal{B}^a(E_+)$. Another way to say this, is to ask whether the *unital* embedding $\mathcal{A}_+ \to \mathcal{A}$ allows for an extension to a *unital* embedding $\mathcal{B}^a(E_+) \to \mathcal{B}^a(E)$ such that that restriction of $u \bullet u^*$ to $\mathcal{B}^a(E_+)$ gives rise to an E_0–semigroup on E_+. A solution of this problem should be related to the question whether it is possible to factorize E into $E_+ \odot E_+^0$. (Like for the full Fock module in Example 14.1.4, also E_+^0 should be some submodule of E, but it need not be the GNS-module E_- of the *past algebra* $\mathcal{A}_-$.) Presently, we do not know whether there is an affirmative answer. Hower, if the answer *is* affirmative, then we will have a canonical way to associate with a white noise $(\mathcal{A}, \alpha, \mathsf{id}, \varphi)$ in the sense of [Küm85] a white noise $(E_+, \vartheta, , \mathsf{i}, \xi)$ of (non-trivial) endomorphisms, and further a (non-trivial) product system. Thus, we may hope that product systems help classifying also white noises in the sense of [Küm85].

14.2 Transition expectations

In [Lie00b] Liebscher proposed a continuous time version of *quantum Markov chains* in the sense of Accardi [Acc74, Acc75]. For $0 \leq t \leq \infty$ let us consider the Hilbert spaces $H_t = G \bar{\otimes} \Gamma(L^2([0,t), \mathfrak{H})$ (with initial space G and another Hilbert space $\mathfrak{H}$). Denoting $\mathcal{B} = \mathcal{B}(G)$, $\mathcal{A}_t = \mathcal{B}(H_t)$ and $\mathcal{A}_t^c = \mathcal{B}(\Gamma(L^2([0,t), \mathfrak{H}))$, we have $\mathcal{A}_t = \mathcal{B} \bar{\otimes}^s \mathcal{A}_t^c$ and $\mathcal{A}_{s+t}^c = \mathcal{A}_s^c \bar{\otimes}^s \mathcal{A}_t^c$, whence, also $\mathcal{A}_{s+t} = \mathcal{A}_s \bar{\otimes}^s \mathcal{A}_t^c$. Liebscher defines a *system of transition expectations* as a family $\big(T_t^L\big)_{t \in \mathbb{R}_+}$ of unital (normal) completely postitive mappings $T_t^L \colon \mathcal{A}_t \to \mathcal{B}$ fulfilling $T_{s+t}^L = T_t^L \circ (T_s \otimes \mathsf{id})$. One can show that the typical system of transition expectations arises via $T_t^L = u_t^* \bullet u_t$ from a family $u = \big(u_t\big)_{t \in \mathbb{R}_+}$ of isometries $u_t \colon G \to H_t$ fulfilling $u_{s+t} = (u_s \otimes \mathsf{id})u_t$. Since this property reminds us of a cocycle property, Liebscher called u a *cocycle of type (H)*.

14.2.1 Remark. The discrete version in the case where $\mathcal{A}_n^c = \mathcal{B}^{\otimes n}$ $(\mathcal{A}_\infty^c = \ldots \otimes \mathcal{B} \otimes \mathcal{B})$ and $\mathcal{A}_n = \mathcal{B} \otimes \mathcal{A}_n^c$ gives us back Accardi's quantum Markov chains. Here the transition expectations T_n^A are determined uniquely by $T_1^A \colon \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$ and the composition property.

14.2.2 Remark. By restriction to $\mathfrak{H} = \{0\}$, i.e. $\mathcal{A}_t^c = \mathbb{C}$, we are concerned with the case of unital CP-semigroups on $\mathcal{B}$.

We know that we may interprete H_t as $E_t \bar{\odot} G$ where $E_t = \Gamma(L^{2,s}([0,t), \overline{\mathfrak{H}_{\mathcal{B}}}^s))$ via Stinespring representation, and that $\mathcal{A}_t = \mathcal{B}^a(E_t)$, whereas $\mathcal{A}_t^c$ is just the $\mathcal{B}$–center $\mathcal{B}^{a,bil}(E_t)$ of $\mathcal{A}_t$; see Examples 6.1.6 and 14.1.3. In this module interpretation u_t is an element in $\mathcal{B}(G, H_t) = E_t$ and the cocycle property means nothing else, but that u is a (unital) unit such that $T_t^L = \langle u_t, \bullet u_t \rangle$.

$E^{\bar{\odot}^s} = \left(E_t\right)_{t\in\mathbb{R}_+}$ is isomorphic to a time ordered product system. If $\xi^{\odot}$ is a unital unit in an arbitrary product system $E^{\odot}$ then $T_t(a_t) = \langle \xi_t, a_t\xi_t \rangle$ defines a family of unital completely positive mappings $\mathcal{B}^a(E_t) \to \mathcal{B}$ fulfilling $T_{s+t}(\vartheta_t(a_s)a_t^c) = T_t(T_s(a_s)a_t^c)$ for all $a_s \in \mathcal{B}^a(E_s)$ and $a_t^c \in \mathcal{B}^{a,bil}(E_t)$. Contrary to the case $\mathcal{B} = \mathcal{B}(G)$, here we do not know, whether $\mathcal{B} \subset \mathcal{B}^a(E_t)$ and $\mathcal{B}^{a,bil}(E_t) \subset \mathcal{B}^a(E_t)$ are in *tensor position* (i.e. whether the subalgebra $\operatorname{span}(\mathcal{B}\mathcal{B}^{a,bil}(E_t))$ of $\mathcal{B}^a(E_t)$ is isomorphic to $\mathcal{B} \otimes \mathcal{B}^{a,bil}(E_t)$ (this is certainly wrong if $\mathcal{B} \cap \mathcal{B}^{a,bil}(E_t) \neq \mathbb{C}\mathbf{1}$), nor do we know, whether this subalgebra is strongly dense in $\mathcal{B}^a(E_t)$. The same questions are open for the (mutually comuting) subalgebras $\mathsf{id}_{E_s} \odot \mathcal{B}^{a,bil}(E_t)$ and $\mathcal{B}^{a,bil}(E_s) \odot \mathsf{id}_{E_t}$ of $\mathcal{B}^{a,bil}(E_{s+t})$.

Following Liebscher and Skeide [LS00a], we present a set of axioms on families $\left(\mathcal{A}_t\right)_{t\in\mathbb{T}}$ and $\left(\mathcal{A}_t^c\right)_{t\in\mathbb{T}}$ of pre–C^*–algebras (there is no reason to consider only the semigroup $\mathbb{R}_+$) and transition expectations $\left(T_t\right)_{t\in\mathbb{T}}$ which allows us to show a reconstruction theorem. More precisely, we want to find a product system $E^{\odot}$ such that on each E_t we have a representation of $\mathcal{A}_t$, and a unit $\xi^{\odot}$ such that $T_t(a_t) = \langle \xi_t, a_t\xi_t \rangle$. The obstacles mentioned just before show that we may not hope to conclude backwards, i.e. to find such families from a given pair $(E^{\odot}, \xi^{\odot})$. The reconstruction will follow very much the lines of Section 11.3. As this construction is purely algebraical, we start also here on an algebraical level, pointing out the places where to put topological conditions like contractivity or normality. We only recall that $\mathcal{B}$ is always a unital C^*–algebra (sometimes a von Neumann algebra).

$\mathcal{B} \otimes \mathcal{A}_t^c$ is a particularly simple example of a $\mathcal{B}$–algebra; see Definition 4.4.1. Of course, a $\mathcal{B}$–algebra is a $\mathcal{B}$–$\mathcal{B}$–module, and as such it can be centered or not. We are interested in $\mathcal{B}$–algebras with a distinguished subalgebra $\mathcal{A}^c \subset C_{\mathcal{B}}(\mathcal{A})$ of the $\mathcal{B}$–center of $\mathcal{A}$ such that $\mathcal{A}$ is (toplogically) spanned by $\mathcal{B}\mathcal{A}^c$. From the discussion above we know that $\mathcal{A}_c$ may be, but need not be all of $C_{\mathcal{B}}(\mathcal{A})$.

14.2.3 Definition. Let $\mathcal{A}^{\otimes} = \left(\mathcal{A}_t\right)_{t\in\mathbb{T}}$ be a family of $*$–$\mathcal{B}$–algebras with a family $\mathcal{A}^{c\otimes} = \left(\mathcal{A}_t^c\right)_{t\in\mathbb{T}}$ of $*$–subalgebras $\mathcal{A}_t^c \subset C_{\mathcal{B}}(\mathcal{A}_t)$ of the $\mathcal{B}$–center of $\mathcal{A}_t$ such that $\operatorname{span}(\mathcal{B}\mathcal{A}_t^c) = \mathcal{A}_t$. We require $\mathcal{A}_0 = \mathcal{B}$ and $\mathcal{A}_0^c = \mathbb{C}$.

Let $\alpha = \left(\alpha_{s,t}\right)_{s,t\in\mathbb{T}}$ be a family of unital homomorphisms $\mathcal{A}_s \otimes \mathcal{A}_t^c \to \mathcal{A}_{s+t}$ such that $\operatorname{span} \alpha_{s,t}(\mathcal{A}_s^c \otimes \mathcal{A}_t^c) = \mathcal{A}_{s+t}^c$. (This implies, in particular, that $\operatorname{span} \alpha_{s,t}(\mathcal{A}_s \otimes \mathcal{A}_t^c) = \mathcal{A}_{s+t}$.)

We require that $\alpha_{0,t}$ is the canonical mapping $b \otimes a_t \mapsto ba_t$ and that $\alpha_{t,0} = \mathsf{id}_{\mathcal{A}_t}$, We define $\alpha^c_{s,t} = \alpha_{s,t} \upharpoonright (\mathcal{A}^c_s \otimes \mathcal{A}^c_t)$. We say $(\mathcal{A}^\otimes, \mathcal{A}^{c\otimes}, \alpha)$ is a *left tensor product system of $*$–$\mathcal{B}$–algebras* with *central tensor product system $\mathcal{A}^{c\otimes}$*, if α fulfill the associativity condition

$$\alpha_{r,s+t} \circ (\mathsf{id} \otimes \alpha^c_{s,t}) \;=\; \alpha_{r+s,t} \circ (\alpha_{r,s} \otimes \mathsf{id}).$$

If we speak about pre–C^*–algebras, we require that the $\alpha_{s,t}$ are contractive. If we speak about von Neumann algebras $\alpha_{s,t}$ should be normal. For C^*–algebras or von Neumann algebras, instead of the linear span we take the closure (in the respective topology) of the linear span.

14.2.4 Definition. Let $T = \big(T_t\big)_{t\in\mathbb{T}}$ be a family of unital completely positive mappings $T_t\colon \mathcal{A}_t \to \mathcal{B}$ with $T_0 = \mathsf{id}_\mathcal{B}$ and

$$T_{s+t} \circ \alpha_{s,t} \;=\; T_t \circ \alpha_{0,t} \circ (T_s \otimes \mathsf{id}).$$

We say T is a
indextransition expectations!system ofhlsystem of transition expectations, if there exists a family $\big(\mathcal{T}_{s+t,t}\big)_{s,t\in\mathbb{T}}$ of mappings $\mathcal{T}_{s+t,t}\colon \mathcal{A}_{s+t} \to \mathcal{A}_t$ such that

$$\mathcal{T}_{s+t,t} \circ \alpha_{s,t} \;=\; \alpha_{0,t} \circ (T_s \otimes \mathsf{id}). \tag{14.2.1}$$

We use the conventions as in Definition 14.2.3 for topological variants.

If $\mathcal{T}$ exists, then it is uniquely determined by (14.2.1). Basically, (14.2.1) tells us that the unital completely positive mapping $ba_s \otimes a_t \mapsto T_s(ba_s)a_t$ $(b \in \mathcal{B}, a_s \in \mathcal{A}^c_s, a_t \in \mathcal{A}^c_t)$ factors through $\alpha_{s,t}$. Therefore, also $\mathcal{T}_{s+t,t}$ is unital and completely positive. If all $\alpha_{s,t}$ are injective (like in [Lie00b]), then we may forget about (14.2.1), at least, from the algebraical point of view. Later on, (14.2.1) shows to be responsible for the possibility to define a representation of $\mathcal{A}^c_t$ on the member E_t of the *GNS-system* $E^\odot$ of T.

Another aspect of (14.2.1), even if the $\alpha_{s,t}$ are injective, is the topological one. (The fact that $\mathcal{A}_s \otimes \mathcal{A}^c_t$ is algebraically isomorphic to (a dense subset of) $\mathcal{A}_{s+t}$, does not mean that some natural topology on the tensor product $\mathcal{A}_s \otimes \mathcal{A}^c_t$ gives us back the correct topology on $\mathcal{A}_{s+t}$.) The topological requirements on $\mathcal{T}$ provide us with all necessary information in order that the construction of the GNS-system is compatible with existing topological structures.

Before we come to the construction of the GNS-system, we draw some general consequences from Definition 14.2.4.

14.2.5 Corollary. *The embeddings $\alpha_{s,t} \upharpoonright (\mathbf{1} \otimes \mathcal{A}^c_t)$ of $\mathcal{A}^c_t$ into $\mathcal{A}_{s+t}$ are injective. In other words, we may consider $\mathcal{A}^c_t$ as a subalgebra of $\mathcal{A}_{s+t}$.*

Proof. We apply (14.2.1) to $\mathbf{1}\otimes a_t$ and obtain $\mathcal{T}_{s+t,t}\circ\alpha_{s,t}(\mathbf{1}\otimes a_t) = \alpha_{0,t}\circ(T_s\otimes\mathsf{id})(\mathbf{1}\otimes a_t) = a_t$. Therefore, $\mathcal{T}_{s+t,t}\circ\alpha_{s,t}\upharpoonright(\mathbf{1}\otimes\mathcal{A}_t^c)$ and *a fortiori* $\alpha_{s,t}\upharpoonright(\mathbf{1}\otimes\mathcal{A}_t^c)$ is injective. ∎

The embedding, in general, does not extend to $\mathcal{A}_t$. Saying that the copy of $\mathcal{B}$ in $\mathcal{A}_t$ is attached to time t, it is, roughly speaking, acting at the wrong time to be imbedded into $\mathcal{A}_{s+t}$ where it should act at time $s+t$. We will see later on very clearly that the different actions of $\mathcal{B}$ at different times correspond to a weak Markov flow. Of course, there is an embedding $\alpha_{s,t}\upharpoonright(\mathcal{A}_s\otimes\mathbf{1})$ of $\mathcal{A}_s$ into $\mathcal{A}_{s+t}$, but (except for $s=0$) it need not be injective.

14.2.6 Corollary. *The $\mathcal{A}_t^c$ with the embeddings $\alpha_{s,t}^c\upharpoonright(\mathbf{1}\otimes\mathcal{A}_t^c)$ form an inductive system with inductive limit $\mathcal{A}^c$. On $\mathcal{A}^c$ we define an E_0–semigroup Θ by setting $\Theta_t(a_s)=\alpha_{s,t}^c(a_s\otimes\mathbf{1})$ (where we identify $a_s\in\mathcal{A}_s^c$ and $\alpha_{s,t}(a_s\otimes\mathbf{1})\in\mathcal{A}_{s+t}^c$ with the corresponding elements in $\mathcal{A}^c$).*

14.2.7 Proposition. $$\mathcal{T}_{s+t,t}\circ\mathcal{T}_{r+s+t,s+t}=\mathcal{T}_{r+s+t,t}.$$

Proof. We have

$$\begin{aligned}
\mathcal{T}_{r+s+t,t}\circ\alpha_{r+s,t}\circ(\alpha_{r,s}\otimes\mathsf{id}) &= \alpha_{0,t}\circ(T_{r+s}\otimes\mathsf{id})\circ(\alpha_{r,s}\otimes\mathsf{id})\\
&= \alpha_{0,t}\circ(T_s\otimes\mathsf{id})\circ(\alpha_{0,s}\otimes\mathsf{id})\circ(T_r\otimes\mathsf{id}\otimes\mathsf{id}) = \mathcal{T}_{s+t,t}\circ\alpha_{s,t}\circ(\alpha_{0,s}\otimes\mathsf{id})\circ(T_r\otimes\mathsf{id}\otimes\mathsf{id})\\
&= \mathcal{T}_{s+t,t}\circ\alpha_{0,s+t}\circ(\mathsf{id}\otimes\alpha_{s,t}^c)\circ(T_r\otimes\mathsf{id}\otimes\mathsf{id}) = \mathcal{T}_{s+t,t}\circ\alpha_{0,s+t}\circ(T_r\otimes\mathsf{id}\otimes\mathsf{id})\circ(\mathsf{id}\otimes\alpha_{s,t}^c)\\
&= \mathcal{T}_{s+t,t}\circ\mathcal{T}_{r+s+t,s+t}\circ\alpha_{r,s+t}\circ(\mathsf{id}\otimes\alpha_{s,t}^c) = \mathcal{T}_{s+t,t}\circ\mathcal{T}_{r+s+t,s+t}\circ\alpha_{r+s,t}\circ(\alpha_{r,s}\otimes\mathsf{id}).
\end{aligned}$$

Since, the range of $\alpha_{r+s,t}\circ(\alpha_{r,s}\otimes\mathsf{id})$ is all of $\mathcal{A}_{r+s+t}$ this shows the statement. ∎

The mappings $\mathcal{T}_{s+t,t}$ have some aspects from a *markovian system of conditional expectations*; see Accardi [Acc78]. Of course, neither $\mathcal{T}_{s+t,t}$ nor $\mathcal{T}_{s+t,t}\upharpoonright(\mathcal{A}_{s+t}^c)$ are conditional expectations. The former are not, because (cf. the discussion before Proposition 14.2.7) $\mathcal{A}_t$ cannot be identified with a subalgebra of $\mathcal{A}_{s+t}$, and the latter are not, because they map into $\mathcal{A}_t$, not $\mathcal{A}_t^c$.

We come to the construction of the GNS-system. Denote by $\breve{E}_{s+t,t}$ the GNS-module of $\mathcal{T}_{s+t,t}$ with cyclic vector $\breve{\xi}_{s+t,t}$ and denote by $\breve{E}_t$ the GNS-module of T_t with cyclic vector $\breve{\xi}_t$. We may consider the $\mathcal{A}_{s+t}$–$\mathcal{A}_t$–module $\breve{E}_{s+t,t}$ also as $\mathcal{A}_s$–$\mathcal{B}$–module (of course not, pre-Hilbert module, because the inner product takes values in $\mathcal{A}_t$, not in $\mathcal{B}$) via the embeddings $\mathcal{A}_s\to\alpha_{s,t}(\mathcal{A}_s\otimes\mathbf{1})\subset\mathcal{A}_{s+t}$ and $\mathcal{B}\to\alpha_{0,t}(\mathcal{B}\otimes\mathbf{1})\subset\mathcal{A}_t$.

14.2.8 Proposition. *The $\mathcal{A}_s$–$\mathcal{B}$–submodule of $\breve{E}_{s+t,t}$ generated by $\breve{\xi}_{s+t,t}$ is isomorphic to the pre-Hilbert $\mathcal{A}_s$–$\mathcal{B}$–module $\breve{E}_s$ and the $\mathcal{A}_s$–$\mathcal{B}$–linear extension of the mapping*

$$\breve{\xi}_{s+t,t}\longmapsto\breve{\xi}_s$$

is the isomorphism.

Proof. It is sufficient to show that

$$\langle \breve{\xi}_{s+t,t}, \alpha_{s,t}(a_s \otimes \mathbf{1})\breve{\xi}_{s+t,t}\rangle \;=\; \mathcal{T}_{s+t,t} \circ \alpha_{s,t}(a_s \otimes \mathbf{1}) \;=\; \alpha_{0,t}(T_s(a_s) \otimes \mathbf{1})$$

for all $a_s \in \mathcal{A}_s$. Then also the $\mathcal{A}_s$–$\mathcal{B}$–linear extension is isometric (of course, it is surjective) and, therefore, well-defined. *A fortiori* the inner product of the submodule of $\breve{E}_{s+t,t}$ takes values in $\alpha_{0,t}(\mathcal{B} \otimes \mathbf{1}) \cong \mathcal{B} \subset \mathcal{A}_t$. ■

We observe that the elements of $\mathcal{A}_t^c$ $(\subset \mathcal{A}_{s+t})$ commute with all elements in the the $\mathcal{A}_s$–$\mathcal{B}$–submodule $\breve{E}_s \subset \breve{E}_{s+t,t}$, and that $\breve{E}_{s+t,t}$ is generated by $\mathcal{A}_t^c$ and $\breve{E}_s$.

14.2.9 Proposition. *Let F be a pre-Hilbert $\mathcal{A}_t$–$\mathcal{C}$–module (which we may also consider as a pre-Hilbert $\mathcal{B}$–$\mathcal{C}$–module). Then*

$$\breve{E}_s \odot F \;=\; \breve{E}_{s+t,t} \odot F.$$

In particular, $\breve{E}_s \odot F$ is a pre-Hilbert $\mathcal{A}_{s+t}$–$\mathcal{C}$–module.

Proof. $\breve{E}_{s+t,t} \odot F$ is spanned by elements of the form $x_s a_t \odot y = x_s \odot a_t y \in \breve{E}_s \odot F$ $(x_s \in \breve{E}_s, a_t \in \mathcal{A}_t^c, y \in F)$. ■

14.2.10 Corollary. *Let $\mathfrak{t} \in \mathbb{J}_t$ and $\mathfrak{s} = \mathfrak{o}(\mathfrak{t}) \in \mathbb{I}_t$ (see Proposition B.3.2). Then*

$$\breve{E}_{\mathfrak{t}} \;:=\; \breve{E}_{t_n} \odot \ldots \odot \breve{E}_{t_1} \;=\; \breve{E}_{s_n,s_{n-1}} \odot \breve{E}_{s_{n-1},s_{n-2}} \odot \ldots \odot \breve{E}_{s_1,0}$$

is a pre-Hilbert $\mathcal{A}_t$–$\mathcal{C}$–module.

14.2.11 Remark. The crucial point here is that, although we construct $\breve{E}_{\mathfrak{t}}$ as multiple tensor product of $\mathcal{B}$–$\mathcal{B}$–modules, it carries a well-defined left action of $\mathcal{A}_t$. The reason why this works can be traced back to the condition in (14.2.1). The message is that an element $a = a_{t_n} \ldots a_{t_1} \in \mathcal{A}_t^c$ which is thought of, roughly speaking, as a product of elements $a_{t_i} \in \mathcal{A}_{t_i}^c$ suitably shifted to the interval $[s_{i-1}, s_i]$ acts as $a(x_{t_n} \odot \ldots \odot x_{t_1}) = a_{t_n} x_{t_n} \odot \ldots \odot a_{t_1} x_{t_1}$. We do not formulate this in a more precise manner. We only want to give an intuitive idea.

Now we are reduced precisely to the situation in Section 11.3. We define two-sided isometric mappings $\beta_{\mathfrak{t}\mathfrak{s}} \colon \breve{E}_{\mathfrak{s}} \to \breve{E}_{\mathfrak{t}}$ and construct an inductive limit E_t. The only difference is that we are concerned with an inductive system of pre-Hilbert $\mathcal{A}_t$–$\mathcal{B}$–modules. Consequently, also the inductive limit E_t is a pre-Hilbert $\mathcal{A}_t$–$\mathcal{B}$–module. Nevertheless, considering E_t as a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module, the E_t form a product system $E^{\odot}$. Also here the $\breve{\xi}_t$ give rise to elements $\xi_t \in E_t$ which form a unital unit $\xi^{\odot}$ for $E^{\odot}$. We collect these and some more fairly obvious results.

14.2.12 Theorem. *Let T be a system of transition expectations for $(\mathcal{A}^{\otimes}, \mathcal{A}^{c\otimes}, \alpha)$. Then there exists a pair $(E^{\odot}, \xi^{\odot})$ consisting of a product system pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules $E^{\odot}$ and a unital unit $\xi^{\odot}$ for $E^{\odot}$, fulfilling the following properties.*

E_t is also a pre-Hilbert $\mathcal{A}_t$–$\mathcal{B}$–module, and generated as such by $E_t^{\{\xi^{\odot}\}}$. The restriction of the left multiplication of $\mathcal{A}_t$ to the subset $\mathcal{B}$ gives back the correct left multiplication of $\mathcal{B}$. In particular, $\mathcal{A}_t^c$ is represented as a subset of $\mathcal{B}^{a,bil}(E_t)$. Finally, $T_t(a) = \langle \xi_t, a\xi_t \rangle$ for $a \in \mathcal{A}_t$.

The pair $(E^{\odot}, \xi^{\odot})$ is determined by these properties up to isomorphism. We call $E^{\odot}$ the GNS-system of T.

Such a product system has a unit, but it need not be generated by it. It is also not sure, whether $E^{\odot}$ will be generated by all of its units. (This fails probably already in the case $\mathcal{B} = \mathcal{B}(G)$, when we tensorize $\mathcal{B}$ with a type II Arveson system $\mathfrak{H}^{\bar{\otimes}}$ and take for T the expectations generated by a unit for $\mathfrak{H}^{\bar{\otimes}}$.) It is interesting to ask, whether the preceding construction allows to find non-type I product systems which are not tensor products with non-type I Arveson systems. We postpone such questions to future work.

14.2.13 Theorem. *On the one-sided inductive limit E for the unit $\xi^{\odot}$ (see Section 11.4), besides the weak Markov flow j of $\mathcal{B}$, we have a family $j^c = \big(j_t^c\big)_{t\in\mathbb{T}}$ of unital representations $a_t \mapsto \mathsf{id} \odot a_t$ of $\mathcal{A}_t^c$. These representations are compatible with the inductive structure of the $\mathcal{A}_t^c$ (i.e. $j_{s+t}^c \circ \alpha_{s+t} \upharpoonright (\mathbf{1} \otimes \mathcal{A}_t^c) = j_t^c$). Therefore, there is a unique unital representation j_∞^c of $\mathcal{A}^c$ on E. (As there is, in general, no natural left action of $\mathcal{B}$ on E, it does not make sense to speak about bilinear operators on E.) Moreover, $j_\infty^c \circ \Theta_t = \vartheta_t \circ j_\infty^c$.*

By $J_t \circ \alpha_{0,t} = m \circ (j_t \otimes j_t^c)$ (where m denotes multiplication in $\mathcal{B}^a(E)$) we define a family $J = \big(J_t\big)_{t\in\mathbb{T}}$ of representations J_t of $\mathcal{A}_t$. These representations fulfill the generalized Markov property

$$p_t J_{s+t}(a) p_t \;=\; J_t \circ \mathcal{T}_{s+t,t}(a).$$

14.2.14 Corollary. *An adapted unitary cocycle $\mathfrak{u}^c$ for Θ (i.e. $\mathfrak{u}_t^c \in \mathcal{A}_t^c$) gives rise to a local cocycle $\mathfrak{u}$ for ϑ via $\mathfrak{u}_t = \mathsf{id} \odot \mathfrak{u}_t^c$.*

Needless, to say that all statements extend to completions or closures under the assumed compatibility conditions. We do not go into details, because it is fairly clear from the corresponding arguments in Chapter 11. We only mention as typical example for the argument that the assumption of normality for $\mathcal{T}_{s+t,t}$ (when $\mathcal{B}$ and $\mathcal{A}_t$ are von Neumann algebras) guarantees that $\overline{\breve{E}}^{s}_{s+t,t}$ is a von Neumann $\mathcal{A}_t$–$\mathcal{B}$–module.

Chapter 15

Outlook

In this chapter we summarize what we achieved about product systems and compare it with existing results in special cases. Then we put it into contrast with open problems for which we have not yet a solution or only a partial solution.

15.1 Existing and open correspondences

By Theorem 14.1.1 we associate with any strict E_0–semigroup ϑ on $\mathcal{B}^a(E)$ a product system $E^\odot$, if at least one unit vector $\xi \in E$ exists. Moreover, we construct an isomorphism of $E \odot E_t$ and E such that ϑ can be recovered as $\vartheta_t(a) = a \odot \mathsf{id}_{E_t}$. Proposition 14.1.2 asserts that $E^\odot$ does not depend (up to isomorphism) on the choice of ξ. Theorem 14.1.5 tells us that two strict E_0–semigroups ϑ and ϑ' on $\mathcal{B}^a(E)$ have the same product system, if and only if they are outer conjugate. (These results require no assumption on some continuity of the t–dependence of ϑ. The form we recover ϑ with the help of the associated product system shows, however, that strictness of each ϑ_t is indispensable.)

Specialization to the case where $E = G$ is some Hilbert space (and strongly continuous normal E_0–semigroups with $\mathbb{T} = \mathbb{R}_+$), gives us back Bhat's construction [Bha96] of product system which is, as discussed in Example 14.1.3, equivalent to the result of Arveson's original construction [Arv89a].

Arveson's result [Arv90b] that any Arveson system arises in that way from an E_0–semigroup is, presently, out of our reach and probably wrong in the stated algebraic generality. First, we must find the correct technical conditions replacing Arveson's measurability assumptions on the product system. For the time being, from our construction in Theorem 14.1.1 we cannot say much more than that, starting from a product system $E^\odot$, its members E_t should be embeddable (as right modules) isometrically into a fixed (pre-)Hilbert $\mathcal{B}$–module E with a unit vector ξ. In this case there would also be supply for measurability conditions on certain *cross sections*.

The following diagram illustrates the situation.

$$
\begin{tikzcd}
(E,\vartheta,\xi) \arrow[rr, shift left, "\text{[Ske00a]}"] \arrow[dr, dotted] & & E^{\odot} \arrow[ll, shift left, "?"] \arrow[dl, dotted] \\
& \text{specialization} \arrow[dl, dotted] \arrow[dr, dotted] & \\
(G,\vartheta,g) \arrow[rr, shift left, "\text{[Bha96] ($\sim$ [Arv89a])}"] & & \mathfrak{H}^{\bar{\otimes}} \arrow[ll, shift left, "\text{[Arv90b]}"]
\end{tikzcd}
$$

If (E,ϑ,ξ) and (E,ϑ',ξ') are strict E_0–semigroups on the same $\mathcal{B}^a(E)$ with possibly different unit vectors ξ,ξ' but isomorphic product sytems $E^{\odot} \cong E'^{\odot}$, then we know from Proposition 14.1.2 and Theorem 14.1.5 that the semigroups are outer conjugate, as with normal E_0–semigroups on $\mathcal{B}(H)$ and Arveson systems. However we know nothing like that, if E is not isomorphic to the module E' where ϑ' lives. In particular, we do not know, whether this may happen at all.

The situation gets considerably more complicated, if the triple (E,ϑ,ξ) is a weak dilation. By Proposition 14.1.6 this is the case, if and only if $\langle\xi,\vartheta_t(\xi\xi^*)\xi\rangle = \mathbf{1}$ for all $t \in \mathbb{T}$, in which case the mappings $T_t = \langle\xi, j_t(\bullet)\xi\rangle$ with $j_t = \vartheta_t(\xi \bullet \xi^*)$ form a unital CP-semigroup T on $\mathcal{B}$ dilated by ϑ, and the j_t form a weak Markov flow j on E for T. The product system $E^{\odot}$ associated with this weak dilation has a unital unit $\xi^{\odot}$ and $T^{\xi} = T$. By Theorem 14.1.8 we may always pass from a dilation $(E,\vartheta,\mathfrak{i},\xi)$ to the weak dilation (E,ϑ,ξ).

Conversely, if we start with a product system $E^{\odot}$ with a unital unit $\xi^{\odot}$, then the one-sided inductive limit from Section 11.4 provides us with a dilation (E,ϑ,ξ) of the unital CP-semigroup T^{ξ}. If $E_\infty = E$, this dilation is called *primary*. In general, the pair $(E^{\odot},\xi^{\odot})$ constructed from an arbitrary weak dilation (E,ϑ,ξ) gives us back the weak dilation via the one-sided inductive limit, if and only if it is primary.

Also starting from a unital CP-semigroup T, the two-sided inductive limit from Section 11.3 provides us with a product system $E^{\odot}$ and a unital unit $\xi^{\odot}$. The product system is the minimal, i.e. $E^{\{\xi\}^{\odot}}$, whence by Proposition 12.1.1 the dilation constructed via the one-sided inductive limit is the minimal one, i.e. ξ is cyclic for the weak Markov flow j.

We illustrate this.

$$
\begin{tikzcd}
(E,\vartheta,\mathfrak{i},\xi) \arrow[dr, shift left, "\text{[Ske00a]}"'] & T \arrow[r, equal] & T^{\xi} \arrow[dr, shift left] & \\
 & (E,\vartheta,\xi) \arrow[ul, shift left, "?"'] \arrow[u, "\text{[Ske00a]}"'] \arrow[r, "\text{[Ske00a]}"] \arrow[dl, "\text{[Ske00a]}"'] \arrow[dr, shift left, "\text{restr.}"'] & (E^{\odot},\xi^{\odot}) \arrow[u] \arrow[r, shift left, "\text{restr.}"] \arrow[d, "\text{[BS00]}"] & (E^{\{\xi\}^{\odot}},\xi^{\odot}) \arrow[ul, shift left, "\text{[BS00]}"'] \arrow[l, shift left, dashed, "E^{\odot}=E^{\{\xi\}^{\odot}}"] \arrow[d, "\text{[BS00]}"] \\
(E,j) \arrow[r, "\text{restr.}"'] & (E_\infty,j) & (E_\infty,\vartheta,\xi) \arrow[l, "\text{restr.}"] \arrow[ul, shift left, dashed, "E=E_\infty"'] \arrow[r, shift right, "\text{restr.}"'] & (E^{min},\vartheta,\xi) \arrow[l, shift right, dashed, "E_\infty=E^{min}"']
\end{tikzcd}
$$

Here the interrupted arrows hold, if and only if the assigned conditions hold. Thus, starting with a dilation (E, ϑ, ξ) and going upwards to the CP-semigroup T, the clockwise way arround the diagram gives back the dilation, if and only if $E = E^{min}$, i.e. if the dilation is the GNS-dilation of T. Restricting the GNS-dilation ϑ on $\mathcal{B}^a(E^{min})$ to $\mathcal{A}_\infty = \mathsf{alg}\, j_{\mathbb{T}}(\mathcal{B})$, we obtain the *minimal* e_0*–dilation* from Bhat [Bha99]. Open is the question, in how far it is possible to turn a weak dilation (E, ϑ, ξ) into other types of dilation $(E, \vartheta, \mathsf{i}, \xi)$, in particular, into unital dilations. There is a partial result by Bhat [Bha01] who provides us with a criterion (cf. Remark 13.2.9) to decide from the form of the generator of a normal uniformly continuous unital CP-semigroup on $\mathcal{B}(G)$, whether the E_0–semigroup of the GNS-dilation coincides with the unital dilation constructed from that generator on the symmetric Fock space with the help of Hudson-Parthasarathy calculus.

On the level of dilation the relation of the diagram with existing work on $\mathcal{B}(G)$ is rather boring. Choosing any unit vector in G we obtain a dilation of the only possible unital CP-semigroup $\mathbb{C}$, namely, id. It is more interesting to think of the construction of a product system from a CP-semigroup as a generalization of Arevson's construction of a product system from an E_0–semigroup on $\mathcal{B}(G)$. The generalization has two aspects. Firstly, passing from E_0–semigroups to CP-semigroups T but still on $\mathcal{B}(G)$. This was done already by Bhat [Bha96] who constructed the unique minimal dilation to an E_0–semigroup on some $\mathcal{B}(H)$ and then applied Arveson's construction. We pointed out that we arrive at the same result, we if construct our product system of von Neumann $\mathcal{B}(G)$–$\mathcal{B}(G)$–modules and then restrict our attention to the subsystem of Hilbert spaces consisting only of the centers as explained in Example 11.1.4. Secondly, we can stay with E_0–semigroups but now on an arbitrary (unital) C^*–algebra $\mathcal{B}$. We discussed this in Example 11.1.3. Restrospectively, it is clear that this is not more than a restriction from our construction for CP-semigroups, and also the extraction of Arveson's results for $\mathcal{B} = \mathcal{B}(G)$ is the same as before. An interesting question is what happens in the case $\mathcal{B} = \mathcal{B}^a(E)$ for some (pre-)Hilbert $\mathcal{C}$–module. What is the relation of the product system $\bigl(\mathcal{B}_t\bigr)_{t\in\mathbb{T}}$ of $\mathcal{B}$–$\mathcal{B}$–modules from Example 11.1.3 and the product system $E^\odot$ of $\mathcal{C}$–$\mathcal{C}$–modules constructed in Theorem 14.1.1? Presently, we do not know the answer. We expect, however, from the example $\mathcal{C} = \mathbb{C}$ that there are interesting relations.

15.2 Type II product systems

In Section 11.3 we associated with each CPD-semigroup $\mathfrak{T}$ on a set S with values in $\mathcal{K}_S(\mathcal{B})$ a GNS-product system $E^\odot$ and a mapping $\xi\colon S \to \mathcal{U}(E^\odot), \sigma \mapsto \xi^{\sigma\odot}$ such that $\mathfrak{T}_t^{\sigma,\sigma'}(b) = \langle \xi_t^\sigma, b\xi_t^{\sigma'}\rangle$ and such that the subset $\xi^{S\odot} \subset \mathcal{U}(E^\odot)$ of units for $E^\odot$ is totalizing. An arbitrary product system $E^\odot$ arises in this way, if and only if it is of type I. Any product system contains a type I subsystem generated by its units

Similarly, type I Arveson systems are those which admit a totalizing set of units. However, Arveson's units have to fulfill a measurability requirement. One can show that an arbitrary normalized unit differs from a measurable unit just by a possibly non-measurable phase function and that to any algebraic normalized unit there exists a measurable structure on the product system for which this unit is measurable; see Liebscher [Lie00a]. The measurability conditions assure that inner products among measurable units are continuous time functions. We see that in the case of Arveson systems we start with nothing and end up with continuous units.

In our case, in general, it is neither possible to normalize a unit (and also for a continuous unit it is not a triviality) nor do we know in how far the CP-semigroup T^ξ for a normalized unit $\xi^\odot$ differs from a measurable (in what sense ever) CP-semigroup. Therefore, we restricted our attention to type I_c product systems which are generated by a set $S \subset \mathcal{U}(E^\odot)$ of units for which the associated CPD-semigroup $\mathfrak{U} \upharpoonright S$ is uniformly continuous. By Theorem 11.6.7 this is equivalent to that inner products among units from S are continuous time functions. In Chapter 13 we showed that type I_c product systems are contained in a time ordered product system and that type I^s_{cn} product systems are time ordered product systems of von Neumann modules (generalizing the fact that type I Arveson systems are time ordered Fock spaces). Example 7.3.7 shows that type I^s_{cn} product systems may have units which are only strongly continuous. We consider it as an interesting question, whether it is possible to extend the preceding result to type I^s_{sn} product systems (i.e. the strongly totalizing set S of units leads to a CPD-semigroup $\mathfrak{U} \upharpoonright S$ which may be only strongly continuous). This would imply that type I^s_{sn} product systems, actually, are type I^s_{cn}. We hope that as many type I^s product systems as possible turn out to be type I^s_{cn} so that, at least, for von Neumann modules there can be a single definition of type I.

Type II Arveson systems are those which admit at least one (measurable) unit (and are not type I). Already Arveson pointed out that life is much easier, if there is a unit. For instance, the construction of an E_0–semigroup from an Arveson system with unit was already mentioned in [Arv89a], whereas the construction for general Arveson systems had to wait until the last one [Arv90b] in a series of four long articles. Zacharias [Zac96, Zac00] shows many properties of Arveson's spectral algebras [Arv90a] (among these their K-theory and their pure infiniteness) if the underlying product systems have at least one unit.

In our case the existence of a unital unit allows to embed all members E_t of a product system $E^\odot$ into one space, namely, the one-sided inductive limit E as constructed in Section 11.4. In Section 12.5 we explain how this can be used to define the analogue of Arveson's spectral algebra. Also a measurable structure drops out immediately, by saying that a family $\big(x_t\big)_{t\in\mathbb{T}}$ of elements $x_t \in E_t$ is measurable, if the function $t \mapsto x_t \in E$ is measurable.

We used a similar definition for continuity of units relative to a fixed (continuous) one already in Section 11.6. By Theorem 11.6.7 the continuity of a set S of units relative to a fixed one implies continuity of any unit in S relative to any other and, therefore, is an intrinsic property of S not refering to the inductive limit constructed from one of the units. This is similar to Section 13.1 where existence of a single central unit among a set of continuous units allowed already to conclude that the generator of the associated CPD-semigroup is a CE-generator.

Let us recall that the type I^s_{cn} product system generated by a single continuous unit is all of a certain time ordered system of von Neumann modules and, therefore, contains not only a unital, but also a central unital unit (for instance, the vacuum of that time ordered system). Conversely, we pointed out that extistence of a central unit in such a system is equivalent to the results by Christensen and Evans [CE79]. (We repeat the question, whether it is possible to show extistence of a central unit without utilizing [CE79], thus, giving an independent proof of [CE79].) Finally, by Example 11.2.6 in the case $\mathcal{B} = \mathcal{B}(G)$ the central units for a product system of von Neumann modules correspond precisely to the Arveson units for the central subsystem of Hilbert spaces. We propose the following defintion of type II product systems and leave a detailed analysis in how far the internal structure (such as independence of the maximal type I subsystems of the choice of a reference unit) to future work. The remaining product systems should be type III, however, until now it is totally unclear, how to impose technical conditions to them (in a reasonable way, of course).

15.2.1 Definition. A product system $E^\odot$ is *type $\underline{II}$*, if it is not type $\underline{\mathrm{I}}$ and has at least one central unital unit. It is *type $\underline{II}_c$* if this unit can be chosen continuous (i.e. T^ξ is uniformly continuous). If we speak about product systems of Hilbert modules and von Neumann modules, then we write *type II* and *type II^s*, respectively. It follows that a product system of von Neumann modules is type II^s_c (or, actually, type II^s_{cn}), if it has a continuous unit.

Liebscher [Lie00a] shows us a way to associate (on-to-one) with an Arveson system a type II Arveson system. We believe that his construction works also for Hilbert modules.

15.3 Index

Two E_0–semigroups on $\mathcal{B}(G)$ may be tensorized to obtain a new E_0–semigroup on $G \bar{\otimes} G$. If G is inifinite-dimensional and separable as in Arveson's set-up, then so is $G \bar{\otimes} G$ and, therefore, again isomorphic to G. (We could try to speak of a tensor product within the category of E_0–semigroups on $\mathcal{B}(G)$. However, because it is not reasonable to distinguish a certain isomorphism of $G \bar{\otimes} G$ and G, the operation of tensor product is rather an operation up to unitary isomorphism.) In general, there is no reason to restrict the dimension of G to a certain value, and we speak of a tensor product of pairs (G, ϑ) and (G', ϑ') giving a new pair $(G \otimes G', \vartheta \otimes \vartheta')$.

The Arveson system $\mathfrak{H}^{\bar{\otimes}}$ of any E_0–semigroup ϑ contains a maximal type I Arveson subsystem $\mathfrak{H}_{\mathrm{I}}^{\bar{\otimes}}$ generated by its (measurable) units which is isomorphic to $\mathbb{I}\Gamma^{\otimes}(\mathfrak{H})$ for some Hilbert space $\mathfrak{H}$ which is determined by the product system up to unitary isomorphism. In other words, we can associate with any Arveson system a number in $n \in \mathbb{N}_0 \cup \{\infty\}$, the so-called *index* being the dimension of $\mathfrak{H}$; see Arveson [Arv89a]. (There is no reason to exclude the cases $n = 0$ and $\mathfrak{H}$ non-separable.) For type I Arveson systems we have $\mathbb{I}\Gamma(\mathfrak{H})^{\otimes} \bar{\otimes} \mathbb{I}\Gamma^{\otimes}(\mathfrak{H}') = \mathbb{I}\Gamma^{\otimes}(\mathfrak{H} \oplus \mathfrak{H}')$ so that the indices are additive under tensor product (thus, justifying the name *index*). One can show (roughly speaking, because units in the tensor product of two Arveson systems must be elementary tensors of units) that the same is true for non-type III systems. Putting by hand the index of a type III system to ∞, we obtain that the index is an invariant of Arveson systems for outer conjugacy of E_0–semigroups on Hilbert spaces, which is additive under tensor product.

Also in our product systems we have that a type II^s_{cn} system $E^{\bar{\odot}^s}$ contains (after having fixed a continuous unit, replacing somehow Arveson's measurability requirements) a maximal type I^s_{cn} subsystem $E_{\mathrm{I}}^{\bar{\odot}^s}$ isomorphic to some $\mathbb{I}\Gamma^{\odot s}(F)$ and F is determined by $E^{\bar{\odot}^s}$ up to (two-sided) isomorphism. The module F (by Proposition 14.1.2 and Theorem 14.1.5 an outer conjugacy invariant of the E_0–semigroup ϑ constructed from any of the unital continuous units) is, however, no longer determined by a simple dimension. If we want to define an index, then we have to consider the whole space F as a candidate. But then we have to ask what is the operation among product systems which sends $\mathbb{I}\Gamma^{\odot}(F)$ and $\mathbb{I}\Gamma^{\odot}(F')$ to $\mathbb{I}\Gamma^{\odot}(F \oplus F')$. Alternatively, we can ask what is the operation among weak dilations (E, ϑ, ξ) which sends $(\mathbb{I}\Gamma^{\odot}(F), \mathcal{S}, \omega)$ and $(\mathbb{I}\Gamma^{\odot}(F'), \mathcal{S}, \omega)$ to $(\mathbb{I}\Gamma^{\odot}(F \oplus F'), \mathcal{S}, \omega)$. Since we do not yet know the answer, we hesitate to call the invariant F of product systems of von Neumann modules (maybe, paired with with a continuous unit) an *index*.

The described problem is typical for independence *over* the algebra $\mathcal{B}$ in the sense that we have to put together two $\mathcal{B}$–algebras in order to obtain a new one which contains the two as $\mathcal{B}$–subalgebras. The proper independence (over $\mathbb{C}$) in Arveson's framework is *tensor indpendence* and the construction is just the tensor product. But already in [Ske98a] we pointed out that a tensor product of $\mathcal{B}$–algebras in general cannot be eqipped with a reasonable multiplication law. An exception are centered $\mathcal{B}$–algebras as considered in Skeide [Ske96, Ske99a]. Also Landsman [Lan99] constructed a different example. For the solution of our problem we expect a module variant of the independence arising by looking at creators and annihilators on the time ordered Fock space (see Example 9.1.4) as introduced by Lu and Ruggeri [LR98]. This independence is also subject to a systematic investigation by Muraki [Mur00].

Part IV

Quantum stochastic calculus

The beginning of quantum stochastic calculus how it is used today is probably the calculus on the *symmetric Fock space* $\Gamma(L^2(\mathbb{R}_+, H))$ by Hudson and Pathasarathy [HP84]. (Cf., however, also the works [AH84] devoted to a calculus on the Fermi Fock space and [BSW82] devoted to the Clifford integral.) The *free* calculus on the *full Fock space* $\mathcal{F}(L^2(\mathbb{R}_+))$ was introduced by Kümmerer and Speicher [KS92, Spe91]. Shortly later Fagnola [Fag91] showed that free calculus fits after very slight modifications into the *representation free* calculus in Accardi, Fagnola and Quaegebeur [AFQ92].

One of the main goals of quantum stochastic calculus is to find unital dilations of unital CP-semigroups on a unital C^*–algebra $\mathcal{B}$ (see Section 10.1). In usual approaches the *initial algebra* $\mathcal{B}$ is taken into account by considering the tensor product of the Fock space by an *initial space* G on which $\mathcal{B}$ is represented. In the calculi in [HP84, KS92, Spe91] the dilation problem has been solved for special CP-semigroups on $\mathcal{B}(G)$, namely, those with (bounded) Linblad generator [Lin76] of *one degree of freedom* (i.e. the one-particle sector is $L^2(\mathbb{R}_+)$ and in each of the possibly infinite sums of Equation (17.1.1) only one summand remains). A general Lindblad generator (for separable G) requires a calculus with arbitrary degree of freedom (with one-particle sector $L^2(\mathbb{R}_+, \mathfrak{H})$). For the symmetric calculus this problem was solved in Mohari and Sinha [MS90] where infinite sums of integrators appear. A similar calculus on the full Fock space was treated in Fagnola and Mancino [FM92]. (However, this calculus is only for one-sided integrals and the conservation integral is only mentioned.)

Here we concentrate on the *free calculus on the full Fock module* as developed in Skeide [Ske00d]. Already in the case of Lindblad generators (i.e. CP-semigroups on $\mathcal{B}(G)$) it has enormous advantages using Hilbert modules just as a language. The initial space disappears. Instead, we consider two-sided Hilbert modules over $\mathcal{B}(G)$. The infinite sums of integrators are replaced by a finite sum (just one summand for creation, annihilation, conservation and for time integral). We explain this in Section 17.1.

However, a calculus on Fock modules does more. It allows to find dilations for (bounded) generators of CP-semigroups on arbitrary C^*–algebras $\mathcal{B}$ whose form was found by Christenson and Evans [CE79]. Recently, Goswami and Sinha [GS99] introduced a calculus on a *symmetric Fock module* (Skeide [Ske98a]), and used it to solve the dilation problem for Christensen-Evans generators.

The one-particle sector is obtained by GNS-construction from the generator and, therefore, it is a Hilbert $\mathcal{B}$–$\mathcal{B}$–module. One problem which had to be faced in [GS99] is that (see Chapter 8) the symmetric Fock module over an arbitrary Hilbert $\mathcal{B}$–$\mathcal{B}$–module does not exist without additional assumptions. One sufficient assumption is that the Hilbert module is *centered*. We know that von Neumann $\mathcal{B}(G)$–$\mathcal{B}(G)$–modules are always centered (see Example 3.3.4). And, indeed, in [GS99] it is one of the first steps to embed the the Hilbert $\mathcal{B}$–$\mathcal{B}$–module which arises by GNS-construction into a bigger Hilbert $\mathcal{B}(G)$–$\mathcal{B}(G)$–module. On the contrary, a full Fock module can be constructed over arbitrary one particle sectors. Therefore, in our case we do not have to enlarge the one-particle sector.

A first attempt for a calculus on a full Fock module was made by Lu [Lu94] where the calculus lives on the Fock module $\mathcal{F}(L^2(\mathbb{R}_+, \mathcal{A}))$ (instead of $\mathcal{F}((L^2(\mathbb{R}_+, E))$. As $\mathcal{A}$ is the simplest $\mathcal{A}$–$\mathcal{A}$–module possible, the module structure of the one-particle sector very simple. In fact, the calculus is *isomorphic* to the calculus on $G \bar{\otimes} \mathcal{F}(L^2(\mathbb{R}_+))$ in [KS92] where $\mathcal{A}$ is represented on G. However, the algebra $\mathcal{A} = \mathcal{B}^a(\mathcal{F}(E))$ is very big and contains the original algebra $\mathcal{B}$ only as a, usually, very small subalgebra. We also mention the abstract calculus by Hellmich, Köstler and Kümmerer [HKK98, Kös00, Hel01] where a one-to-one correspondence between additive and multiplicative *adapted* cocycles with respect to an abstract white noise is established. These results are, however, restricted to the set-up of von Neumann algebras with faithful normal (invariant) states.

Our approach to calculus is inspired very much by [KS92] and we borrowed also some essential ideas from [Spe91] as far as conservation integrals are concerned. [KS92] develops stochastic integration for creation and annihilation processes. All limits there are norm limits. Taking into account also conservation integrals destroys norm convergence. In [Spe91] this problem is solved with the help of a kernel calculus. We follow, however, the ideas in Skeide [Ske98b] and use the $*$–strong topology, dealing always with concrete operators.

The basic idea in [KS92, Spe91] is probably to use the graduation on the Fock space in order to define a new norm. It is this idea which is responsible for the fact that we are in a position to find a calculus of bounded operators. In Appendix A.3 we present this idea in a general set-up and we proof the necessary generalizations for strong limits.

For the full Fock module basic operators on it we use the notations from Chapter 6. In particular, the generalized creators and annihilators as introduced in [Ske98b] simplify notation considerably. Our notion of adaptedness (defined in Section 6.3, again following [Ske98b]) is simpler and more general than the original notion in [KS92]. Also here the generalized creators play a crucial role in drawing consequences of adaptedness in a transparent way.

In [KS92] the theory is developed for processes which belong to some L^4–space. This is in some sense the most general class possible. Here we consider $*$–strongly continuous processes. This is sufficient, because all integrals lead to processes belonging to this class. Additionally, our restriction has the advantage that all integrals are limits of Riemann-Stieltjes sums. On the other hand, our theory is dealing with very general integrators. (Whereas the integrators in [KS92, Spe91] are the simplest possible.) In fact, our integrators are so general that the differential equation resolving the dilation problem has not a single coefficient. The function spaces from which we take our processes and integrators are introduced in Appendix B.

In Section 16.1 we show existence of integrals for the considered class of processes and integrators. In Section 16.3 we show that conservation integrals are essentially non-continuous. In Section 16.4 we establish the Ito formula. As the techniques used here depend highly on the class of processes and integrators, these sections differ considerably from the corresponding Sections in [KS92]. In particular, the results in Section 16.3 are much more involved than the corresponding results in [Spe91].

In Section 16.2 we show existence and uniqueness of solutions of differential equations. In Section 16.6 we establish that solutions of particular differential equations, those with "stationary independent increments", have cocycles as solutions. In Section 16.5 we state necessary and sufficient conditions for unitarity of the solution and in Section 16.7 we use the results to solve the dilation problem for a general Christensen-Evans generator. The ideas to all proofs in these sections are taken directly from [KS92, Spe91]. It is noteworthy that, actually, the proofs here, although more general, are formally simpler than the original proofs. (This is due to absence of coefficients in our differential equations.)

In Section 17.1 we explain that the calculus on the full Fock space $G \bar{\otimes} \mathcal{F}(L^2(\mathbb{R}, \mathfrak{H}))$ ([KS92, Spe91] treated only the case $\mathfrak{H} = \mathbb{C}$) is contained in our set-up. In Section 17.2 we show that the calculus on the *boolean Fock module* is included. In particular, we show that the (non-conservative) CP-semigroups T on $\mathcal{B}$ which may be dilated with the help of a boolean calculus are precisely those having the form $T_t(b) = b_t b b_t^*$ where $b_t = e^{t\jmath}$ ($\jmath \in \mathcal{B}, \mathsf{Re}\,\jmath \leq 0$) is a semigroup of contractions in $\mathcal{B}$. Finally, we extend our uniqueness results from Sections 16.3 and 16.4 to differential equations with arbitrarily many summands. This is done by the same trick as in Section 13.5, simple for modules, but completely out of reach for Hilbert spaces.

Convention: In the sequel, in constructs like 'quantum stochastic calculus' or 'quantum stochastic differential equation', etc., we leave out the words 'quantum stochastic'.

Since we are interested mainly in adapted processes, the steps in a Riemann-Stieltjes sum take their value from the left border. Consequently, we do not consider all step functions, but only those where indicator functions to left closed and right open intervals are involved. The limits of such functions are precisely the (strongly) right continuous functions with left (strong) limit in each point, the so-called *(strong) càdlàg* functions (**c**ontinue **à d**roite, **l**imitée **à g**auche).

The full Fock module and operators on it are defined in Chapter 16. Results related to the grading on the Fock space are collected Appendix A.3. The function spaces used throughout and their basic properties, are introduced in Appendix B. Particularly important is Appendix B.4 where we investigate the general properties of integrals over càdlàg functions like (B.4.1) without refering to the "stochastic structure". Existence of all integrals in Chapter IV is covered either by Observation B.4.3, or by Proposition B.4.5. Adaptedness is explained in Section 6.3.

Convention: The notions from Appendix B will be used, usually, without mention. The results in Appendix B.4 form a vivid part of Chapter 16.

Chapter 16

Calculus on full Fock modules

16.1 Integrals

In this section we define the $*$–algebra $\mathfrak{A}_1$ of adapted processes and define for them stochastic integrals with respect to creation, annihilation, conservation processes, and the time integral. We use, however, a condensed notation where, formally, only conservation integrals appear, however, where the class of processes which are allowed as integrands is bigger. This condensed notation does not contain more or less information. It allows, however, for more economic proofs.

Let $\mathcal{B}$ be a unital C^*–algebra and let F be a $\mathcal{B}$–$\mathcal{B}$–Hilbert module. We work on the full Fock module $\mathcal{F} = \mathcal{F}(E_\infty)$. Recall Observation 6.3.8 for E_∞ and related notions. See Definitions 6.2.1 and 6.2.2 for the graded structure of $\mathcal{F}$ and $\mathbb{B} = \mathbb{B}^a(\mathcal{F})$. We abbreviate also $\mathcal{F}_g = \mathcal{F}_g(E_\infty)$ and $\mathcal{F}_1 = \mathcal{F}_1(E_\infty)$.

16.1.1 Remark. The arguments of our integrators are in $\mathfrak{L}^\infty_{loc}(\mathbb{R}, F)$ and $\mathfrak{L}^\infty_{loc}(\mathbb{R}, \mathbb{B}^a(F))$, respectively. They enter integrals only as restrictions to some compact interval $K = [\tau, \mathfrak{T}]$. For a simple function $x \in \mathfrak{L}^\infty_{loc}(\mathbb{R}, \mathcal{F})$ (or a step function; see Remark B.4.8) we have

$$\|x\|^K \;\leq\; \|x\|_2^K \;\leq\; \sqrt{\mathfrak{T} - \tau}\,\|x\|_{ess}^K \;\leq\; \sqrt{\mathfrak{T} - \tau}\,\|x\|_\infty^K$$

so that $\mathfrak{L}^\infty(K, F)$"$\subset$"$L^\infty(K, F) \subset L^2_B(K, F) \subset L^2(K, F)$ (of course, the "embedding" $\mathfrak{L}^\infty(K, F) \to L^\infty(K, F)$ is not faithful). Moreover, the elements of $\mathfrak{L}^\infty(K, \mathbb{B}^a(F))$ act as bounded operators on $L^2(K, F)$ and leave invariant $\mathfrak{L}^\infty(K, F), L^\infty(K, F)$. See Appendix B.1 for details.

As explained in the beginning, we express all our integrals in such a way that they formally look like conservation integrals. So we use only one integrator function $p \in \mathfrak{L}^\infty_{loc}(\mathbb{R}, \mathbb{B}_1)$

with $p_t = p(I\!I_{(-\infty,t]})$. The form of Riemann-Stieltjes sums as in (B.4.1) with two processes as integrands reminds us of an inner product with values in $\mathcal{B}^a(\mathcal{F})$. We make this explicit.

Let $K = [\tau, \mathfrak{T}]$ be compact intervall and let P be a partition in $\mathbb{P}_K$. By setting

$$\big(A, B\big)_P = \sum_{k=1}^{N} A_{t_{k-1}}\, dp_{t_k}\, B_{t_{k-1}},$$

we define a $\mathcal{B}^a(\mathcal{F})$–valued, $\mathcal{B}^a(\mathcal{F})$–$\mathcal{B}^a(\mathcal{F})$–***bilinear*** (i.e. $\big(aA, Ba'\big)_P = a\big(A, B\big)_P a'$ for all $a, a' \in \mathcal{B}^a(\mathcal{F})$) mapping on the $\mathcal{B}^a(\mathcal{F})$–$\mathcal{B}^a(\mathcal{F})$–module of all mappings $\mathbb{R} \to \mathcal{B}^a(\mathcal{F})$. By the following lemma this mapping is ***positive*** (i.e. $\big(A^*, A\big)_P \geq 0$) so that we may speak of a bilinear (not a sesquilinear) inner product. Of course, $\big(A, B\big)_P^* = \big(B^*, A^*\big)_P$.

16.1.2 Lemma. *For all functions $A, B \colon \mathbb{R} \to \mathcal{B}^a(\mathcal{F})$ we have*

$$\big(A, B\big)_P = \big(A, \mathbf{1}\big)_P \big(\mathbf{1}, B\big)_P. \tag{16.1.1}$$

PROOF. This follows immediately from $dp_{t_k} dp_{t_\ell} = dp_{t_k}\, \delta_{k\ell}$. ■

16.1.3 Corollary. *We have the Cauchy-Schwarz inequality*

$$\begin{aligned} \big|\big(A, B\big)_P\big|^2 &= \big(\mathbf{1}, B\big)_P^* \big(A, \mathbf{1}\big)_P^* \big(A, \mathbf{1}\big)_P \big(\mathbf{1}, B\big)_P \\ &\leq \big\|\big(A, \mathbf{1}\big)_P^* \big(A, \mathbf{1}\big)_P\big\|\ \big(\mathbf{1}, B\big)_P^* \big(\mathbf{1}, B\big)_P = \big\|\big(A, A^*\big)_P\big\|\ \big(B^*, B\big)_P. \end{aligned}$$

Lemma 16.1.2 may be considered as a particularly simple example for an *Ito formula.* We see that in order to analyze under which circumstances the two-sided integral $\int A_t\, dp_t\, B_t$ exists, it is sufficient to understand under which circumstances the one-sided integrals $\int dp_t\, B_t$ and $\int A_t\, dp_t$ exist. Of course, the two typs are adjoints of eachother (put $A = B^*$). Therefore, if we show existence of both one-sided integrals as a strong limit, actually, we show that both exist as $*$–strong limits. If, additionally, the nets $\big(A, \mathbf{1}\big)_P$ and $\big(\mathbf{1}, B\big)_P$ are bounded, then also the net $\big(A, B\big)_P$ converges $*$–strongly.

Lemma 16.1.2 holds for arbitrary processes A, B. In order to show convergence of the inner product $\big(A, B\big)_P$, we have to restrict our processes to smaller classes.

16.1.4 Definition. The ***$*$–algebra of processes*** $\mathfrak{P} = \mathcal{C}^s(\mathbb{R}, \mathcal{B}^a(\mathcal{F})) \cap \mathcal{C}^s(\mathbb{R}, \mathcal{B}^a(\mathcal{F}))^*$ consists of all families $A = \big(A_t\big)_{t \in \mathbb{R}}$ of elements $A_t \in \mathcal{B}^a(\mathcal{F})$ which are $*$–strongly continuous as mappings $t \mapsto A_t$. Let K be a compact intervall. Then $\mathfrak{P}^K$ is nothing but the C^*–algebra $\mathcal{C}^s(K, \mathcal{B}^a(\mathcal{F})) \cap \mathcal{C}^s(K, \mathcal{B}^a(\mathcal{F}))^*$

We decompose $\mathfrak{P}$ into the ***homogeneous subspaces*** $\mathfrak{P}^{(n)} = \{A \in \mathfrak{P} \colon A_t \in \mathcal{B}^{(n)}\ (t \in \mathbb{R})\}$ $(n \in \mathbb{Z})$. By $\mathfrak{P}_g$ we denote the algebraic direct sum over all $\mathfrak{P}^{(n)}$ and by $\mathfrak{P}_1$ we denote its completion with respect to the ℓ^1–norm $\|\bullet\|_1$ as defined in Appendix A.3. We use similar notations for $\mathfrak{P}^K$.

The $*$–*algebra of adapted processes* $\mathfrak{A}$ consists of all $A \in \mathfrak{P}$ such that A_t is adapted to E_t. By $\mathfrak{A}^\tau \subset \mathfrak{P}$ we denote the $*$–algebra of those processes where A_t is adapted to E_t at least for $t \geq \tau$. We set $\mathfrak{A}^{(n)} = \mathfrak{A} \cap \mathfrak{P}^{(n)}$, $\mathfrak{A}_g = \mathfrak{A} \cap \mathfrak{P}_g$, and $\mathfrak{A}_1 = \mathfrak{A} \cap \mathfrak{P}_1$. We use similar notations for $\mathfrak{A}^K$ and $\mathfrak{A}^\tau$.

We are interested in showing existence of the following four limits over $\mathbb{P}_K$. Firstly, $\big(A, B\big)_P$ where A, B are adapted. This corresponds to the usual conservation integral. In order to include also an argument $T \in \mathcal{L}^\infty_{loc}(\mathbb{R}, \mathcal{B}^a(F))$ for the integrator, we consider the slightly more general $\big(A, p(T)B\big)_P = \big(Ap(T), B\big)_P$. Secondly, $\big(A, \ell^*(I\!I_K x)B\big)_P$ where $x \in \mathcal{L}^\infty_{loc}(\mathbb{R}, F)$ (so that $I\!I_K x \in E_\infty$), and, thirdly, its adjoint. These correspond to the usual creation integral and annihilation integral, respectively. Fourthly, $\big(A\ell(I\!I_K x), \ell^*(I\!I_K y)B\big)_P$ where $x, y \in \mathcal{L}^\infty_{loc}(\mathbb{R}, F)$. This corresponds to the integral with respect to the operator valued measure $\mu^{x,y}([s,t]) = \langle x, I\!I_{[s,t]} y\rangle$.

We start with the last one. Obviously

$$\ell(I\!I_K x)\, p_t\, \ell^*(I\!I_K y) = \langle x, I\!I_{[\tau,t]} y\rangle = \int_\tau^t \langle x(s), y(s)\rangle\, ds$$

($t \in K$) where by Cauchy-Schwarz inequality $\langle x(t), y(t)\rangle \chi_K(t)$ is a bounded function. In other words, the assumptions of Proposition B.4.5 are fulfilled and we are concerned with a generalized time integral.

16.1.5 Corollary. *The integral*

$$\int_\tau^{\mathfrak{T}} A_t\, d\mu_t^{x,y}\, B_t := \lim_{P \in \mathbb{P}_K} \big(A\ell(I\!I_K x), \ell^*(I\!I_K y)B\big)_P$$

exists $*$–equistrongly for all $A, B \in \mathfrak{P}$, and it concides with $\int_\tau^{\mathfrak{T}} A_t \langle x(t), y(t)\rangle B_t\, dt$.

Moreover,

$$\left\| \big(A\ell(I\!I_K x), \ell^*(I\!I_K y)B\big)_P \right\| \leq (\mathfrak{T} - \tau)\, \|x\|^K\, \|y\|^K\, \|A\|^K\, \|B\|^K$$

and, in particular,

$$\left\| \big(\mathbf{1}, \ell^*(I\!I_K x)A\big)_P \right\| = \left\| \big(A^*\ell(I\!I_K x), \mathbf{1}\big)_P \right\| \leq \sqrt{\mathfrak{T} - \tau}\, \|x\|^K\, \|A\|^K \tag{16.1.2}$$

for all $A, B \colon \mathbb{R} \to \mathcal{B}^a(\mathcal{F})$.

Let us come to the remaining integrals.

16.1.6 Lemma. *Let $A\colon \mathbb{R} \to \mathcal{B}^{(n)}$ $(n \in \mathbb{Z})$ be an adapted function. Then*

$$\left\|\big(A,\mathbf{1}\big)_P\right\| = \left\|\big(\mathbf{1},A^*\big)_P\right\| \le \|A\|^K.$$

PROOF. Observe that $A_t^*A_t \in \mathcal{B}^{(0)}$. Therefore, by Corollary 6.3.7 and the relations in Proposition 6.1.3 we have

$$\Big|\sum_{k=1}^N A_{t_{k-1}}\,dp_{t_k}\Big|^2 = \sum_{k,\ell=1}^N dp_{t_k}\,A^*_{t_{k-1}}A_{t_{\ell-1}}\,dp_{t_\ell} = \sum_{k,\ell=1}^N dp_{t_k}\,\mathbb{E}_0(A^*_{t_{k-1}}A_{t_{\ell-1}})\,dp_{t_\ell}$$
$$= \sum_{k=1}^N dp_{t_k}\,\mathbb{E}_0(A^*_{t_{k-1}}A_{t_{k-1}})\,dp_{t_k} \le \big(\|A\|^K\big)^2\sum_{k=1}^N dp_{t_k} \le \big(\|A\|^K\big)^2. \blacksquare$$

16.1.7 Proposition. *All Riemann-Stieltjes sums $(\bullet,\bullet)_P$ are bounded in ℓ^1–norm on the considered class of processes. More precisely, let $A,B \in \mathfrak{A}_1^\tau$, $x,y \in \mathcal{L}^\infty_{loc}(\mathbb{R},F)$, and $T \in \mathcal{L}^\infty_{loc}(\mathbb{R},\mathcal{B}^a(F))$. Then*

$$\left\|\big(A,p(T)B\big)_P\right\|_1 = \left\|\big(Ap(T),B\big)_P\right\|_1 \le \|T\|^K\,\|A\|_1^K\,\|B\|_1^K \tag{16.1.3a}$$

$$\left\|\big(A,\ell^*(I\!I_Kx)B\big)_P\right\|_1 = \left\|\big(B^*\ell(I\!I_Kx),A^*\big)_P\right\|_1 \le \sqrt{\mathfrak{T}-\tau}\,\|x\|^K\,\|A\|_1^K\,\|B\|_1^K \tag{16.1.3b}$$

$$\left\|\big(A\ell(I\!I_Kx),\ell^*(I\!I_Ky),B\big)_P\right\|_1 \le (\mathfrak{T}-\tau)\,\|x\|^K\,\|y\|^K\,\|A\|_1^K\,\|B\|_1^K \tag{16.1.3c}$$

PROOF. By Lemma A.3.1 it is sufficient to show the estimates for homogeneous processes A,B. By Corollary 16.1.3 and Lemma 16.1.6 we find

$$\left\|\big(A,p(T)B\big)_P\right\| \le \|A\|^K\left\|\big(\mathbf{1},p(T)B\big)_P\right\| = \|A\|^K\left\|\big(p(T),B\big)_P\right\|$$
$$= \|A\|^K\left\|p(T)\,\big(\mathbf{1},B\big)_P\right\| \le \|T\|^K\,\|A\|^K\,\|B\|^K.$$

This shows (16.1.3a). Equations (16.1.3c) and (16.1.3c) follow in a similar manner from Corollary 16.1.5. $\blacksquare$

16.1.8 Theorem. *Let $A,B \in \mathfrak{A}_1^\tau$, $x,y \in \mathcal{L}^\infty_{loc}(\mathbb{R},F)$, and $T \in \mathcal{L}^\infty_{loc}(\mathbb{R},\mathcal{B}^a(F))$. Then the conservation integral*

$$\int_\tau^{\mathfrak{T}} A_t\,dp_t(T)\,B_t := \lim_{P\in\mathbb{P}_K}\big(A,p(T)B\big)_P, \tag{16.1.4a}$$

the creation integral

$$\int_\tau^{\mathfrak{T}} A_t\,d\ell_t^*(x)\,B_t := \lim_{P\in\mathbb{P}_K}\big(A,\ell^*(I\!I_Ky)B\big)_P, \tag{16.1.4b}$$

and the **annihilation integral**

$$\int_{\tau}^{\mathfrak{T}} A_t \, d\ell_t(x) \, B_t := \lim_{P \in \mathbb{P}_K} \big(A\ell(I\!I_K x), B\big)_P \tag{16.1.4c}$$

exist, like the (generalized) **time integral** $\int_{\tau}^{\mathfrak{T}} A_t \, d\mu_t^{x,y} \, B_t = \int_{\tau}^{\mathfrak{T}} A_t \langle x(t), y(t) \rangle B_t \, dt$*, as $*$–equistrong limits in the $*$–strong topology of $\mathcal{B}_1$.*

Moreover, for all four integrators the process M defined by setting

$$M_t = \begin{cases} \int_{\tau}^{t} A_s \, dI_s \, B_s & \text{for } t \geq \tau \\ 0 & \text{otherwise} \end{cases} \tag{16.1.5}$$

is an element of $\mathfrak{A}_1^{\tau}$.

Proof. By Proposition 16.1.7 the assumptions of Lemma A.3.2 are fulfilled so that we may reduce to homogeneous elements. Moreover, all nets are bounded. Therefore, as explained after Corollary 16.1.3, it is sufficient to show strong convergence in each of the cases $\big(\mathbf{1}, B\big)_P$ and $\big(\mathbf{1}, \ell^*(I\!I_K x)B\big)_P$ and the respective adjoints. (Of course, the case $\big(\mathbf{1}, p(T)B\big)_P = p(T)\big(\mathbf{1}, B\big)_P$ is included in the case $\big(\mathbf{1}, B\big)_P$.)

By Lemma 6.3.5, in $\big(\mathbf{1}, B\big)_P$ we may replace B by the process $\widehat{\ell}(B^*\omega)$ which is $\|\bullet\|_1$–continuous by Corollary 6.2.7. Therefore, by Proposition 16.1.7 we are in the first case of Proposition B.4.3. This even settles norm convergence of both $\big(\mathbf{1}, B\big)_P$ and $\big(B^*, \mathbf{1}\big)_P$.

Strong convergence of $\big(\mathbf{1}, \ell^*(I\!I_K x)B\big)_P$ is settled by the strong analogue

$$\big\|\big(\mathbf{1}, \ell^*(I\!I_K x)B\big)_P Z\big\| \leq \sqrt{\mathfrak{T} - \tau} \, \|x\|^K \, \|B\|_Z^K$$

of (16.1.2) for all Z in the whole domain $\mathcal{F}$.

For the case $\big(B^*\ell(I\!I_K x), \mathbf{1}\big)_P$ we choose $Z = z \odot Z'$ where $z \in \mathfrak{L}^{\infty}_{loc}(\mathbb{R}, F) \cap E_{\infty}$, and $Z' \in \mathcal{F}$. We find

$$\ell(I\!I_K x) \, dp_{t_k} Z = \langle I\!I_K x \, , \, dI\!I_{t_k} \, z \rangle Z'.$$

Therefore, by Corollary 16.1.5

$$\big(B^*\ell(I\!I_K x), \mathbf{1}\big)_P Z = \big(B^*\ell(x), \ell^*(z)\mathbf{1}\big)_P Z' \longrightarrow \int_{\tau}^{\mathfrak{T}} B_t^* \, d\mu_t^{x,z} \, \mathbf{1} \, Z' \tag{16.1.6}$$

equiuniformly. Since the net $\big(B^*\ell(x), \mathbf{1}\big)_P$ is bounded, and since the $z \odot Z'$ form a total subset of $\mathcal{F}$, we obtain equistrong convergence on $\mathcal{F}$.

Clearly, M_t is adapted to E_t. And, clearly, by Proposition 16.1.7 the time, creation, and annihilation integrals depend even continuously on their upper bound. To see strong continuity (once again, this is sufficient by symmetry under adjoint) of the conservation integral, we also may consider $\int_\tau^\mathfrak{T} B_t^*\,dp_t$ and $\int_\tau^\mathfrak{T} dp_t\,B_t$ separately. The former case is clear by (16.1.6). For the latter the idea is the same, but we need a more refined argument. We choose $B \in \mathfrak{A}^{\tau(-n)}$ where $n \geq 0$ (otherwise $\widehat{\ell}(B^*\omega) = 0$), and $Z = Z^{(n)} \odot z \odot Z'$ where $Z^{(n)} \in E^{(n)}$ and z, Z' as before. We find

$$\left\|\big(\mathbf{1}, B\big)_P Z\right\|^2 = \left\|\langle Z, \big(B^*, B\big)_P Z\rangle\right\|$$
$$= \left\|\langle Z', \big(\zeta^*\ell(I\!I_K z), \ell^*(I\!I_K z)\zeta\big)_P Z'\rangle\right\| \leq (\mathfrak{T} - \tau)\big(\|\zeta\|^K\big)^2 \|Z'\|^2$$

for all $P \in \mathbb{P}_K$, where ζ is the adapted process $t \mapsto \langle \omega, B_t Z^{(n)}\rangle \in \mathcal{B} \subset \mathcal{B}_1$. For P sufficiently fine, $\big(\mathbf{1}, B\big)_P Z$ is close to $\int_\tau^\mathfrak{T} dp_t\,B_t Z$. This implies strong continuity on a total subset, hence, everywhere. ■

16.1.9 Remark. In the sequel, we will use shorthand notations like

$$\int_\tau^\mathfrak{T} A\,dI\,B = \int_\tau^\mathfrak{T} A_t\,dI_t\,B_t, \qquad \int_\tau^t A\,dI\,B = M_t, \quad \text{and} \quad \int A\,dI\,B = M,$$

if no confusion can arise. But keep in mind that $M_t = 0$ for $t \leq \tau$.

16.1.10 Remark. As the proof shows, many statements in Theorem 16.1.8 can be specified further. Additionally, weakening the convergence to $*$–strong convergence, all integrals exist also if the processes are only $*$–strong càdlàg functions.

Conversely, if we restrict to continuous integrands, then also the creation, annihilation, and time integral converge in norm. Therefore, if we omit the conservation integral (which is essentially non-continuous; see Lemma 16.3.2), then we may restrict like in [KS92] to a theory of continuous processes where everything converges in norm.

We also mention that for most statements it is not necessary to factorize according to Lemma 16.1.2. We emphasize, however, that convergence of the annihilation integral becomes much more complicated without this factorization.

16.1.11 Remark. In [Lie98] Liebscher considers a generalization of the usual conservation integral in the calculus on the symmetric Fock space by Hudson and Parthasarathy [HP84, Par92]. In this generalization the conservation integral is explained not only for time functions T, but for all operators $T \in \mathcal{B}^a(E_\infty)$. Unlike the usual behaviour in symmetric calculus, the integrators do no longer commute with the processes. Consequently, in [Lie98] there are two types of conservation integrals, one with the process on the right of the integrator, and one with the process on the left. One of the two possibilities is so complicated that its existence is guaranteed (explicitly) only for simple integrands.

A literal translation into our free setup encourages to consider limits of $\bigl(p(T), A\bigr)_P$ and of $\bigl(A, p(T)\bigr)_P$. However, by the particularly simple rules in Proposition 6.1.3 we find

$$\bigl(p(T), A\bigr)_P = p(T)\bigl(\mathbf{1}, A\bigr)_P \quad \text{and} \quad \bigl(A, p(T)\bigr)_P = \bigl(A, \mathbf{1}\bigr)_P p(T).$$

Convergence of these expressions becomes a trivial application of Theorem 16.1.8. We ask two questions. Firstly, could it be possible to translate this back into the symmetric framework? Secondly, is it possible to treat limits of expressions with two integrands like $\bigl(Ap(T), B\bigr)_P$ and $\bigl(A, p(T)B\bigr)_P$? (Of course, $\bigl(A, p(T)B\bigr)_P = \bigl(A, \mathbf{1}\bigr)_P\bigl(\mathbf{1}, p(T)B\bigr)_P$ still holds. However, as $p(T)$ does no longer commute with dp_t, we cannot treat $\bigl(\mathbf{1}, p(T)B\bigr)_P$ as before.) Presently, we do not know the answers.

16.2 Differential equations

In this section we show that a quite general class of quantum stochastic differential equations has unique solutions. A typical differential equation has the form

$$dW = W\,dM \qquad\qquad W_\tau = w \tag{16.2.1}$$

where $dM = A^0\,dI\,B^0 + A^+\,d\ell^*(x)\,B^+ + A^-\,d\ell(y)\,B^- + A^1\,dp(T)\,B^1$ (as in Theorem 16.4.4 below) and w is an operator on $\mathcal{F}$ adapted to E_τ. (Of course, also the adjoint equation is considered.) A solution of such a differential equation is a process $W \in \mathfrak{A}_1^\tau$ fulfilling

$$W_t = w + \int_\tau^t W\,dM.$$

The standard procedure already used in the calculus on the symmetric Fock space [HP84] is *successive approximation*. We also follow this approach. However, thanks to the fact that we are dealing with bounded operators, we are able as in [KS92, Spe91] to show convergence by an application of *Banach's fix point theorem*. As in [KS92] for a calculus without conservation integrals we may apply the fix point theorem directly. If conservation integrals are involved, we need a triple iteration (cf. [Spe91]). In both cases we will meet more general types of differential equations, when we consider unitarity conditions. Therefore, we decided to keep the description from the beginning as general as possible.

16.2.1 Definition. A *general integral* is a linear mapping $\mathcal{I}\colon \mathfrak{A}_1^\tau \to \mathcal{B}^a(\mathcal{F})$ which is contained in the linear span of mappings of one of the following forms:

$$
\begin{aligned}
W &\longmapsto \int_\tau^{\mathcal{T}} WA\,dI\,B & W &\longmapsto \int_\tau^{\mathcal{T}} \mathbb{E}_0(W)A\,dI\,B \\
W &\longmapsto \int_\tau^{\mathcal{T}} AW\,dI\,B & W &\longmapsto \int_\tau^{\mathcal{T}} A\mathbb{E}_0(W)\,dI\,B \\
W &\longmapsto \int_\tau^{\mathcal{T}} A\,dI\,WB & W &\longmapsto \int_\tau^{\mathcal{T}} A\,dI\,\mathbb{E}_0(W)B \\
W &\longmapsto \int_\tau^{\mathcal{T}} A\,dI\,BW & W &\longmapsto \int_\tau^{\mathcal{T}} A\,dI\,B\mathbb{E}_0(W)
\end{aligned}
$$

where dI is one of the integrators dl, $d\ell^*(x)$, $d\ell(x)$, or $dp(T)$ ($l' \in \mathfrak{L}^\infty_{loc}(\mathbb{R}, \mathcal{B})$, $x \in \mathfrak{L}^\infty_{loc}(\mathbb{R}, F)$, $T \in \mathfrak{L}^\infty_{loc}(\mathbb{R}, \mathcal{B}^a(F))$ and $A, B \in \mathfrak{A}_1$, or

$$W \longmapsto \int_\tau^{\mathcal{T}} A\,dI^{\mathbb{E}_0(W)}\,B$$

where the argument of the integrator $dI^{\mathbb{E}_0(W)}$ depends linearly (or anti-linearly for the annihilator) and continuously (in the repective norms) on $\mathbb{E}_0(W)$. We write $\mathcal{I}_\tau^{\mathcal{T}}$, if we want to indicate the end points of the involved time interval. By $\mathcal{I}_\tau(W)$ we denote the process $t \mapsto \mathcal{I}_\tau^t(W)$

A *special general integral* is a general integral where the appearing conservation integrals are subject to the restriction that the parameters A, B in each conservation integral take values only in $\mathcal{B} \subset \mathcal{B}^a(\mathcal{F})$.

The definition of a general integral is motivated by the way how processes enter the Ito formula (cf. (16.4.1) below). Whereas the restriction for the special general integral is necessary, if we want to apply the following refined version of *Banach's fix point theorem*. Already in the calculus on the full Fock space Speicher [Spe91] has shown that there exist differential equations with general conservation integrals which do not have a solution even, if we allow for unbounded operators.

16.2.2 Proposition. *Let $\mathcal{I}$ be a general integral. Assume that for each compact interval K there exist constants $0 < C < 1$ and $d > 0$ such that*

$$\left\|\mathcal{I}_t^{t+\delta} \circ \mathcal{I}_t \circ \mathcal{I}_t(W)\right\|_1 \leq C\,\|W\|_1$$

for all $t \in K$ and $0 \leq \delta \leq d$. Then for all $\tau \in \mathbb{R}$ and $w \in \mathcal{B}^a(\mathcal{F})$ adapted to E_τ the differential equation

$$W_t = w + \mathcal{I}_\tau^t(W) \tag{16.2.2}$$

has a unique solution in $\mathfrak{A}_1^\tau$.

Proof. For $s \in [t, t+d]$ and W_t adapted to E_t we find a solution by successive approximation, i.e. we set $W_s^0 = W_t$ and $W_s^{n+1} = W_t + \mathfrak{I}_t^s(W^n)$ for $n \geq 1$. Then as in the proof of Banach's fix point theorem the W_s^n form a Cauchy sequence in ℓ^1–norm whose limit is the unique solution (16.2.2) on $[t, t+d]$. By splitting a compact interval into finitely many intersecting intervals of length d, we construct a unique solution on each compact interval K. In this way, we obtain for each $t \in \mathbb{R}$ a solution on $[\tau, t]$. By uniqueness the solution restricted to a subinterval $[\tau, s]$ must coincide with the solution constructed on this subinterval so that we obtain a unique solution on $[\tau, \infty)$. Finally, we extend this solution by the constant w to times smaller than τ and obtain the unique solution on $\mathbb{R}$ which is by construction in $\mathfrak{A}_1^\tau$. ∎

16.2.3 Theorem. *Let $\mathfrak{I}$ be a special general integral. Then the differential equation* (16.2.2) *with $w \in \mathcal{B}^a(\mathcal{F})$ adapted to E_τ has a unique solution in $\mathfrak{A}_1^\tau$.*

Proof. We show that the assumptions of Proposition 16.2.2 are fulfilled. By Lemma A.3.1 it is enough to understand this for each of the (finitely many) homogeneous parts of the operator $\mathfrak{I}_t^{t+\delta} \circ \mathfrak{I}_t \circ \mathfrak{I}_t \colon \mathfrak{A}_1^\tau \to \mathcal{B}_1$ and for homogeneous W. In the iterated integral $\mathfrak{I}_t^{t+\delta} \circ \mathfrak{I}_t \circ \mathfrak{I}_t(W)$ we have two types of summands. Either at least one time, creation, or annihilation integral is involved. Then existence of suitable constants C, d follows from (16.1.3b, 16.1.3c). Or we have an iterated conservation integral. In this case, we conclude from the fact that dp commutes with all functions taking values in $\mathcal{B}$ and from $dp_{t_k} dp_{t_\ell} = 0$ for $k \neq \ell$ that the triple conservation integral is 0. ∎

16.3 0–Criteria

In this section we prepare for Theorem 16.4.4 which asserts in how far the coefficients in a stochastic differential equation are unique. The main result is Lemma 16.3.2 which tells us that conservation integrals are essentially strongly continuous. This allows to separate them from the other types of integrals in Theorem 16.1.8 (which are continuous by Proposition 16.1.7) by looking at their continuity properties.

All results in this section, besides Proposition 16.3.3, may be considered as consequences of Lemma 16.1.2 which by computations as in the the proof of Lemma 16.1.6 gives rise to a particularly simple case of an Ito formula for homogeneous integrands in one-sided conservation integrals. For a full prove of Theorem 16.4.4 we need the full Ito formula for creation and annihilation integrals. Therefore, it is postponed to the end of the following section.

16.3.1 Proposition. *Let $K = [\tau, \mathfrak{T}]$ be a compact interval, and let $f \in \mathcal{L}^\infty(K, F)$. Then there exists a $t_0 \in K$ such that $\operatorname*{ess\,sup}_{t \in [t_0, t_0+\delta] \cap K} \|f(t)\| = \operatorname*{ess\,sup}_{t \in K} \|f(t)\|$ for all $\delta > 0$.*

PROOF. If the contrary was true, then we could cover K with finitely many open intervals on which the ess sup of f is strictly less than its ess sup on K. ■

Lemma B.1.6 and Observation B.1.7 tell us that $L^\infty(K, \mathcal{B}^a(F))$ is a C^*–subalgebra of $\mathcal{B}^a(L^2(K, F))$.

16.3.2 Lemma. *Let $A, B \in \mathfrak{A}_1^\tau$ and $T \in \mathcal{L}^\infty_{loc}(\mathbb{R}, \mathcal{B}^a(F))$. Then for the process M defined by setting $M_t = \int_\tau^t A \, dp(T) \, B$ the following conditions are equivalent.*

1. *$M = 0$.*

2. *M is continuous.*

3. *$\operatorname*{ess\,sup}_{t \in [\tau, \infty)} \big\| \mathbb{E}_0(A_t^* A_t) T_t \mathbb{E}_0(B_t B_t^*) \big\| = 0$.*

PROOF. $1 \Rightarrow 2$. Obvious.

$2 \Rightarrow 3$. We conclude indirectly. So let us assume that there is a compact interval $K = [\tau, \mathfrak{T}]$ ($\mathfrak{T} \geq \tau$) such that $C = \operatorname*{ess\,sup}_{t \in K} \big\| \mathbb{E}_0(A_t^* A_t) T_t \mathbb{E}_0(B_t B_t^*) \big\| > 0$. By Proposition 16.3.1 we may choose $t_0 \in [\tau, \mathfrak{T})$ such that

$$\operatorname*{ess\,sup}_{t \in [t_0, t_0+\delta] \cap K} \big\| \mathbb{E}_0(A_t^* A_t) T_t \mathbb{E}_0(B_t B_t^*) \big\| = C \tag{16.3.1}$$

for all $\delta > 0$. Of course, this implies $\|A\|^{[t_0, t_0+\delta] \cap K} > 0$, $\|B\|^{[t_0, t_0+\delta] \cap K} > 0$, and

$$\operatorname*{ess\,sup}_{t \in [t_0, t_0+\delta] \cap K} \|T_t\| \geq \frac{C}{\left(\|A\|^{[t_0, t_0+\delta] \cap K} \, \|B\|^{[t_0, t_0+\delta] \cap K} \right)^2} > 0.$$

Necessarily, we have $\big\| \mathbb{E}_0(A_{t_0}^* A_{t_0}) \big\| > 0$ and $\big\| \mathbb{E}_0(B_{t_0} B_{t_0}^*) \big\| > 0$. Otherwise, by continuity of $\mathbb{E}_0(A_t^* A_t)$ and $\mathbb{E}_0(B_t B_t^*)$, we obtain a contradiction to (16.3.1).

If we choose δ sufficiently small, then the following assertions become true. (For simplicity, we assume $t_0 + \delta \in K$.) Firstly, $\int_{t_0}^{t_0+\delta} dM$ is close to $B_{t_0} \, p(\mathbb{I}_{[t_0, t_0+\delta]} T) \, A_{t_0}$, because the norm of the partition $(t_0, t_0 + \delta)$ is δ, therefore, small. Consequently,

$$p(\mathbb{I}_{[t_0, t_0+\delta]}) \, A_{t_0}^* \left(\int_{t_0}^{t_0+\delta} dM \right) B_{t_0}^* \, p(\mathbb{I}_{[t_0, t_0+\delta]})$$

is close to $\mathbb{E}_0(A_{t_0}^* A_{t_0}) \, p(\mathbb{I}_{[t_0, t_0+\delta]} T) \, \mathbb{E}_0(B_{t_0} B_{t_0}^*)$.

Secondly, $\mathbb{E}_0(A^*_{t_0}A_{t_0})\,T_t\,\mathbb{E}_0(B_{t_0}B^*_{t_0})$ is close to $\mathbb{E}_0(A^*_tA_t)\,T_t\,\mathbb{E}_0(B_tB^*_t)$ for all $t\in[t_0,t_0+\delta]$, because $\mathbb{E}_0(A^*_tA_t)$ and $\mathbb{E}_0(B_tB^*_t)$ are continuous. Therefore, by Proposition 6.1.3

$$\begin{aligned}&\big\|\mathbb{E}_0(A^*_{t_0}A_{t_0})\,p(\mathbb{I}_{[t_0,t_0+\delta]}T)\,\mathbb{E}_0(B_{t_0}B^*_{t_0})\big\|\\&\quad=\Big\|p\Big(\mathbb{E}_0(A^*_{t_0}A_{t_0})\,(\mathbb{I}_{[t_0,t_0+\delta]}T)\,\mathbb{E}_0(B_{t_0}B^*_{t_0})\Big)\Big\|=\big\|\mathbb{E}_0(A^*_{t_0}A_{t_0})\,(\mathbb{I}_{[t_0,t_0+\delta]}T)\,\mathbb{E}_0(B_{t_0}B^*_{t_0})\big\|\\&\qquad=\operatorname*{ess\,sup}_{t\in[t_0,t_0+\delta]}\big\|\mathbb{E}_0(A^*_{t_0}A_{t_0})\,T_t\,\mathbb{E}_0(B_{t_0}B^*_{t_0})\big\|\end{aligned}$$

is close to C. As C does not depend on the choice of δ, $\|M_{t_0+\delta}-M_{t_0}\|$ is bounded below by a non-zero positive number. Therefore, M is not continuous at t_0.

$3\Rightarrow 1$. Again, we conclude indirectly. So let us assume that $M_t\neq 0$ for some $t>\tau$. We may write $A=\sum_{n\in\mathbb{N}_0}A^{(n)}$ and $B=\sum_{n\in\mathbb{N}_0}B^{(-n)}$. (The components with $n<0$ do not contribute.)

Observe that $\mathbb{E}_0(A^*A)=\sum_{n\in\mathbb{N}_0}\mathbb{E}_0(A^{(n)^*}A^{(n)})$ and, similarly, for $\mathbb{E}_0(BB^*)$. By Proposition A.7.3(2) it is sufficient to show that the element $\mathbb{E}_0(A^{(n)^*}A^{(n)})\,T\,\mathbb{E}_0(B^{(-m)}B^{(-m)^*})\neq 0$ in the C^*–algebra $L^\infty([\tau,t],\mathcal{B}^a(F))$ for some $n,m\in\mathbb{N}_0$.

As $M_t\neq 0$, there exist n and m such that

$$\int_\tau^t A^{(n)}\,dp(T)\,B^{(-m)}=\Big(\int_\tau^t A^{(n)}\,dp\Big)\,p(T)\,\Big(\int_\tau^t dp\,B^{(-m)}\Big)\neq 0. \tag{16.3.2}$$

By Proposition A.7.3(1) we have $\int_\tau^t dp\,A^{(n)^*}\int_\tau^t A^{(n)}\,dp(T)\,B^{(-m)}\int_\tau^t B^{(-m)^*}\,dp\neq 0$. By computations similar to the proof of Lemma 16.1.6 we find

$$\begin{aligned}&\int_\tau^t dp\,A^{(n)^*}\int_\tau^t A^{(n)}\,dp(T)\,B^{(-m)}\int_\tau^t B^{(-m)^*}\,dp\\&\quad=\lim_{P\in\mathbb{P}_{[\tau,t]}}\big(\mathbf{1},A^{(n)^*}\big)_P\big(A^{(n)},\mathbf{1}\big)_P\,p(T)\,\big(\mathbf{1},B^{(-m)}\big)_P\big(B^{(-m)^*},\mathbf{1}\big)_P\\&\quad=\int_\tau^t\mathbb{E}_0(A^{(n)^*}A^{(n)})\,dp(T)\,\mathbb{E}_0(B^{(-m)}B^{(-m)^*})\\&\quad=\int_\tau^t dp\big(\mathbb{E}_0(A^{(n)^*}A^{(n)})\,T\,\mathbb{E}_0(B^{(-m)}B^{(-m)^*})\big)\\&\quad=p\big(\mathbb{I}_{[\tau,t]}\mathbb{E}_0(A^{(n)^*}A^{(n)})\,T\,\mathbb{E}_0(B^{(-m)}B^{(-m)^*})\big)\neq 0.\end{aligned}\tag{16.3.3}$$

Equality of the last integral and the integral before follows, because it is true for step functions, and because both $\mathbb{E}_0(A^{(n)^*}A^{(n)})$ and $\mathbb{E}_0(B^{(-m)}B^{(-m)^*})$ may be approximated equiuniformly by step functions. Since $T\mapsto p(T)$ is an isometry by Proposition 6.1.3, we arrive at

$$\operatorname*{ess\,sup}_{s\in[\tau,t]}\big\|\mathbb{E}_0(A_s^{(n)^*}A_s^{(n)})\,T_s\,\mathbb{E}_0(B_s^{(-m)}B_s^{(-m)^*})\big\|\neq 0.\ \blacksquare$$

In order to proceed, we need to know, when time integrals are 0.

16.3.3 Proposition. *Let $A, B \in \mathfrak{A}_1^\tau$ and $x, y \in \mathfrak{L}^\infty_{loc}(\mathbb{R}, F)$. Then $M_t = \int_\tau^t A\, d\mu^{x,y}\, B = 0$, if and only if*

$$\operatorname*{ess\,sup}_{t\in[\tau,\infty)} \big\| A_t \langle x(t), y(t)\rangle B_t \big\| = 0.$$

Proof. By changing the function $A_t \langle x(t), y(t)\rangle B_t$ on a (measurable) null-set, we may achieve that $\operatorname{ess\,sup} \|\bullet\| = \sup \|\bullet\|$. Now the statement follows by Corollary 16.1.5. ■

16.3.4 Lemma. *Let $A, B \in \mathfrak{A}_1^\tau$ and $x \in \mathfrak{L}^\infty_{loc}(\mathbb{R}, F)$. Then*

$$\operatorname*{ess\,sup}_{t\in[\tau,\infty)} \big\| B_t^* \langle x(t), \mathbb{E}_0(A_t^* A_t) x(t)\rangle B_t \big\| = 0 \quad \textit{implies} \quad M_t = \int_\tau^t A\, d\ell^*(x)\, B = 0.$$

An analogue statement is true for annihilation integrals.

Proof. Of course, $\operatorname*{ess\,sup}_{t\in[\tau,\infty)} \big\| B_t^* \langle x(t), \mathbb{E}_0(A_t^* A_t) x(t)\rangle B_t \big\| = 0$ implies

$$\operatorname*{ess\,sup}_{t\in[\tau,\infty)} \big\| B_t^* \langle x(t), \mathbb{E}_0({A_t^{(n)}}^* A_t^{(n)}) x(t)\rangle B_t \big\| = 0$$

for all $n \in \mathbb{Z}$. By computations similar to (16.3.3) we find

$$\Big| \int_\tau^t A^{(n)}\, d\ell^*(x)\, B \Big|^2 = \int_\tau^t B^*\, d\mu^{x, \mathbb{E}_0({A_t^{(n)}}^* A_t^{(n)}) x}\, B$$

which is 0 by Proposition 16.3.3 so that $\int_\tau^t A^{(n)}\, d\ell^*(x)\, B = 0$ for all $n \in \mathbb{Z}$. Therefore, $M_t = \int_\tau^t A\, d\ell^*(x)\, B = 0$. ■

16.3.5 Remark. The converse direction of Lemma 16.3.4 is done best by using the Ito formula. We postpone it to the following section. Notice, however, that computations like (16.3.3) already constitute an Ito formula in a particularly simple case.

16.4 Ito formula

We start by introducing explicitly the notation which turns all integrals into conservation integrals, formally. For that goal, we consider the formal "operators" $\widehat{\ell^*}(X)$ and $\widehat{\ell}(X)$ where either $X = \omega$ (whence $\widehat{\ell^*}(X) = \widehat{\ell}(X) = \mathbf{1}$), or $X = x \in \mathfrak{L}^\infty_{loc}(\mathbb{R}, F)$. This notation is formal in the sense that $\ell^*(x)$ and $\ell(x)$, in general, are not elements of $\mathcal{B}^a(\mathcal{F})$. In integrals they appear, however, only in combinations like $p(\mathit{I\!I}_K)\ell^*(x) = \ell^*(\mathit{I\!I}_K x)$ which are perfectly well-defined.

In this notation all integrals in Theorem 16.1.8 including the time integral can be written in the form

$$\int_\tau^{\mathfrak{T}} A\widehat{\ell}(X)\,dp(T)\,\widehat{\ell}^*(Y)B$$

for suitable choices of X, Y, and T. By slight abuse of notation, we say $A\widehat{\ell}(X) \in \mathfrak{A}_1^\tau\widehat{\ell}(\mathcal{F}^{01})$ and $\widehat{\ell}^*(X)B \in \widehat{\ell}^*(\mathcal{F}^{01})\mathfrak{A}_1^\tau$ where $\mathcal{F}^{01} = \mathcal{B} \oplus \mathfrak{L}^\infty_{loc}(\mathbb{R}, F)$.

Of course, for creation, annihilation, or time integral we are reduced to $T = \mathbf{1}$. However, in the cases $X = x$, or $Y = y$, the operator $p(T)$ in $dp(T) = p(T)dp$ may be absorbed either into the creator on the right, or the annihilator on the left by Proposition 6.1.3.

16.4.1 Theorem. *Let M, M' be processes in $\mathfrak{A}_1^\tau$ given by integrals*

$$M_t = \int_\tau^t A\,dp(T)\,B \quad \textit{and} \quad M'_t = \int_\tau^t A'\,dp(T')\,B'$$

where $A, A' \in \mathfrak{A}_1^\tau\widehat{\ell}(\mathcal{F}^{01})$, $B, B' \in \widehat{\ell}^*(\mathcal{F}^{01})\mathfrak{A}_1^\tau$ and $T, T' \in \mathfrak{L}^\infty(\mathbb{R}, \mathcal{B}^a(F))$. Then the product $MM' \in \mathfrak{A}_1^\tau$ is given by

$$M_tM'_t = \int_\tau^t A\,dp(T)\,BM' + \int_\tau^t MA'\,dp(T')\,B' + \int_\tau^t A\,dp\big(T\mathbb{E}_0(BA')T'\big)\,B' \tag{16.4.1}$$

where $\mathbb{E}_0(BA')$ is the function $t \mapsto \mathbb{E}_0(B_tA'_t) \in \mathcal{B} \subset \mathcal{B}^a(\mathcal{F})$.

In differential notation $dM = A\,dp(T)\,B$ and $d(MM') = dM\,M' + M\,dM' + dM\,dM'$ we find the Ito formula

$$dM\,dM' = A\,dp\big(T\mathbb{E}_0(BA')T'\big)\,B'.$$

Proof. Let us fix the compact interval $K = [\tau, \mathfrak{T}]$. The nets $\big(A, p(T)B\big)_P$ and $\big(A'p(T'), B'\big)_P$ converge $*$–strongly uniformly over $\mathbb{P}_{K'}$ for all compact intervalls $K' = [\tau, t] \subset K$ to M_t and M'_t, respectively, by Theorem 16.1.8. By Proposition 16.1.7 all nets are bounded uniformly for all $K' \subset K$. Therefore,

$$\big(A, p(T)B\big)_P\big(A'p(T'), B'\big)_P \xrightarrow{*\text{–equistrongly}} M_tM'_t.$$

Splitting the double sum over k and ℓ into the parts where $k > \ell$, $k < \ell$ and $k = \ell$, we find

$$\begin{aligned}\big(A, p(T)B\big)_P&\big(A'p(T'), B'\big)_P\\ &= \Big[\sum_{1\le\ell<k\le N} + \sum_{1\le k<\ell\le N}\Big] A_{t_{k-1}}\,dp_{t_k}(T)\,B_{t_{k-1}}A'_{t_{\ell-1}}\,dp_{t_\ell}(T')\,B'_{t_{\ell-1}}\\ &\qquad + \sum_{k=1}^N A_{t_{k-1}}\,dp_{t_k}(T)\,B_{t_{k-1}}A'_{t_{k-1}}\,dp_{t_k}(T')\,B'_{t_{k-1}}.\end{aligned} \tag{16.4.2}$$

We will show that the first summand and the third summand of (16.4.2) converge strongly to the first summand and the third summand, respectively, of (16.4.1), establishing in this way that also the second summand of (16.4.2) converges strongly. Looking at the adjoint, we have formally the same sums, except that the first and the second summand have changed their roles. This shows that not only the limits are $*$–strong limits, but also that the limit of the second summand of (16.4.2) is the second summand of (16.4.1).

Let $Z \in \mathcal{F}_1$. By Theorem 16.1.8

$$\Big\|\Big(M'_{t_{k-1}} - \sum_{\ell=1}^{k-1} A'_{t_{\ell-1}}\, dp_{t_\ell}(T')\, B'_{t_{\ell-1}}\Big)Z\Big\|_1 < \varepsilon$$

for all k, if only the norm of $P \in \mathbb{P}_K$ is sufficiently small. Therefore, strong versions of (16.1.3a) and (16.1.3b) (depending on whether $B \in \mathfrak{A}_1^\tau$ or $B \in \ell(y)\mathfrak{A}_1^\tau$) tell us that the first summand in (16.4.2) converges strongly to the first summand in (16.4.1).

For the last summand of (16.4.1) we assume concretely that $A = \bar{A}\widehat{\ell}(X)$ and $B = \widehat{\ell}^*(Y)\bar{B}$ ($\bar{A}, \bar{B} \in \mathfrak{A}_1; X, Y \in \mathcal{F}^{01}$), and similarly for A', B'. For the case $Y = X' = \omega$ we find from Corollary 6.3.6 and the proof of Theorem 16.1.8 convergence in norm to the correct limit. In the remaining cases $\mathbb{E}_0(B_t A'_t)$ is 0. Let us check, whether this is also true for the limit of the last summand of (16.4.2). For instance, assume that $Y = y \in \mathfrak{L}^\infty_{loc}(\mathbb{R}, F)$. We find

$$\Big\|\sum_{k=1}^N A_{t_{k-1}}\, dp_{t_k}(T)\, B_{t_{k-1}} A'_{t_{k-1}}\, dp_{t_k}(T')\, B'_{t_{k-1}}\Big\|$$
$$\le \big\|\big(A, p(T)\big)_P\big\| \Big\|\sum_{k=1}^N dp_{t_k}\, B_{t_{k-1}} A'_{t_{k-1}}\, dp_{t_k}\Big\| \big\|\big(p(T'), B'\big)_P\big\|.$$

For the square modulus of the sum we find by computations as in Lemma 16.1.6

$$\sum_{k=1}^N dp_{t_k}\, A'^*_{t_{k-1}} B^*_{t_{k-1}}\, dp_{t_k}\, B_{t_{k-1}} A'_{t_{k-1}}\, dp_{t_k}$$
$$\le \sum_{k=1}^N \|\langle y, dp_{t_k}\, y\rangle\|\, \big\|A'^*_{t_{k-1}} \bar{B}^*_{t_{k-1}} \bar{B}_{t_{k-1}} A'_{t_{k-1}}\big\|\, dp_{t_k} \le \max_{1\le k\le N}\big(\|\langle y, dp_{t_k}\, y\rangle\|\big)\big\|A'^* \bar{B}^* \bar{B} A'\big\|.$$

As the first factor tends to 0, we find covergence to 0 in norm. ■

16.4.2 Corollary. *Let $M_t = \int_0^t A\, dI\, B$ and $M'_t = \int_0^t A'\, dI'\, B'$ be integrals as in Theorem 16.1.8. Then $dM\, dM' = A\, dI''\, B'$ where dI'' has to be chosen according to the Ito table*

$dI \backslash dI'$	$d\mu^{x',y'}$	$d\ell^*(x')$	$d\ell(x')$	$dp(T')$
$d\mu^{x,y}$	0	0	0	0
$d\ell^*(x)$	0	0	0	0
$d\ell(x)$	0	$d\mu^{x,\mathbb{E}_0(BA')x'}$	0	$d\ell\big(T'^*\mathbb{E}_0(BA')^* x\big)$
$dp(T)$	0	$d\ell^*\big(T\mathbb{E}_0(BA')x'\big)$	0	$dp\big(T\mathbb{E}_0(BA')T'\big)$.

16.4.3 Remark. It is easy to see that the Ito formula extends also to more general time integrals $\int A\,dl\,B$ where l is an integrator with a locally bounded density $l' \in \mathfrak{L}^{\infty}_{loc}(\mathbb{R}, \mathcal{B})$. Of course, also Proposition 16.3.3 remains true replacing $\langle x(t), y(t)\rangle$ with a more general density l'.

16.4.4 Theorem. *Let $A^i, B^i \in \mathfrak{A}_1^{\tau}$ $(i = 0, +, -, 1)$, $x, y \in \mathfrak{L}^{\infty}_{loc}(\mathbb{R}, F)$, $T \in \mathfrak{L}^{\infty}_{loc}(\mathbb{R}, \mathcal{B}^a(F))$, and let l be an integrator with locally bounded density $l' \in \mathfrak{L}^{\infty}_{loc}(\mathbb{R}, \mathcal{B})$. Let*

$$M_t = \int_{\tau}^{t} dM^0 + \int_{\tau}^{t} dM^+ + \int_{\tau}^{t} dM^- + \int_{\tau}^{t} dM^1$$

be a sum of integrals where $dM^0 = A^0\,dl\,B^0$, $dM^+ = A^+\,d\ell^*(x)\,B^+$, $dM^- = A^-\,d\ell(y)\,B^-$, and $dM^1 = A^1\,dp(T)\,B^1$. Then the following conditions are equivalent.

1. $M = 0$.
2. $\displaystyle\int dM^0 = \int dM^+ = \int dM^- = \int dM^1 = 0.$
3. $$\operatorname*{ess\,sup}_{t \in [\tau,\infty)} \left\| A_t^0\, l'_t\, B_t^0 \right\| = 0 \qquad \operatorname*{ess\,sup}_{t \in [\tau,\infty)} \left\| \mathbb{E}_0(A_t^{1*} A_t^1)\, T_t\, \mathbb{E}_0(B_t^1 B_t^{1*}) \right\| = 0$$
$$\operatorname*{ess\,sup}_{t \in [\tau,\infty)} \left\| B_t^{+*} \langle x(t), \mathbb{E}_0(A_t^{+*} A_t^+) x(t)\rangle\, B_t^+ \right\| = 0$$
$$\operatorname*{ess\,sup}_{t \in [\tau,\infty)} \left\| A_t^- \langle y(t), \mathbb{E}_0(B_t^- B_t^{-*}) y(t)\rangle\, A_t^{-*} \right\| = 0.$$

Proof. By Proposition 16.3.3 and Lemmata 16.3.2 and 16.3.4 we have $3 \Rightarrow 2$ and, of course, we have $2 \Rightarrow 1$.

So let us assume $M = 0$. In particular, M is continuous. Since $\int dM^0 + \int dM^+ + \int dM^-$ is continuous by Proposition 16.1.7, so is $\int dM^1$. By Lemma 16.3.2 we conclude that $\int dM^1 = 0$, and that the condition in 3 concerning the conservation integral is fulfilled.

Writing down the Ito formulae for M^*M and MM^*, and taking into account that $M = M^* = 0$ and that the conservation part is absent, we find that $\int dM^*\,dM = \int B^{+*}\,d\mu^{x,\mathbb{E}_0(A^{+*}A^+)x}\,B^+ = 0$ and $\int dM\,dM^* = \int A^-\,d\mu^{y,\mathbb{E}_0(B^-B^{-*})y}\,A^{-*} = 0$. Therefore, by Proposition 16.3.3 also the conditions in 3 concerning creation and annihilation part must be fulfilled.

Since all parts except the time integral are known to be 0, also the time integral must be 0. Again by Proposition 16.3.3 we find that also the last condition in 3 must be fulfilled. This is $1 \Rightarrow 3$. ■

16.5 Unitarity conditions

We are interested in finding necessary and sufficient conditions under which a solution U of a differential equation like (16.2.2) is unitary. Usually, this is done by writing down what the Ito formula asserts for

$$d(U^*U) = dU^*\,U + U^*\,dU + dU^*\,dU \tag{16.5.1a}$$

and

$$d(UU^*) = dU\,U^* + U\,dU^* + dU\,dU^*. \tag{16.5.1b}$$

If the coefficients of $d(U^*U)$ and $d(UU^*)$ are 0, then this is certainly sufficient to conclude that U is unitary. To have necessity we must conclude backwards from $\int d(U^*U) = \int d(UU^*) = 0$ that also all coefficients vanish. Presently, however, we have only the criterion Theorem 16.4.4, where each type of integrators $dI, d\ell^*, d\ell, dp$ appears not more than once. Unfortunately, even in differential equations of the simpler form 16.2.1 the Ito formula yields, in general, more summands of the same type which cannot be summed up to a single one. We explain in Section 17.3 how we can treat such sums, but we do not need them.

Here we consider differential equations *without coefficients*. This means that there are no processes A, B arround the integrators. At the first sight, this looks poor. However, we allow for rather arbitrary arguments in the integrators. As we explain in Section 17.1, this is already sufficient to include the case of a calculus on a full Fock space with initial space and arbitrarily many degrees of freedom. (In [KS92, Spe91] only the Fock space over $L^2(\mathbb{R})$ is considered which, roughly speaking, corresponds to one degree of freedom. In the unitarity conditions in [Spe91] at least some of the processes arround the integrators may vary over $\mathfrak{A}_1$. So, at least in the cases were [Spe91] applies the conditions given there are more general.) The proof of the following theorem is very much along the lines of the corresponding proof in [Spe91].

16.5.1 Theorem. *Let $x, y \in \mathcal{L}^\infty_{loc}(\mathbb{R}, F)$, $T \in \mathcal{L}^\infty_{loc}(\mathbb{R}, \mathcal{B}^a(F))$, and let l be an integrator with locally bounded density $l' \in \mathcal{L}^\infty_{loc}(\mathbb{R}, \mathcal{B})$. Then the unique solution in $\mathfrak{A}_1^0$ of the differential equation*

$$dU = U\big(dp(T) + d\ell^*(x) + d\ell(y) + dl\big) \qquad\qquad U_0 = \mathbf{1} \tag{16.5.2}$$

is unitary, if and only if the following conditions are fulfilled.

1. *$T(t) + \mathbf{1}$ is unitary almost everywhere on $\mathbb{R}_+$.*

2. *$x(t) + T(t)y(t) + y(t) = 0$ almost everywhere on $\mathbb{R}_+$.*

3. $l'(t) + l'^*(t) + \langle x(t), x(t)\rangle = 0$ *almost everywhere on* $\mathbb{R}_+$.

PROOF. From

$$dU^* = \bigl(dp(T^*) + d\ell^*(y) + d\ell(x) + dl^*\bigr)U^* \qquad\qquad U_0^* = \mathbf{1}$$

we find for (16.5.1a,16.5.1b) the explicit expressions

$$\begin{aligned} d(U^*U) = &\bigl(dp(T^*) + d\ell^*(y) + d\ell(x) + dl^*\bigr)U^*U \\ &+ U^*U\bigl(dp(T) + d\ell^*(x) + d\ell(y) + dl\bigr) \\ &+ dp(T^*\mathbb{E}_0(U^*U)T) + d\ell^*(T^*\mathbb{E}_0(U^*U)x) + d\ell(T^*\mathbb{E}_0(U^*U)x) + d\mu^{x,\mathbb{E}_0(U^*U)x} \end{aligned} \tag{16.5.3a}$$

and

$$\begin{aligned} d(UU^*) =&\, U\Bigl(dp(T) + d\ell^*(x) + d\ell(y) + dl \\ &+ dp(T^*) + d\ell^*(y) + d\ell(x) + dl^* \\ &+ dp(TT^*) + d\ell^*(Ty) + d\ell(Ty) + d\mu^{y,y}\Bigr)U^* \\ =&\, U\Bigl(dp(T + T^* + TT^*) + d\ell^*(x + y + Ty) + d\ell(x + y + Ty) \\ &+ (dl + dl^* + d\mu^{y,y})\Bigr)U^*. \end{aligned} \tag{16.5.3b}$$

If U is unitary, then $\mathbb{E}_0(U^*U) = \mathbf{1}$ and (16.5.3a) simplifies to

$$0 = dp(T + T^* + T^*T) + d\ell^*(x + y + T^*x) + d\ell(x + y + T^*x) + (dl + dl^* + d\mu^{x,x}).$$

By Theorem 16.4.4 we find $(T + T^* + T^*T)(t) = 0$ (i.e. $(T + \mathbf{1})(t)$ is an isometry), $(x + y + T^*x)(t) = 0$, and $l(t) + l^*(t) + \langle x(t), x(t)\rangle = 0$ for almost all $t \in \mathbb{R}_+$. Equation (16.5.3b) implies (notice that U and U^* dissappear in all suprema in Theorem 16.4.4, if U is unitary) that also $(T + T^* + TT^*)(t) = 0$ for almost all $t \in \mathbb{R}_+$. In other words, $(T + \mathbf{1})(t)$ is a unitary, such that also $(x + y + Ty)(t) = (T + \mathbf{1})(x + y + T^*x)(t) = 0$ and $d\mu^{y,y} = d\mu^{x,x}$.

Conversely, if the three conditions are fulfilled, then by (16.5.3b), $d(UU^*) = 0$. Together with the initial condition $(UU^*)_0 = \mathbf{1}$ we find that U is a coisometry. Whereas, U^*U fulfills the differential equation (16.5.3a) also with initial condition $(U^*U)_0 = \mathbf{1}$. One easily checks that $U^*U = \mathbf{1}$ is a solution of (16.5.3a). By Theorem 16.2.3 this solution is unique. Therefore, U is unitary. ■

It is noteworthy that, although our differential equation has no coefficients A and B, we needed Lemma 16.3.2 in full generality in order to be able to conclude from (16.5.3b) to $T + T^* + TT^* = 0$.

A more common way to write down a differential equation with unitary solution is

$$dU = U\big(dp(W - \mathbf{1}) + d\ell^*(Wy) - d\ell(y) + (i\,dH - \frac{1}{2}\,d\mu^{y,y})\big) \qquad U_0 = \mathbf{1},$$

where W is unitary, y is arbitrary, and H is self-adjoint.

16.6 Cocycles

Let us return for a moment to the differential equation in the form (16.5.2) (without unitarity conditions). U_t is adpted to E_t and the differentials $dp, d\ell^*, d\ell, dI$ are adapted to the complement of E_t. As pointed out by Speicher [Spe98] this means that in the sense of Voiculescu [Voi95] U_t and the differentials are *freely independent with amalgamation over* $\mathcal{B}$ in the vacuum conditional expectation. In other words, U is a process with independent (right) multiplicative increments.

If we choose constant functions $T(t) = \mathfrak{T}$, $x(t) = \xi$, $y(t) = \zeta$, and $l'(t) = \jmath$ (with $\mathfrak{T} \in \mathcal{B}^a(F)$, $\xi, \zeta \in F$, $\jmath \in \mathcal{B}$), then U has even *stationary* increments. The goal of this section is to show that in this case U is a cocycle with respect to the time shift automorphism group on $\mathcal{B}^a(\mathcal{F})$. The results by Hellmich, Köstler and Kümmerer [HKK98, Kös00, Hel01] indicate that (at least, when $\mathcal{B}$ is a von Neumann algebra with a faithful normal state) for unitary cocycles U also the converse is true.

In the sequel, we identify a constant function in some $\mathcal{L}^\infty$–space with its constant value. It should be clear from the context, whether we refer to the constant function or its value.

See Example 6.1.5 for the time shift automorphism group $\mathcal{S}$ and Definition 10.3.1 for cocycles. The proof of the following theorem is like in [KS92]. We just do not require that the cocycle is unitary.

16.6.1 Theorem. *Let $\mathfrak{T} \in \mathcal{B}^a(F)$, $\xi, \zeta \in F$, and $\jmath \in \mathcal{B}$. Then the solution of*

$$dU = U\big(dp(\mathfrak{T}) + d\ell^*(\xi) + d\ell(\zeta) + \jmath\,dt\big) \qquad U_0 = \mathbf{1} \qquad (16.6.1)$$

is an adapted left cocycle for the time shift $\mathcal{S}$.

Proof. Thanks to the *stationarity* of the differentials (i.e. the arguments of the integrators do not depend on time) we have the following substitution rule

$$\mathcal{S}_s\Big(\int_0^t A_{t'}\big(dp_{t'}(\mathfrak{T}) + d\ell^*_{t'}(\xi) + d\ell_{t'}(\zeta) + \jmath\,dt'\big)B_{t'}\Big)$$
$$= \int_s^{s+t} \mathcal{S}_s(A_{t'-s})\big(dp_{t'}(\mathfrak{T}) + d\ell^*_{t'}(\xi) + d\ell_{t'}(\zeta) + \jmath\,dt'\big)\mathcal{S}_s(B_{t'-s})$$

which is easily verified by looking at the definitions of the integrals in Theorem 16.1.8. We insert this for U_t and find

$$U_s \mathcal{S}_s(U_t) = U_s + \int_s^{s+t} U_s \mathcal{S}_s(U_{t'-s}) \big(dp_{t'}(\mathfrak{T}) + d\ell^*_{t'}(\xi) + d\ell_{t'}(\zeta) + \jmath\, dt' \big). \tag{16.6.2}$$

In other words, the process $U'_t = U_s \mathcal{S}_s(U_{t-s})$ fulfills for $t \in [s, \infty)$ the same differential equation as U_t with the same initial condition $U'_s = U_s$, i.e. $U_t = U'_t$ for $t \geq s$. ■

Notice that the initial condiditon $U_0 = \mathbf{1}$ (or at least a condition like $U_s \mathcal{S}_s(U_0) = U_s$ for all s) is indispensable. Otherwise, the first summand in (16.6.2) was $U_s \mathcal{S}_s(U_0)$ so that we gain the wrong initial value.

16.7 Dilations

In this section we construct unital dilations $(\mathcal{F}, \mathcal{S}^U, \mathsf{id}, \omega)$ (see Definition 10.5.1) of a uniformly continuous unital CP-semigroup T on a suitably chosen Fock module $\mathcal{F} = \mathcal{F}(E_\infty)$. Here id means the canonical (unital) embedding of $\mathcal{B}$ into the operators on the $\mathcal{B}$–$\mathcal{B}$–module $\mathcal{F}$. We find the dilating E_0–semigroup by *perturbing* the time shift $\mathcal{S}$ (which leaves invariant $\mathcal{B}$) by an adapted unitary cocycle U. In other words, we dilate T to the automorphism semigroup $\mathcal{S}^U$ (which, of course, may be extended to an automorphism group). As usual, U is the solution of a differential equation. Because U is adapted also to $E_{\mathbb{R}_+}$, we may restrict as in Example 14.1.4 $\mathcal{S}^U$ to an E_0–semigroup on $\mathcal{B}^a(\mathcal{F}(E_{\mathbb{R}_+})$. This is more similar to the approach in [HP84].

Let $T = \big(T_t\big)_{t \in \mathbb{R}_+}$ be a unital CP-semigroup on a unital C^*–algebra $\mathcal{B}$ which is uniformly continuous or, equivalently, which has a bounded CE-generator $\mathcal{L}$, i.e. $\mathcal{L}$ has the form (5.4.3). (If $\mathcal{B}$ is a von Neumann algebra and T is normal, then this is automatic. Otherwise, we always achieve this by passing to the enveloping von Neumann algebra $\mathcal{B}^{**}$. See Appendix A.6 for these results obtained by Christensen and Evans [CE79].) Since T is unital, $\mathcal{L}(\mathbf{1}) = 0$ and $\mathcal{L}$ can be written as in (16.7.1).

16.7.1 Theorem. *Let $T = \big(T_t\big)_{t \in \mathbb{R}_+}$ be a unital CP-semigroup on a unital C^*–algebra $\mathcal{B}$ with bounded generator $\mathcal{L}$ of Christensen-Evans form, i.e. there is a Hilbert $\mathcal{B}$–$\mathcal{B}$–module F with a cyclic vector $\zeta \in F$, and a self-adjoint element $h \in \mathcal{B}$ such that*

$$\mathcal{L}(b) = \langle \zeta, b\zeta \rangle - \frac{b\langle \zeta, \zeta \rangle + \langle \zeta, \zeta \rangle b}{2} + i[h, b]. \tag{16.7.1}$$

Let w be a unitary in $\mathcal{B}^a(F)$. Let U be the adapted unitary left cocycle obtained as the unique solution of the differential equation

$$dU = U\big(dp(w - \mathbf{1}) + d\ell^*(w\zeta) - d\ell(\zeta) + (ih - \tfrac{1}{2}\langle \zeta, \zeta \rangle)\, dt \big) \qquad U_0 = \mathbf{1}. \tag{16.7.2}$$

Then $\mathcal{S}^U$ is a dilation of T, i.e. $T_t = \mathbb{E}_0 \circ \mathcal{S}^U_t \upharpoonright \mathcal{B}$.

Conversely, if F is a Hilbert $\mathcal{B}$–$\mathcal{B}$–module and $\zeta \in F$, $h \in \mathcal{B}$, then by setting $T_t = \mathbb{E}_0 \circ \mathcal{S}^U_t \upharpoonright \mathcal{B}$, where U is the adapted unitary left cocycle fulfilling (16.7.2)*, we define a uniformly continuous unital CP-semigroup T whose generator $\mathcal{L}$ is given by* (16.7.1).

Proof. It is enough to show that for U given by (16.7.2) the family $T_t(b) = \mathbb{E}_0 \circ \mathcal{S}^U_t(b)$ fulfills $T'_t(b) = T_t \circ \mathcal{L}(b)$.

As $\mathcal{S}$ leaves invariant $\mathcal{B}$, we have $\mathcal{S}^U_t(b) = U_t b U^*_t$. Applying, for fixed $b \in \mathcal{B}$, the Ito formula to this product of integrals, we find

$$\begin{aligned} d\mathcal{S}^U(b) =& dU\, bU^* + Ub\, dU^* + dU\, b\, dU^* \\ =& U\Big(\big(dp(w-\mathbf{1}) + d\ell^*(w\zeta) - d\ell(\zeta) + (ih - \tfrac{1}{2}\langle\zeta,\zeta\rangle)\, dt\big)b \\ &+ b\big(dp(w^*-\mathbf{1}) - d\ell^*(\zeta) + d\ell(w\zeta) - (ih + \tfrac{1}{2}\langle\zeta,\zeta\rangle)\, dt\big) \\ &+ dp((w-\mathbf{1})b(w^*-\mathbf{1})) - d\ell^*((w-\mathbf{1})b\zeta) - d\ell((w-\mathbf{1})b^*\zeta) + \langle\zeta, b\zeta\rangle\, dt\Big)U^*. \end{aligned}$$

By Lemma 6.3.5 in all summands containing dp or $d\ell^*$ we may replace U on the left by $\widehat{\ell^*}(U\omega)$ and in all summands containing dp or $d\ell$ we may replace U^* on the right by $\widehat{\ell}(U\omega)$. It follows that applying the vacuum conditional expectation only the time differentials survive. As $\mathbb{E}_0 \colon \mathcal{B}^a(\mathcal{F}) \to \mathcal{B}$ is continuous in the $*$–strong topology on $\mathcal{B}^a(\mathcal{F})$ and the uniform topology on $\mathcal{B}$, it follows that

$$\begin{aligned} T_t(b) - b &= \int_0^t \mathbb{E}_0\Big(U_s\Big(\big((ih - \tfrac{1}{2}\langle\zeta,\zeta\rangle)b - b(ih + \tfrac{1}{2}\langle\zeta,\zeta\rangle) + \langle\zeta, b\zeta\rangle\big)\, ds\Big)U^*_s\Big) \\ &= \int_0^t \mathbb{E}_0\big(U_s\mathcal{L}(b)U^*_s\, ds\big) = \int_0^t T_s \circ \mathcal{L}(b)\, ds. \blacksquare \end{aligned}$$

Chapter 17

Restrictions and extensions

17.1 The case $\mathcal{B} = \mathcal{B}(G)$

Let T be a normal unital CP-semigroup T on a von Neumann algebra $\mathcal{B} \subset \mathcal{B}(G)$ with bounded generator $\mathcal{L}$ (which is σ–weak by Observation 5.4.9). All our operators in the calculus extend to the strong closure of the Fock module (and it plays no role, if we close the one-particle sector first) and we find a normal strongly continuous dilation of T on this closure.

Now since we know that our results extend to von Neumann modules, we restrict our attention to the special case $\mathcal{B} = \mathcal{B}(G)$. By Example 6.1.6 the Stinespring representation (see Definition 2.3.6) is an isomophism between the von Neumann algebras $\mathcal{B}^a(\mathcal{F}^s)$ and $\mathcal{B}(G \bar{\otimes} \mathcal{F}(\mathfrak{H})$ where $\mathfrak{H}$ is the center of F ($\cong \mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$ by Example 3.3.4). Applying Example 4.1.15 to the cyclic vector ζ, we find

$$\zeta = \sum_{i \in \mathcal{I}} b_i \otimes e_i$$

with unique coefficients $b_i = \langle \mathbf{1} \otimes e_i, \zeta \rangle \in \mathcal{B}(G)$. We recover the well-known Lindblad form

$$\mathcal{L}(b) = \sum_{i \in \mathcal{I}} b_i^* b b_i - \frac{b \sum_{i \in \mathcal{I}} b_i^* b_i + \sum_{i \in \mathcal{I}} b_i^* b_i b}{2} + i[h, b] \tag{17.1.1}$$

of the generator [Lin76]. Also the unitary operator w appearing in the differential equation (16.7.2) can be expanded according to the basis. We find a matrix $\big(b_{ij}\big)_{i,j \in \mathcal{I}}$ of elements in $\mathcal{B}(G)$ such that $w(\mathbf{1} \otimes e_i) = \sum_{j \in \mathcal{I}} b_{ij} \otimes e_j$. Expressing all ingredients of (16.7.2) in this way, we find an expansion of our integrators into “basic” integrators $dp(e_i e_j^*)$, $d\ell^*(e_i)$, and $d\ell(e_i)$ as used in [MS90, Par92] in the *calculus on the symmetric Fock space with arbitrary degree* *of freedom.* The Mohari-Sinha regularity conditions

$$\sum_{i\in\mathcal{I}} b_i^* b_i < \infty \quad \text{and} \quad \sum_{i\in\mathcal{I}} b_{ij}^* b_{ij} < \infty \text{ for all } j \in \mathcal{I}$$

mean just that ζ is a well defined in $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$ and that w is a well-defined operator at least on the $\mathcal{B}(G)$–linear span of all $\mathbf{1} \otimes e_i$. If the constant in the above condition for b_{ij} does not depend on j then w is a bounded operator on $\mathcal{B}(G, G \bar{\otimes} \mathfrak{H})$.

In other words, for $\mathcal{B} = \mathcal{B}(G)$ our calculus can be interpreted as a calculus with arbitrary (even uncountable) degree of freedom on the tensor product of the initial space G and the full Fock space $\mathcal{F}(L^2(\mathbb{R}) \bar{\otimes} \mathfrak{H})$. In [Spe91] only the case $\mathfrak{H} = \mathbb{C}$ is treated, which corresponds to one degree of freedom. Let us compare. Although we follow in many respects directly the ideas in [KS92, Spe91], we can say that our calculus is both formally simpler and more general. It is formally simpler, because our differential equation for U contains no coefficients. (Of course, the coefficients are hidden in the much more general arguments of the integrators.) And our calculus is more general, because it allows to find dilations for arbitrary CE-generators on arbitrary C^*–algebras $\mathcal{B}$. As a special case we showed in this section how the calculus for an arbitrary Lindblad generator is contained, which on a Fock space — symmetric or full — requires a calculus with arbitrary degree of freedom.

Recently, in [GS99] a calculus on the *symmetric Fock* $\mathcal{B}(G)$*–*$\mathcal{B}(G)$*–module*

$$\Gamma^s(\mathcal{B}(G) \bar{\otimes}^s L^2(\mathbb{R}_+, \mathfrak{H}) = \mathcal{B}\big(G, G \bar{\otimes} \Gamma(L^2(\mathbb{R}_+, \mathfrak{H}))\big)$$

has been constructed. This calculus allowed for the first time to dilate an arbitrary CE-generator on a von Neumann algebra (and also the construction of Evans-Hudson flows, which we do not consider at all). The construction of the one-particle sector in [GS99] is, however, less canonical in the following sense. The completely positive part of the generator $\mathcal{L}$ gives rise only to a $\mathcal{B}$–$\mathcal{B}$–module F (see the proof of Theorem A.6.3). Before finding the $\mathcal{B}(G)$–$\mathcal{B}(G)$–module $\mathcal{B}(G) \bar{\otimes}^s \mathfrak{H}$, from which the symmetric Fock module can be constructed, it is necessary to extend the module structure from $\mathcal{B}$ (which is rarely centered) to $\mathcal{B}(G)$ (which is always centered). Also the techniques in [GS99] refer more to Hilbert spaces, which do not play a role in our treatment.

17.2 Boolean calculus

There are several possibilities to translate the concept of *independence* from classical (or commutative) probability to quantum (or non-commutative) probability. The minimal requirement for a notion of non-commutative independence is probably that used by Kümmerer [Küm85], where (speaking about unital $*$–algebras and states instead of von Neumann algebras and faithful normal states) two (unital) $*$–subalgebras $\mathcal{A}_i$ $(i = 1, 2)$ of a $*$–algebra $\mathcal{A}$ are independent in a state φ on $\mathcal{A}$, if $\varphi(a_j a_k) = \varphi(a_j)\varphi(a_j)$ $(k \neq j; a_i \in \mathcal{A}_i; i = 1, 2)$.

A more specific notion of non-commutative independence as introduced in [Sch95] requires that the values of φ on alternating monomials in $\mathcal{A}_1$ and $\mathcal{A}_2$ may be obtained from a *universal product* $\varphi_1\varphi_2$ of the restrictions $\varphi_i \upharpoonright \mathcal{A}_i$, where a universal product is a state on the *free product* (with identification of units) $\mathcal{A}_1 * \mathcal{A}_2$ (i.e. the *coproduct* of $\mathcal{A}_1$ and $\mathcal{A}_2$ in the category of unital $*$–algebras) fulfilling conditions like associativity and functoriality (i.e. the construction *commutes* with unital $*$–homomorphisms). The conditions are motivated by the fact that, when interpreted classically (i.e. in the context of commutative unital $*$–algebras) there is only one such universal product, namely, the tensor product of φ_1 and φ_2.

In the non-commutative context, besides the tensor product (corresponding to *tensor independence*), we have the *free product* of states which corresponds to *free independence* introduced by Voiculescu [Voi87]. Speicher [Spe97] showed that under stronger (from the combinatorial point of view very natural) assumptions there are only those two universal products. In [BGS99] Ben Ghorbal and Schürmann show how the original set-up from [Sch95] can be reduced to [Spe97].

Allowing for non-unital $*$–algebras, there is a third universal product, namely, the *boolean product* introduced by von Waldenfels [Wal73] which corresponds to *boolean independence*. (Actually, there is a whole family of such products labelled by a scaling parameter; see [BGS99]. We consider only the simplest choice of this parameter.) The boolean product is in some sense the simplest possible product, as it just factorizes on alternating monomials, i.e. the boolean product sends a monomial $a_1 a_2 \ldots$ where two neighbours are from different algebras just to the product $\varphi_j(a_1)\varphi_k(a_2)\ldots$ where a_i must be evaluated in the appropriate state, i.e. $j = 1$ for $a_1 \in \mathcal{A}_1$ and $j = 2$ for $a_1 \in \mathcal{A}_2$, and so on.

Each type of independence has its own type of Fock space which is suggested by the GNS-construction for the respective product states; see [Sch95] for details. For tensor indepence this is the symmetric Fock space. (This is mirrored by the well-known factorization $\Gamma(H_1 \oplus H_2) = \Gamma(H_1) \bar{\otimes} \Gamma(H_2)$.) For free independence this is the full Fock space. (This is mirrored by the fact that $\mathcal{F}(H_1 \oplus H_2)$ is the *free product* of $\mathcal{F}(H_1)$ and $\mathcal{F}(H_2)$ with their respective vaccua as reference vector.) The **boolean Fock space** over H is just $\mathcal{F}_b(H) = \mathbb{C}\Omega \oplus H$. (Here the composition law is just the direct sum of the one-particle sectors. We may view this as a direct sum of $\mathcal{F}_b(H_i) = \mathbb{C}\Omega_i \oplus H_i$ $(i = 1, 2)$ with *identification of the reference vectors* Ω_i.)

The primary goal of this section is to discover a calculus on the boolean Fock space. Like for the symmetric and the full Fock space the solution of a differential equation like (16.6.1) should be a process with stationary boolean independent multiplicative increments;

cf. Section 16.6. The way we find this calculus is to assign to a Hilbert space H (i.e. a Hilbert $\mathbb{C}$–module) a suitable $\widetilde{\mathbb{C}}$–$\widetilde{\mathbb{C}}$–module structure (where $\widetilde{\mathbb{C}}$ denotes the *unitization* of the unital C^*–algebra $\mathbb{C}$). Then the full Fock module over this $\widetilde{\mathbb{C}}$–module turns out to be (up to one vector) the boolean Fock space. However, for a couple of reasons we find it convenient to start from the beginning with amalgamated versions. Firstly, the C^*–algebra $\widetilde{\mathbb{C}}$ is a source of continuous confusion of the several different copies of $\mathbb{C}$ which appear in this context. Secondly, the examples without amalgamation are rather poor and can easily be computed by hand. Last but not least, we classify the uniformly continuous contractive, but, not necessarily unital CP-semigropus T on a unital C^*–algebra $\mathcal{B}$ which may be dilated with the help of an almagamated boolean calculus, as those which are of the form $T_t(b) = e^{t\jmath^*}be^{t\jmath}$ for suitable $\jmath \in \mathcal{B}$.

As usual, $\mathcal{B}$ is a unital C^*–algebra. Let E be a Hilbert $\mathcal{B}$–module. We equip E with the structure of a Hilbert $\widetilde{\mathcal{B}}$–$\widetilde{\mathcal{B}}$–module as in Example 1.6.5. By Example 4.2.14 we have $E \odot E = \{0\}$ so that

$$\mathcal{F}(E) \;=\; \widetilde{\mathcal{B}} \oplus E.$$

On $\mathcal{F}(E)$ we may define the central projection $q = \mathbf{1}^r \colon x \mapsto x\mathbf{1}$. The range of q is the Hilbert $\mathcal{B}$–module $\mathcal{F}_b(E) = \mathcal{B} \oplus E$; cf. also Section 12.3. Its orthogonal complement is the one-dimensional subspace spanned by the element $\widetilde{\mathbf{1}} - \mathbf{1}$ of $\widetilde{\mathcal{B}} \subset \mathcal{F}(E)$. We may think of $\mathcal{F}_b(E)$ as the ***boolean Fock module*** over E. (This may be justified by giving a formal definition of *boolean independence with amalgamation over* $\mathcal{B}$ paralleling that of Voiculescu [Voi95] for free independence and that of Skeide [Ske96, Ske99a] for tensor independence.)

17.2.1 Proposition. *q is a central projection in $\mathcal{B}^a(\mathcal{F}(E))$. Moreover, $q\mathcal{B}^a(\mathcal{F}(E))$ is an ideal in $\mathcal{B}^a(\mathcal{F}(E))$ which is isomorphic to $\mathcal{B}^a(\mathcal{F}_b(E))$ and has codimension 1. Consequently, $\mathcal{B}^a(\mathcal{F}(E)) \cong \mathcal{B}^a(\mathcal{F}_b(E))^\sim$. In other words,*

$$\mathcal{B}^a(\mathcal{F}(E)) = \mathbb{C}(\mathbf{1}_\mathcal{F} - q) \oplus \mathcal{B}^a(\mathcal{F}_b(E)) = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathcal{B}^a(\mathcal{F}_b(E)) \end{pmatrix}$$

acting on $\mathcal{F}(E) = \left(\begin{smallmatrix} \mathbb{C}(\widetilde{\mathbf{1}}-\mathbf{1}) \\ \mathcal{F}_b(E) \end{smallmatrix}\right)$, where $\mathbf{1}_\mathcal{F}$ denotes the unit in $\mathcal{B}^a(\mathcal{F}(E))$.

Proof. Let $a \in \mathcal{B}^a(\mathcal{F}(E))$. Then $qax = (ax)\mathbf{1} = a(x\mathbf{1}) = aqx$. From this the remaining statements are obvious. ■

As $\mathcal{F}_b(E) = \left(\begin{smallmatrix} \mathcal{B} \\ E \end{smallmatrix}\right)$, we may decompose also $\mathcal{B}^a(\mathcal{F}_b(E)) = \left(\begin{smallmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{smallmatrix}\right)$. We find

$$\mathcal{B}^a(\mathcal{F}(E)) = \begin{pmatrix} \mathbb{C} & 0 & 0 \\ 0 & \mathcal{B} & E^* \\ 0 & E & \mathcal{B}^a(E) \end{pmatrix} \quad \text{acting on} \quad \mathcal{F}(E) = \begin{pmatrix} \mathbb{C}(\widetilde{\mathbf{1}} - \mathbf{1}) \\ \mathcal{B} \\ E \end{pmatrix}. \tag{17.2.1}$$

Now let E and F be Hilbert $\mathcal{B}$–modules both equipped with the Hilbert $\widetilde{\mathcal{B}}$–$\widetilde{\mathcal{B}}$–module structure as described above. Then

$$\mathcal{F}(E \oplus F) = \mathcal{F}(E) \odot (\widetilde{\mathcal{B}} \oplus F \odot \mathcal{F}(E \oplus F)) = (\widetilde{\mathcal{B}} \oplus E) \odot (\widetilde{\mathcal{B}} \oplus F) = \mathcal{F}(E) \odot \mathcal{F}(F)$$

(which, of course, equals $\widetilde{\mathcal{B}} \oplus E \oplus F$ as $E \odot F = \{0\}$).

17.2.2 Proposition. *$q \in \mathcal{B}^a(\mathcal{F}(E \oplus F))$ is not adapted to E.*

Proof. Let $X = (1, 0, 0) \in \mathcal{F}(E)$ and $Y = (1, 0, y) \in \mathcal{F}(F)$ $(y \neq 0)$ as in (17.2.1). Then

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \odot \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix}$$

in $\mathcal{F}(E \oplus F)$. Applying q to this vector, we obtain $(0, 0, y)$. However, as

$$\begin{pmatrix} \beta \\ b \\ x \end{pmatrix} \odot \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} \beta \\ 0 \\ \beta y + x \end{pmatrix},$$

there is no vector $X' \in \mathcal{F}(E)$ such that $X' \odot Y = q(X \odot Y)$. *A fortiori* there is no operator a on $\mathcal{F}(E)$ such that $q = a \odot \mathsf{id}$. ■

This property makes the definition of adaptedness to E of operators on $\mathcal{F}_b(E \oplus F)$ a little bit delicate. In [Ske00d] we proposed to say an operator $a \in \mathcal{B}^a(\mathcal{F}_b(E \oplus F))$ is adapted to E, if it is adapted to E in $\mathcal{B}^a(\mathcal{F}(E \oplus F))$. (Then $q = \mathbf{1}_{\mathcal{F}_b(E \oplus F)}$ is not adapted, whereas, all the operators $p(T)$ $(T \in \mathcal{B}^a(E))$, $\ell^*(x)$ $(x \in E)$ and, consequently also $\ell(x)$ would be adapted in this sense.) We owe thanks to U. Franz (private communication) for the suggestion of the following definition. We say an operator $\mathcal{B}^a(\mathcal{F}_b(E \oplus F))$ is ***adapted to*** E, if it can be written as qa for an operator $a \in \mathcal{B}^a(\mathcal{F}(E \oplus F))$ adapted to E. Now $q = q\mathbf{1}_{\mathcal{F}}$ is adapted. Clearly, a unitary u on $\mathcal{F}(E \oplus F)$ gives rise to a unitary qu on $\mathcal{F}_b(E \oplus F)$. This assures that our calculus, including all statements concerning unitarity conditions, reduces properly to the boolean Fock module.

Now we want to see how operators in $\mathcal{B}^a(\mathcal{F}_b(E \oplus F))$ adapted to E in $\mathcal{B}^a(\mathcal{F}(E \oplus F))$ look like and how an arbitrary operator adapted in $\mathcal{B}^a(\mathcal{F}_b(E \oplus F))$ to E is related to the former. Denote by p_E, p_F the projections onto $E, F \subset \mathcal{F}_b(E \oplus F) \subset \mathcal{F}(E \oplus F)$. Clearly, $p_E = p_E \odot \mathsf{id}$ on $\mathcal{F}(E \oplus F) = \mathcal{F}(E) \odot \mathcal{F}(F)$ so that p_E is adapted to E in $\mathcal{B}^a(\mathcal{F}(E \oplus F))$ and $qp_E = p_E$. Denote by $\widetilde{\omega}$ the vacuum of $\mathcal{F}(E \oplus F)$ and by ω that of $\mathcal{F}_b(E \oplus F)$ and define $\Omega = \widetilde{\omega} - \omega$. Observe that $\widetilde{\omega}\widetilde{\omega}^* - \omega\omega^* = \Omega\Omega^* = \mathbf{1}_{\mathcal{F}} - q$. Obviously, $\omega\omega^* \odot \mathsf{id} = \omega\omega^* \odot \omega\omega^* = \omega\omega^*$ so that also the vacuum projection $\omega\omega^*$ on $\mathcal{F}_b(E \oplus F)$ is adapted to E both in $\mathcal{B}^a(\mathcal{F}(E \oplus F))$ and in $\mathcal{B}^a(\mathcal{F}_b(E \oplus F))$. It follows that $\Omega\Omega^* + p_F = \mathbf{1}_{\mathcal{F}} - \omega\omega^* - p_E$ is adapted to E in $\mathcal{B}^a(\mathcal{F}(E \oplus F))$.

An operator on $\mathcal{F}_b(E \oplus F)$ adapted to E differs from an operator a on $\mathcal{F}(E \oplus F)$ adapted to E just by the component $\Omega\Omega^* a = \Omega\Omega^* \alpha$ $(\alpha = \langle \Omega, a\Omega \rangle)$ of a along $\Omega\Omega^*$. It follows that

$a-(\Omega\Omega^*+p_F)\alpha \in \mathcal{B}^a(\mathcal{F}_b(E\oplus F))$ is adapted to E in $\mathcal{B}^a(\mathcal{F}(E\oplus F))$. In other words, each operator $qa \in \mathcal{B}^a(\mathcal{F}_b(E\oplus F))$ with a adatped to E in $\mathcal{B}^a(\mathcal{F}(E\oplus F))$ can be written as a sum of the operator $a-(\Omega\Omega^*+p_F)\alpha \in \mathcal{B}^a(\mathcal{F}_b(E\oplus F))$ is adapted to E in $\mathcal{B}^a(\mathcal{F}(E\oplus F))$ and $p_F\alpha \in \mathcal{B}^a(\mathcal{F}_b(E\oplus F))$. The component α is unique, because $qa = qa'$ implies $a-a' \in \mathbb{C}\Omega\Omega^*$ and the only adapted element in this space is 0. One may verify that $qa \mapsto \alpha$ defines a character. Therefore, the operators in $\mathcal{B}^a(\mathcal{F}_b(E\oplus F))$ adapted to E both in $\mathcal{B}^a(\mathcal{F}(E\oplus F))$ and in $\mathcal{B}^a(\mathcal{F}_b(E\oplus F))$ form an ideal of codimension 1 in those operators adapted to E only in $\mathcal{B}^a(\mathcal{F}_b(E\oplus F))$.

17.2.3 Theorem. *An operator in $\mathcal{B}^a(\mathcal{F}_b(E\oplus F))$ is adapted to E in $\mathcal{B}^a(\mathcal{F}_b(E\oplus F))$ is the sum of a unique operator in $\mathcal{B}^a(\mathcal{F}_b(E))$ exended by 0 to $F \subset \mathcal{F}_b(E\oplus F)$ and a unique multiple of p_F.*

PROOF. It remains to show the statement for an operator $a \in \mathcal{B}^a(\mathcal{F}_b(E\oplus F))$ adapted to E in $\mathcal{B}^a(\mathcal{F}(E\oplus F))$ so that the second summand is zero. One easily checks that the subspace F of $\mathcal{F}_b(E\oplus F) \subset \mathcal{F}(E\oplus F)$ consists of all elements of the form $(\widetilde{\omega}-\omega)\odot y$. Then for $a' \in \mathcal{F}_b(E)$ we have $(a'\odot \mathsf{id})((\widetilde{\omega}-\omega)\odot y) = 0$, because $a'(\widetilde{\omega}-\omega) = 0$. Conversely, if $ay = 0$ for all $y \in F$, then a restricts to an operator a' on $\mathcal{F}_b(E)$ such that $a'\odot \mathsf{id} = a$, as before. ■

Now we concentrate on $\mathcal{F} = \mathcal{F}(L^2(\mathbb{R},E))$. We remark that it does not matter, whether we first construct $L^2(\mathbb{R},E))$ for the Hilbert $\mathcal{B}$–module E and then turn it into a Hilbert $\widetilde{\mathcal{B}}$–$\widetilde{\mathcal{B}}$–module, or conversely. (Of course, E contains not one non-zero element commuting with any non-zero element of $\widetilde{\mathcal{B}}$. Thus, E is extremely uncentered.)

The truncated structure of our Fock module or, what is the same, the trivial action of $\mathcal{B}$ on E reduces the possibilities for integrals. In a creation integral $\int F\,d\ell^*(x)\,G$ only the component of F along $\mathbf{1}_{\mathcal{F}}-q$ contributes. Absorbing the numerical time dependence of the multiple of $\mathbf{1}_{\mathcal{F}}-q$ into G, we may replace F by $\mathbf{1}_{\mathcal{F}}$. The opposite statement is true for annihilation integrals. Particularly boring are conservation integrals where only integrals of the form $\int_\tau^{\mathfrak{T}} f(t)\,dp_t(T)$ with a numerical function $f \in \mathcal{C}(\mathbb{R})$ survive. Considering f as multiplication operator on E_∞, we just obtain $p(fT\mathbb{I}_{[\tau,\mathfrak{T}]}) = \int_\tau^{\mathfrak{T}} dp_t(fT)$. This means that in all non-zero places of the Ito table the processes G and F', which are "sandwiched" between the differentials, dissappear.

On the remaining sides of the integrators we may insert the vacuum projection $\omega\omega^*$ without changing the value of the integral. Thus, we have $\int d\ell^*(x)\,G = \int d\ell^*(x)\,\widehat{\ell}(G^*\omega)$, $\int F\,d\ell(x) = \int \widehat{\ell}^*(F\omega)\,d\ell(x)$ and $\int F\,d\mu^{x,y}\,G = \int \widehat{\ell}^*(F\omega)\,d\mu^{x,y}\,\widehat{\ell}(G^*\omega)$.

Let U be a left adapted cocycle obtained as solution of the differential equation as in

Theorem 16.6.1 (with $\jmath \in \mathcal{B}$, not in $\widetilde{\mathcal{B}}$). We write this in integral form and obtain

$$U_t = \mathbf{1}_{\mathcal{F}} + \int_0^t U_s\bigl(dp_s(\mathtt{T}) + d\ell_s^*(\xi) + d\ell_s(\zeta) + \jmath\, ds\bigr). \tag{17.2.2}$$

Let us multiply (17.2.2) by $\mathbf{1} \in \mathcal{B} \subset \mathcal{B}^a(\mathcal{F})$ from both sides. (Due to the trivial action of $\mathcal{B}$ this corresponds more or less to the vaccum conditional expectation on the boolean Fock module. Thus, the result may be interpreted as an element of $\mathcal{B}$.) Then all integrals except the time integral cancel. We obtain

$$\mathbf{1}U_t\mathbf{1} = \mathbf{1} + \int_0^t \mathbf{1}U_s\mathbf{1}\jmath\, ds.$$

In other words, setting $b_t = \mathbf{1}U_t\mathbf{1} \in \mathcal{B}$, we find $b_t = e^{t\jmath}$. This means that we obtain a CP-semigroup T of the very special form

$$T_t(b) = \langle \omega, U_t b U_t^{-1}\omega\rangle = b_t b b_t^*.$$

(Notice that we did not even require U_t to be unitary.)

Conversely, let E be a Hilbert $\mathcal{B}$–module equipped with the usual Hilbert $\widetilde{\mathcal{B}}$–$\widetilde{\mathcal{B}}$–module structure. If $T_t(b) = \langle \zeta_t, b\zeta_t\rangle$ ($\zeta_t \in \mathcal{F}$) defines a semigroup on $\mathcal{B}$, then for the components $b_t^* = \mathbf{1}\zeta_t \in \mathcal{B}$ of ζ_t, we necessarily have $b_s b_t b b_t^* b_s^* = b_{s+t} b b_{s+t}^*$ for all $b \in \mathcal{B}; s,t \in \mathbb{R}_+$. Of course, this does not necessarily mean that $b_s b_t = b_{s+t}$. Constructing the GNS-system of T we may show that the b_t can be chosen accordingly. Together with the assumption that T has a bounded generator we arrive at the same conclusion.

And yet another way to look at it is to start with a uniformly continuous unital CP-semigroup $\widetilde{T}$ on $\widetilde{\mathcal{B}}$. Then the GNS-module E of the part $\langle \zeta, \bullet\zeta\rangle$ of the generator $\mathcal{L}$ has the desired $\widetilde{\mathcal{B}}$–$\widetilde{\mathcal{B}}$–module structure, if and only if $\langle \zeta, b\zeta\rangle = 0$ for all $b \in \mathcal{B}$. In this case, the restriction of $\mathcal{L}$ to $\mathcal{B}$ has the form $\mathcal{L}(b) = \jmath b + b\jmath^*$ where $\mathsf{Re}\,\jmath = -\frac{1}{2}\langle \zeta, \zeta\rangle \in \mathcal{B}$ and $\mathsf{Im}\,\jmath = \mathbf{1}h \in \mathcal{B}$. Once again, one verifies directly by differentiation that $T_t(b) = e^{t\jmath} b e^{t\jmath^*}$ has this generator. Additionally, we see that $\widetilde{T}$ is the unital extension of a contractive uniformly continuous CP-semigroup T from $\mathcal{B}$ to $\widetilde{T}$.

Contractive CP-semigroups on $\mathbb{C}$ have the form $T_t(z) = e^{-tc}z$ ($c \geq 0$). We discuss this special case and its truncated Fock $\widetilde{\mathbb{C}}$–$\widetilde{\mathbb{C}}$–module $L^2(\mathbb{R}_+) \oplus \widetilde{\mathbf{1}}$ at length in Example 12.3.7.

17.3 More general 0–criteria

The main goal of this section is to extend Theorem 16.4.4 to expressions with more than one summand for each type of integral. The trick is the same as in Theorem 13.1.2 Section 13.5 where we tensorize all modules with M_n in order to write n summands as a single expression.

As explained for the time ordered Fock module in Section 13.5 we have

$$\mathcal{F}(L^2(\mathbb{R}, M_n \otimes F)) \;=\; M_n \otimes \mathcal{F}(L^2(\mathbb{R}, F)).$$

We use this to generalize Theorem 16.4.4 to finitely many summands. We do not discuss this in any detail, but explain only the general idea. The resulting criteria are similar to their analogues in Speicher [Spe91].

Let A_i, B_i $(i = 1, \ldots, n)$ be processes in $\mathfrak{A}_1^0$. Let I_i $(i = 1, \ldots, n)$ be integrators of the same type. Define the (adapted for $t \geq 0$) processes $\widehat{A} = \big(A_j\delta_{1j}\big)_{ij}$ and $\widehat{B} = \big(B_i\delta_{i1}\big)_{ij}$ on $M_n \otimes \mathcal{F}(L^2(\mathbb{R}, F))$ and the integrator function $\widehat{I} = \big(I_i\delta_{ij}\big)_{ij}$. Then $\int \widehat{A}\, d\widehat{I}\, \widehat{B}$ is the (adapted for $t \geq 0$) process on $M_n \otimes \mathcal{F}(L^2(\mathbb{R}, F))$ whose 1–1–entry is $\sum\limits_{i=1}^{n} \int A_i \, dI_i \, B_i$ and which is 0 in any other place. Applying the appropriate criterion from Theorem 16.4.4 we find (as in [Spe91]) a matrix of functions each of which must be 0 separately. These criteria are not very handy, so we dispense with a general formulation.

Part V

Appendix

The appendix consists of several parts. In Appendices A and B we collect mainly preliminary results, which are more less well-known, but maybe cannot be found in literature in this condensed form, adapted precisely to our needs. Whereas, Appendices C and D contain original material.

Appendix A contains preliminary results about semi-norms, (pre–)C^*–algebras and normal mappings on von Neumann algebras. Of course, it is not our goal to repeat all basic facts which we are using throughout these notes. As far as we are concerned with standard results in C^*–algebra theory and elementary fact about von Neumann algebras, we refer the reader to a text book, for instance, like Murphy [Mur90]. Most of these facts and some results about normal mappings can be found also in the fairly self-contained appendix of Meyer's book [Mey93]. (This appendix meets also our intention to use only elementary methods available to non-experts.) Of course, any other text book like Sakai [Sak71] or [Tak79] serves as reference. We concentrate rather on emphasizing those results which turn over from C^*–algebras to pre–C^*–algebras and those which need a revision before we may use them. Some original ideas (basically, extensions from Kümmerer and Speicher [KS92]) about graded Banach spaces, which we need for Part IV, are collected in Appendix A.3.

In Appendix B we collect the function spaces (in particular, those with values in Banach spaces) used throughout and state their basic properties. Appendix B.1 gives a fairly complete account of what we need to know about functions on measure spaces. Certainly, everything here is standard, but spread over literature. As it is tedious to collect everthing from books like Diestel and Uhl [DU77], we decided to include it in a compact form. Appendix B.2 collects a few properties of continuous functions on locally compact spaces. In these notes we need mainly functions on $\mathbb{R}$ (or $\mathbb{R}^d$) which are treated in Appendix B.4. The treatement of the *generalized time integral* and some of the extension results may be non-standard. Appendix B.3 introduces the basic lattices we need (one of which is not very usual).

In Appendix C we extend the notion of two-sided Hilbert modules to modules over P^*–algebras ($*$–algebras with an extended, but still algebraic positivity structure) as introduced in Accardi and Skeide [AS98]. We need this in Chapters 8 and 9. In Appendix D we present our results from Skeide [Ske98a] about the full Fock module arising from the stochastic limit of the free electron. This is an important example from physics, but the lengthy computations would interupt the general text where we dicided to limit ourselves to a short description of the emerging structure in short examples.

Appendix A

Miscelaneous preliminaries

A.1 Semi-normed spaces

A.1.1 Definition. A *seminorm* on a vector space V is a mapping $\|\bullet\|: V \to \mathbb{R}$ fulfilling

$$\|x + y\| \leq \|x\| + \|y\| \quad (x, y \in V) \quad \text{and} \quad \|zx\| = |z|\,\|x\| \quad (z \in \mathbb{C}, x \in V).$$

A *norm* is a seminorm for which $\|x\| = 0$ implies $x = 0$. A *(semi-)normed space* is a vector space with a (semi-)norm. By $\mathcal{N}_V$ we denote the subset $\{x \in V : \|x\| = 0\}$ of *length-zero elements* in a seminormed space V. One easily checks that $V/\mathcal{N}_V$ with $\|x + \mathcal{N}_V\| = \|x\|$ is a normed space.

A.1.2 Definition. Let V, W be two seminormed spaces. On the space $\mathcal{L}(V, W)$ of linear mappings $T: V \to W$ we define the function

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$$

with values in $[0, \infty]$. By $\mathcal{B}(V, W) = \{T \in \mathcal{L}(V, W) : \|T\| < \infty\}$ we denote the space of *bounded* linear mappings $V \to W$.

Obviously, $\mathcal{B}(V, W)$ is a seminormed space, and $\mathcal{B}(V, W)$ is normed, if and only if W is normed.

A.1.3 Lemma. *Let $T \in \mathcal{B}(V, W)$. Then $T\mathcal{N}_V \subset \mathcal{N}_W$. In other words, T gives rise to a (unique) bounded operator in $\mathcal{B}(V/\mathcal{N}_V, W/\mathcal{N}_W)$ also denoted by T.*

Proof. We conclude indirectly. Suppose $T \in \mathcal{L}(V, W)$ and x is an element in V such that $\|x\| = 0$, but $\|Tx\| = 1$. Then $\|\lambda x\| = 0$, but $\|T\lambda x\| = \lambda$ so that $\sup_{\|y\| \leq 1} \|Ty\| \geq \sup_{\lambda} \|T\lambda x\| = \infty$. Therefore, T is not an element of $\mathcal{B}(V, W)$. ■

A.1.4 Corollary. $\|T\| = \sup_{\|x\|\leq 1} \|Tx\| = \sup_{\|x\|<1} \|Tx\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|\neq 0} \frac{\|Tx\|}{\|x\|}$.

We mention that $\|ST\| \leq \|S\| \, \|T\|$ also for mappings between seminormed spaces. And if W is a *Banach space* (i.e. a complete normed space), then so is $\mathcal{B}(V, W)$. In this case, each mapping in $T \in \mathcal{B}(V, W)$ gives rise to a (unique) mapping in $\mathcal{B}(\overline{V/\mathcal{N}_V}, W)$.

A.1.5 Remark. If T is an isometric mapping between semiinner product spaces V and W, then it is bounded, in particular. This provides us with a simple and powerful tool to establish well-definedness of isometric mappings $V/\mathcal{N}_V \to W/\mathcal{N}_W$, even if the mapping is not adjointable.

A.1.6 Remark. Let W be a subspace of a normed space V. One easily verifies that $x \mapsto \|x\|_W := \inf_{y \in W} \|x + y\|$ defines a seminorm on V. The space of length-zero elements for this norm is precisely the closure of W in V. Therefore, V/W is a seminormed space with seminorm $\|\bullet\|_W$. It is normed, if and only if W is closed in V. If $V \supset W$ are Banach spaces, then so is V/W. Indeed, suppose that $\left(x_n\right)_{n\in\mathbb{N}}$ is a sequence in V such that $\left(x_n + W\right)$ is a Cauchy sequence in V/W and find a monotone function $k \mapsto N_k$ on $\mathbb{N}$ such that $\|x_n - x_m\|_W < \frac{1}{(k+1)^2}$ for all $n, m \geq N_k$. Set $z_n = x_{N_n} - x_{N_{n-1}}$ for $n \in \mathbb{N}$ with $N_0 = 0$, $x_0 = 0$. For each $n \in \mathbb{N}$ choose $y_n \in z_n + V$ such that $\|y_n\| < \|z_n\|_W + \frac{1}{n^2} < \frac{2}{n^2}$. Then the series $\sum_{n=1}^{\infty} y_n$ converges to some $x \in V$, and $\lim_{n\to\infty} (x_n + W) = x + W$ in V/W, because

$$\|x - x_{N_n}\|_W \;=\; \Big\| \sum_{k=n+1}^{\infty} y_k + \sum_{k=1}^{n} (y_k - z_k) \Big\|_W \;\leq\; \Big\| \sum_{k=n+1}^{\infty} y_k \Big\| \;\to\; 0.$$

A.2 Direct sums of normed spaces

A.2.1 Definition. Let $\left(V^{(t)}\right)_{t\in\mathbb{L}}$ be a family of normed spaces. We say a norm on $V = \bigoplus_{t\in\mathbb{L}} V^{(t)}$ is *admissible*, if its restriction to $V^{(t)}$ coincides with the norm of $V^{(t)}$, and if

$$\left\|x^{(t)}\right\| \leq \|x\| \tag{A.2.1}$$

for each $x = \left(x^{(s)}\right) \in V$ and each $t \in \mathbb{L}$.

A.2.2 Remark. The admissible norms are precisely those for which all *canonical projections* $\mathfrak{p}^{(t)} \colon V \to V^{(t)}, \mathfrak{p}^{(t)}(x) = x^{(t)}$ have norm 1. Define the norm $\|\bullet\|_1$ of the ℓ^1*–direct sum*, by setting

$$\|x\|_1 = \sum_{t\in\mathbb{L}} \left\|x^{(t)}\right\|.$$

Clearly, $\|\bullet\|_1$ is admissible and $\|x\|_1 \geq \|x\|$ for any other admissible norm. By V_1 we denote the ℓ^1*–completion* of V. Obviously, for any admissible norm $\overline{V}$ is contained in V_1.

A.2.3 Proposition. *V is complete, if and only if $\mathbb{L}$ is a finite set and each $V^{(t)}$ is complete.*

PROOF. "$\Rightarrow$". Each Cauchy sequence in $x_n^{(t)}$ in $V^{(t)}$ embeds as a Cauchy sequence into V and the projection onto $V^{(t)}$ is continuous. Therefore, if V is complete, then so is $V^{(t)}$ for all $t \in \mathbb{L}$. On the other hand, if $\#\mathbb{L}$ is not finite, then V is certainly not complete.

"$\Leftarrow$". If $\#\mathbb{L}$ is finite, then we have

$$\frac{\|x\|_1}{\#\mathbb{L}} \leq \|x\| \leq \|x\|_1 .$$

In other words, any admissible norm is equivalent to $\|\bullet\|_1$. A sequence $\big(x_n\big)_{n\in\mathbb{N}}$ in V is a Cauchy sequence in $\|\bullet\|_1$, if and only if each sequence $\big(x_n^{(t)}\big)_{n\in\mathbb{N}}$ in $V^{(t)}$ is a Cauchy sequence. Therefore, for finite $\#\mathbb{L}$ the direct sum V is complete, if and only if all $V^{(t)}$ are complete. ■

A.2.4 Remark. V_1 consists precisely of those families $\big(x^{(t)}\big)$ for which that norms $\|x^{(t)}\|$ are absolutely summable over $\mathbb{L}$. See B.1.17 and cf. also Example 1.5.5.

A.3 Operators on graded Banach spaces

Direct sums as considered in Appendix A.2, in the first place, are vector spaces with an $\mathbb{L}$*–graduation*. Sometimes, if we want to emphasize this graduation, we write V_g for the algebraic direct sum. This is particularly useful in Chapter 16 where we start with a Banach space V which has graded subspace V_g (usually dense in V in some topology). According to our needs in Chapter 16, we consider only $\mathbb{Z}$–graduations and we assume that the *graded subspaces* $V^{(n)}$ of V are Banach spaces. We call the elements of $V^{(n)}$ *homogeneous* of *degree* n. (We do not exclude $0 \in V^{(n)}$ $(n \in \mathbb{Z})$.)

The following lemma differs from a result in [KS92] just by a slightly more general formulation. Together with Lemma A.3.2 it shows us that all limits of bilinear mappings in Chapter 16 have to be computed only when evaluated at homogeneous elements.

A.3.1 Lemma. *Let $\big(V^{(n)}\big)_{n\in\mathbb{Z}}$, $\big(C^{(n)}\big)_{n\in\mathbb{Z}}$, and $\big(D^{(n)}\big)_{n\in\mathbb{Z}}$ be families of Banach spaces. Suppose that $j\colon C_g \times D_g \to V_g$ is an* even *bilinear mapping (i.e. $j(C^{(n)}, D^{(m)}) \subset V^{(n+m)}$ for all $n, m \in \mathbb{Z}$), and that $M > 0$ is a constant such that*

$$\|j(c,d)\|_1 \leq M \|c\|_1 \|d\|_1 \tag{A.3.1}$$

for all homogeneous $c \in C_1$, $d \in D_1$. Then j extends to a (unique) bilinear mapping $C_1 \times D_1 \to V_1$, also denoted by j, such that (A.3.1) *is fulfilled for all $c \in C_1$, $d \in D_1$. (In other words, j is* bounded*.)*

Proof. We show that (A.3.1) extends to arbitrary $c \in C_g$, $d \in D_g$. (Of course, such a mapping j extends by means of continuity to a unique bilinear mapping on $C_1 \times D_1$ also fulfilling (A.3.1).) Indeed,

$$\|j(c,d)\|_1 = \Big\|\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} j(c^{(m)}, d^{(n-m)})\Big\|_1 \leq M \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \|c^{(m)}\| \, \|d^{(n-m)}\| = M \|c\|_1 \|d\|_1 \, . \blacksquare$$

Let V be a Banach space with a family $\big(V^{(n)}\big)_{n\in\mathbb{Z}}$ of mutually linearly independent Banach subspaces so that we may identify V_1 as a subspace of V. Consider the Banach space $\mathcal{B}(V)$ of bounded (linear) operators on V. For each $n \in \mathbb{Z}$ we denote by

$$\mathcal{B}(V)^{(n)} = \big\{a \in \mathcal{B}(V) \colon aV^{(m)} \subset V^{(n+m)}\big\}$$

the Banach space of all bounded operators on V which are homogeneous of degree n. As $V^{(n)}$ is typically the n–particle sector of a Fock module, we call $\mathcal{B}(V)^{(n)}$ the space of operators with *offset n in the number of particles*.

A.3.2 Lemma. *Let $\big(j_\lambda\big)_{\lambda\in\Lambda}$ be a net of even bilinear mappings $j_\lambda \colon C_1 \times D_1 \to \mathcal{B}(V)_1$ (indexed by some directed set Λ) all fulfilling* (A.3.1) *with a constant $M > 0$ which is independent of λ. Furthermore, suppose that for all homogeneous $c \in C_1$, $d \in D_1$ the net $j_\lambda(c,d)$ converges strongly in $\mathcal{B}(V)$ (of course, to a homogeneous element in $\mathcal{B}(V)$). Then the mapping $(c,d) \mapsto \lim_\lambda j_\lambda(c,d)$ on $C_g \times D_g$ fulfills* (A.3.1) *and, therefore, extends by Lemma A.3.1 to a mapping $j \colon C_1 \times D_1 \to \mathcal{B}(V)_1$. Moreover, $j_\lambda(c,d)$ converges strongly in $\mathcal{B}(V)$ to $j(c,d)$ for all $c \in C_1$, $d \in D_1$.*

Proof. Let $c \in C_1$, $d \in D_1$, $v \neq 0$ in V and $\epsilon > 0$. We may choose $c_g \in C_g$, $d_g \in D_g$ such that

$$\|j_\lambda(c,d) - j_\lambda(c_g,d_g)\|_1 < \frac{\epsilon}{3\,\|v\|} \quad \text{and} \quad \|j(c,d) - j(c_g,d_g)\|_1 < \frac{\epsilon}{3\,\|v\|}$$

for all $\lambda \in \Lambda$. Furthermore, choose $\lambda_0 \in \Lambda$ such that

$$\|j(c_g,d_g)v - j_\lambda(c_g,d_g)v\| < \frac{\epsilon}{3}$$

for all $\lambda \geq \lambda_0$. $\blacksquare$

Lemmata A.3.1 and A.3.2 have obvious generalizations to multi-linear even mappings. This can be shown by direct generalization of the above proofs. A less direct but more elegant method makes use of the *projective tensor product* of Banach spaces. Almost everything what can be said about the projective norm on the algebraic tensor product of two Banach spaces is collected in the remarkable Proposition T.3.6 in [WO93, Appendix T] which we repeat here for convenience.

Let C, D be normed spaces and define the following two numerical functions on $B = C \otimes D$.

$$\begin{aligned}\gamma(b) &= \inf\Big\{\sum \|c_i\|\,\|d_i\| : \sum c_i \otimes d_i = b\Big\} \\ \lambda(b) &= \sup\Big\{\|(\varphi \otimes \psi)(b)\| : \varphi \in \mathfrak{L}(C, \mathbb{C}), \|\varphi\| \leq 1, \psi \in \mathfrak{L}(D, \mathbb{C}), \|\psi\| \leq 1\Big\}\end{aligned}$$

Recall that a seminorm α on B is called ***cross*** (***subcross***), if $\alpha(c \otimes d) = \|c\|\,\|d\|$ $(\alpha(c \otimes d) \leq \|c\|\,\|d\|)$.

A.3.3 Proposition. [WO93]. *The functions γ and λ are cross norms on B, and γ majorizes any other subcross seminorm on B.*

PROOF. (i) Of course, λ is a seminorm. The functionals $\varphi \otimes \psi$ separate the points of B. (This follows from the same statement for finite dimensional vector spaces and an application of the *Hahn-Banach theorem*.) Therefore, λ is a norm.

(ii) Obviously, λ is subcross. On the other hand, by the *Hahn-Banach theorem* there are norm-one functionals φ, ψ with $\varphi(c) = 1 = \psi(d)$ so that λ is cross.

(iii) γ is a seminorm (because in the norm of a sum the infimum is taken over more possibilities than in the sum of the norms of each summand) and, obviously, γ is subcross.

(iv) If α is a subcross seminorm, then $\alpha\big(\sum c_i \otimes d_i\big) \leq \sum \|c_i\|\,\|d_i\|$, whence, $\alpha(b) \leq \gamma(b)$.

(v) γ majorizes λ and, therefore, is a norm. In particular, $\|c\|\,\|d\| \geq \gamma(c \otimes d) \geq \lambda(c \otimes d) = \|c\|\,\|d\|$ so that γ is a cross norm. ∎

γ is called the ***projective norm*** and the completion $C \otimes_\gamma D$ with respect to γ is the ***projective tensor product***. The projective tensor product carries over the universal property of the algberaic tensor product to that of Banach spaces; cf. Proposition C.4.2.

A.3.4 Corollary. *If $j \colon C \times D \to V$ (V a Banach space) is a bounded bilinear mapping, then there exists a unique linear mapping $\widehat{j} \colon C \otimes_\gamma D \to V$ fulfilling $\widehat{j}(c \otimes d) = j(c, d)$ and $\|\widehat{j}\| = \|j\|$.*

PROOF. $\big\|\widehat{j}\big(\sum c_i \otimes d_i\big)\big\| \leq \|j\| \sum \|c_i\|\,\|d_i\|$. Taking the infimum over $\sum c_i \otimes d_i = b$, we find $\|\widehat{j}(b)\| \leq \|j\|\,\gamma(b)$. ∎

A.3.5 Proposition. *In the notations of Lemma A.3.1 we equip $B = C_g \otimes_\gamma D_g$ with a graduation by defining the graded subspaces $B^{(n)} = \overline{\bigoplus_{m \in \mathbb{Z}}} C^{(m)} \otimes_\gamma D^{(n-m)}$. Then $\|\bullet\|_1 = \gamma$.*

Proof. Of course, $\|\bullet\|_1$ majorizes γ. On the other hand, a simple computation as in the proof of Lemma A.3.1 shows that $\|\bullet\|_1$ is subcross. Consequently, it must coincide with the projective norm. ■

Now the multi-linear analogues of the lemmata follow easily by induction, when translated into statements on linear mappings on multiple projective tensor products.

Clearly, Lemmata A.3.1 and A.3.2 (and their multi-linear extensions) remain also true in the case, when j is *homogeneous of degree* ℓ, i.e. when $j(C^{(n)}, D^{(m)}) \subset V^{(n+m+\ell)}$ $(n, m \in \mathbb{Z})$.

A.3.6 Corollary. *The convergence in Lemma A.3.2 is also strongly in* $\mathcal{B}(V_1)$.

Proof. $(c, d, v) \mapsto j_\lambda(c, d)v$ is a 3–linear mapping on $C_1 \times D_1 \times V_1$. Therefore, we may also replace $v \in V_1$ by an element $v_g \in V_g$ which is close to v in $\|\bullet\|_1$. ■

A.4 Banach algebras

A.4.1 Definition. A *Banach algebra* is a Banach space $\mathcal{A}$ with a bilinear multiplication $(a, a') \mapsto aa'$ such that the norm is *submultiplicative*, i.e. $\|aa'\| \leq \|a\| \|a'\|$. If $\mathcal{A}$ is not complete, then we speak of a *normed* algebra. If $\mathcal{A}$ is unital, then we assume that $\|\mathbf{1}\| = 1$. If $\|\bullet\|$ is only a seminorm, then the quotient $\mathcal{A}/\mathcal{N}_\mathcal{A}$ is a normed algebra.

A *Banach $*$–algebra* (a *normed $*$–algebra*) is a Banach algebra (a normed algebra) $\mathcal{A}$ with an isometric *involution* (i.e. an anti-linear, self-inverse mapping on $\mathcal{A}$).

The most important (unital) Banach algebra is that of bounded operators on a Banach space B. However, unlike C^*–algebras, by far not all Banach algebras admit a faithful representation as a subalgebra of some $\mathcal{B}(B)$.

The most obvious property of (unital) Banach algebras (sufficient to understand Appendix A.5) is that they allow for a *power series calculus*, i.e. if $f(z) = \sum_{n=0}^{\infty} c_n z^n$ is a power series with radius of convergence r say, then $f(a)$ is a well-defined element of $\mathcal{A}$, whenever $\|a\| < r$. In this context, 'calculus' means that $f(a)+g(a) = (f+g)(a)$, $f(a)g(a) = (fg)(a)$, and $f(g(a)) = (f \circ g)(a)$, whenever both sides exist. $z \mapsto f\left(\frac{rza}{\|a\|}\right)$ is an analytic, $\mathcal{A}$–valued function the disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$. The most important series is the *geometric* or *Neumann* series $\sum_{n=0}^{\infty} a^n$ which exists for $\|a\| < 1$ and, as usual, gives us $(\mathbf{1} - a)^{-1}$. This allows to show that the invertible elements in a Banach algebra form an open subset. (Indeed, let $\in \mathcal{A}$ be invertible and choose $b \in \mathcal{A}$ with $\|a - b\| < \|a^{-1}\|^{-1}$, whence $\|\mathbf{1} - a^{-1}b\| < 1$. Therfore, $a^{-1}b$ and a are invertible so that also b is invertible.)

A.5 Semigroups in Banach algebras

A.5.1 Definition. Let $\mathcal{A}$ be a unital Banach algebra and $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{T} = \mathbb{N}_0$. A *semigroup in* $\mathcal{A}$ is a family $T = \big(T_t\big)_{t\in\mathbb{T}}$ of elements $T_t \in \mathcal{A}$ such that $T_0 = \mathbf{1}$ and $T_sT_t = T_{s+t}$. If $\mathcal{A} = \mathcal{B}(B)$ is the algebra of bounded operators on a Banach space B (with composition $\circ$ as product), then we say T is a *semigroup on* B.

A semigroup $T = \big(T_t\big)_{t\in\mathbb{R}_+}$ in $\mathcal{A}$ is *uniformly continuous*, if

$$\lim_{t\to 0} \|T_t - \mathbf{1}\| \;=\; 0.$$

A semigroup T on B is a C_0*–semigroup*, if $t \mapsto T_t(b)$ is continuous for all $b \in B$. If B is itself a Banach space of operators on another Banach space (for instance, if B is a von Neumann algebra), then T is *strongly continuous*, if $T_t(b)$ is strongly continuous in B for all $b \in B$.

A.5.2 Proposition. *A uniformly continuous semigroup T is continuous.*

PROOF. Let us fix a compact interval $K = [0, \mathfrak{T}]$. Since $T_{t+\delta} - T_t = (T_\delta - \mathbf{1})T_t$, there exists for all $t \in K$ and all $\varepsilon > 0$ a $\delta_t > 0$ such that $\|T_s - T_t\| < \frac{\varepsilon}{2}$ for all $s \in [t, t+\delta_t)$. Choose finitely many t_i with corresponding δ_n such that the open intervals $(t_i, t_i + \delta_i)$ cover K. Let $t \in (t_i, t_i+\delta_i)$ for some i. Then $\|T_s - T_t\| \leq \|T_s - T_{t_i}\| + \|T_{t_i} - T_t\| < \varepsilon$ for $s \in (t_i, t_i+\delta_i)$. ■

A.5.3 Proposition. *T is differentiable everywhere. Denoting by $\mathcal{L} = T_0' \in \mathcal{A}$ the* generator *of T, the formula*

$$T_t \;=\; e^{t\mathcal{L}}$$

establishes a one-to-one correspondence between elements $\mathcal{L}$ in $\mathcal{A}$ and uniformly continuous semigroups T. In particular, any uniformly continuous semigroup extends to a uniformly continuous group indexed by $\mathbb{R}$.

PROOF. Let $\mathcal{L} \in \mathcal{A}$. Clearly, the series

$$e^{t\mathcal{L}} \;=\; \sum_{n=0}^{\infty} \frac{t^n\mathcal{L}^n}{n!}$$

converges absolutely and defines an entire function of t. The semigroup property may be checked as usual by the *Cauchy product formula.* The function is arbitrarily continuously differentiable and fulfills the differential equation $T_t' = \mathcal{L}T_t$ with initial condition $T_0 = \mathbf{1}$ or, equivalently, $T_t = \mathbf{1} + \int_0^t \mathcal{L}T_s\,ds$. Uniqueness of the solutions of such integral equations follows as in the proof of Proposition 16.2.2 with the help of *Banach's fix point theorem.*

Conversely, if T_t is a uniformly continuous semigroup and, therefore, integrable, then we have $\frac{d}{dt}\int_{t_0}^t T_s\,ds = T_t$ and a similar formula for the lower bound. In particular, for any $\varepsilon > 0$ we have

$$\Big\|T_0 - \frac{1}{\tau}\int_0^\tau T_s\,ds\Big\| \;<\; \varepsilon,$$

for τ sufficiently small. Since T_0 is invertible and the invertibles in a Banach algebra form an open subset, we find that $\int_0^\tau T_s\,ds$ is invertible for $\tau > 0$ sufficiently small. Define $I_\tau(t) = \int_t^{\tau+t} T_s\,ds = T_t\int_0^\tau T_s\,ds$. Then $I_\tau(t)$ is differentiable with respect to t and so is $T_t = I_\tau(t)\big[\int_0^\tau T_s\,ds\big]^{-1}$. Denote $\mathcal{L} = T_0'$. By the semigroup property we have $T_t' = \mathcal{L}T_t$. Once again, $e^{t\mathcal{L}}$ is the unique solution of this differential equation, fulfilling the initial condition $T_0 = \mathbf{1}$. ■

A.5.4 Proposition. *Let T be a uniformly continuous (semi)group with generator $\mathcal{L}$. Then*

$$T_t \;=\; \lim_{n\to\infty}\big(\mathbf{1} + \tfrac{t}{n}\mathcal{L}\big)^n \;=\; \lim_{n\to\infty}\big(\mathbf{1} - \tfrac{t}{n}\mathcal{L}\big)^{-n}.$$

Proof. We have the following (very rough) estimate

$$\|T_\varepsilon - \mathbf{1} - \varepsilon\mathcal{L}\| \;\le\; \varepsilon^2\,\|\mathcal{L}\|^2 \tag{A.5.1}$$

for $|\varepsilon|$ sufficiently small. This yields

$$\begin{aligned}\big\|T_t - \big(\mathbf{1} + \tfrac{t}{n}\mathcal{L}\big)^n\big\| \;&=\; \big\|T_{\frac{t}{n}}^n - \big(\mathbf{1} + \tfrac{t}{n}\mathcal{L}\big)^n\big\| \;=\; \Big\|\sum_{i=1}^{n-1} T_{\frac{t}{n}}^{i-1}(T_{\frac{t}{n}} - \mathbf{1} - \tfrac{t}{n}\mathcal{L})(\mathbf{1} + \tfrac{t}{n}\mathcal{L})^{n-i}\Big\|\\ &\le\; (n-1)\big(\tfrac{t}{n}\big)^2 \max_{i=1,\dots,n}\big(\big\|T_{\frac{t}{n}}^{i-1}\big\|, \big\|(\mathbf{1} + \tfrac{t}{n}\mathcal{L})^{n-i}\big\|\big) \;\le\; \tfrac{n-1}{n}\tfrac{t^2}{n}e^{\frac{n-1}{n}t\|\mathcal{L}\|}\end{aligned}$$

which, clearly, goes to 0 as n increases to ∞. This proves the first limit. The second limit follows from $T_t = (T_{-t})^{-1}$ and the observation that $\mathbf{1} + \frac{t}{n}\mathcal{L}$ is invertible for n sufficiently big. ■

A.5.5 Proposition. (Trotter product formula.) *Let T, S be uniformly continuous semigroups with generators $\mathcal{L}, \mathcal{K}$, respectively. Denote by $S^\circ T = \big(S^\circ T_t\big)$ the uniformly continuous semigroup with generator $\frac{\mathcal{L}+\mathcal{K}}{2}$. Then*

$$S^\circ T_t \;=\; \lim_{n\to\infty}(S_{\frac{t}{2n}}T_{\frac{t}{2n}})^n.$$

Proof. We have

$$\big\|S^\circ T_\varepsilon - S_{\frac{\varepsilon}{2}}T_{\frac{\varepsilon}{2}}\big\| \;\le\; \big\|S^\circ T_\varepsilon - \mathbf{1} - \varepsilon\tfrac{\mathcal{L}+\mathcal{K}}{2}\big\| + \big\|\mathbf{1} + \varepsilon\tfrac{\mathcal{L}+\mathcal{K}}{2} - S_{\frac{\varepsilon}{2}}T_{\frac{\varepsilon}{2}}\big\| \;\le\; \varepsilon^2 C$$

where $C > 0$ is a suitable constant. Now the statement follows precisely as in the preceding proof. ■

By the same proof we obtain the following variant.

A.5.6 Corollary. *Let $0 \leq \varkappa \leq 1$ and denote by $S^\varkappa T = \big(S^\varkappa T_t\big)$ the uniformly continuous semigroup with generator $\varkappa\mathcal{L} + (1-\varkappa)\mathcal{K}$. Then*

$$S^\varkappa T_t \;=\; \lim_{n\to\infty} \big(S_{\frac{\varkappa t}{n}} T_{\frac{(1-\varkappa)t}{n}}\big)^n.$$

A.5.7 Remark. Corollary A.5.6 tells us how we may construct semigroups to convex combinations of a pair of generators. Liebscher [Lie00a] remarks that (by a proof similar to Lemma 7.4.1(3)) also affine combination (i.e. $\varkappa \in \mathbb{C}$) are possible.

A.5.8 Remark. If T is a semigroup, but $T_0 \neq \mathbf{1}$, then all preceding statements make sense, if we restrict to the Banach subalgebra of $\mathcal{A}$ generated by T_t ($t \in \mathbb{R}_+$). In this subalgebra T_0 is a unit. So the statements are true replacing everywhere $\mathbf{1}$ with T_0. Reembedding the semigroup into $\mathcal{A}$, we obtain $T_t = T_0 e^{t\mathcal{L}} = (T_0 - \mathbf{1}) + e^{t\mathcal{L}}$.

A.6 Generators of CP-semigroups

The form of generators of uniformly continuous CP-semigroups was found by Christensen and Evans [CE79] for arbitrary, even non-unital, C^*–algebras $\mathcal{B}$. We quote the basic result [CE79, Theorem 2.1] rephrased in the lanuguage of *derivations* with values in a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module F, i.e. a linear mapping $d\colon \mathcal{B} \to F$ fulfilling

$$d(bb') \;=\; bd(b') + d(b)b'.$$

Then we repeat the cohomological discussion of [CE79] which allows to find the form of the generator in the case of von Neumann algebras.

A.6.1 Lemma. *Let d be a bounded derivation from a pre–C^*–algebra $\mathcal{B}$ ($\subset \mathscr{B}^a(G)$) to a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module F ($\subset \mathscr{B}^a(G, F \odot G)$). Then there exists $\zeta \in \overline{\mathsf{span}}^s\, d(\mathcal{B})\mathcal{B}$ ($\subset \overline{F}^s \subset \mathscr{B}(\overline{G}, \overline{F \odot G})$) such that*

$$d(b) \;=\; b\zeta - \zeta b. \tag{A.6.1}$$

Observe that $\overline{\mathsf{span}}^s\, d(\mathcal{B})\mathcal{B}$ is a two-sided submodule of $\overline{F}^s$. Indeed, we have $bd(b') = d(bb') - d(b)b'$ so that we have invariance under left multiplication.

Recall that a derivation of the form as in (A.6.1) is called *inner*, if $\zeta \in F$. Specializing to a von Neumann algebra $\mathcal{B}$ we reformulate as follows.

A.6.2 Corollary. *Bounded derivations from a von Neumann algebra $\mathcal{B}$ to a von Neumman $\mathcal{B}$–$\mathcal{B}$–module are inner (and, therefore, normal).*

Specializing further to the von Neumann module $\mathcal{B}$, we find the older result that bounded derivations on von Neumann algebras are inner; see e.g. [Sak71]. We remark, however, that Lemma A.6.1 is proved by reducing it to the statements for von Neumann algebras. The proofs of both depend heavilly on the decomposition of von Neumann algebras into different types.

In the sequel, we restrict to normal CP-semigroups von Neumann algebas. As an advantage (which is closely related to self-duality of von Neumann modules) we end up with simple statements as in Corollary A.6.2 instead of the involved ones in Lemma A.6.1. The more general setting does not give more insight (in fact, the only insight is that satisfactory results about the generator are only possible in the context of von Neumann algebras), but just causes unpleasant formulations.

A.6.3 Theorem. [CE79]. *Let T be a normal uniformly continuous CP-semigroup on a von Neumann algebra $\mathcal{B}$ with generator $\mathcal{L}$. Then there exist a von Neumann $\mathcal{B}$–$\mathcal{B}$–module F, an element $\zeta \in F$, and an element $\beta \in \mathcal{B}$ such that $\mathcal{L}$ has the Christensen-Evans form* (5.4.3)*, and such that F is the strongly closed submodule of F generated by the derivation $d(b) = b\zeta - \zeta b$. Moreover, F (but not the pair (β, ζ)) is determined by $\mathcal{L}$ up to (two-sided) isomorphism.*

Proof. We proceed similarly as for the GNS-construction, and try to define an inner product on the $\mathcal{B}$–$\mathcal{B}$–module $\mathcal{B}\otimes\mathcal{B}$ with the help of $\mathcal{L}$. However, since $\mathcal{L}$ is only conditionally completely positive, we can define this inner product not for all elements in this module, but only for those elements in the two-sided submodule generated by elements of the form $b \otimes \mathbf{1} - \mathbf{1} \otimes b$. This is precisely the subspace of all $\sum\limits_i a_i \otimes b_i$ for which $\sum\limits_i a_i b_i = 0$ with inner product

$$\Big\langle \sum_i a_i \otimes b_i, \sum_j a_j \otimes b_j \Big\rangle \;=\; \sum_{i,j} b_i^* \mathcal{L}(a_i^* a_j) b_j. \tag{A.6.2}$$

We divide out the length-zero elements and denote by F the strong closure.

By construction, F is a von Neumann $\mathcal{B}$–$\mathcal{B}$–module and it is generated as a von Neumann module by the bounded derivation $d(b) = (b \otimes \mathbf{1} - \mathbf{1} \otimes b) + \mathcal{N}_F$. By Corollary A.6.2 there exists $\zeta \in F$ such that $d(b) = b\zeta - \zeta b$. Moreover, we have

$$\mathcal{L}(bb') - b\mathcal{L}(b') - \mathcal{L}(b)b' + b\mathcal{L}(\mathbf{1})b' \;=\; \langle \zeta, bb'\zeta\rangle - b\langle \zeta, b'\zeta\rangle - \langle \zeta, b\zeta\rangle b' + b\langle \zeta, \zeta\rangle b'$$

from which it follows that the mapping $D\colon b \mapsto \mathcal{L}(b) - \langle \zeta, b\zeta\rangle - \frac{b(\mathcal{L}(\mathbf{1}) - \langle \zeta,\zeta\rangle) + (\mathcal{L}(\mathbf{1}) - \langle \zeta,\zeta\rangle)b}{2}$ is a bounded hermitian derivation on $\mathcal{B}$. Therefore, there exists $h = h^* \in \mathcal{B}$ such that $D(b) = ibh - ihb$. Setting $\beta = \frac{\mathcal{L}(\mathbf{1}) - \langle \zeta,\zeta\rangle}{2} + ih$ we find $\mathcal{L}(b) = \langle \zeta, b\zeta\rangle + b\beta + \beta^* b$.

Let F' be another von Neumann module with an element ζ' such that the derivation $d'(b) = b\zeta' - \zeta' b$ generates F and such that $\mathcal{L}(b) = \langle \zeta', b\zeta' \rangle + b\beta' + \beta'^* b$ for some $\beta' \in \mathcal{B}$. Then the mapping $d(b) \mapsto d'(b)$ extends as a two-sided unitary $F \to F'$, because the inner product (A.6.2) does not depend on β. ■

A.7 Pre–C^*–algebras

A.7.1 Definition. A normed $*$–algebra $\mathcal{A}$ is a *pre–C^*–algebra*, if the norm is a *C^*–norm*, i.e. if $\|a^*a\| = \|a\|^2$. A *C^*–algebra* is a complete pre–C^*–algebra.

The completion of a pre–C^*–algebra is a C^*–algebra. Also the quotient of a C^*–seminorm may be divided out, but we will not speak about semi–C^*–algebras. It is rather our goal to collect those well-known properties of C^*–algebras which extend to pre–C^*–algebras and to separate them from those which must be revised.

C^*–algebras admit a faithful representation on a (pre-)Hilbert space, and so do pre–C^*–algebras. (Just complete and restrict the faithful representation of the completion to the original pre–C^*–algebra.) Usually, our pre–C^*–algebras will be $*$–algebras of bounded operators on a pre-Hilbert space or module and come along equipped with the operator norm of this *defining* representation.

An element a in a C^*–algebra $\mathcal{A}$ is *positive*, if it can be written in the form $a = b^*b$ for a some $b \in \mathcal{A}$. We may even choose b itself positive, i.e. b is the *positive square root* $\sqrt{a}$ of a (and as such it is determined uniquely). For a self-adjoint element $a \in \mathcal{A}$ one checks that $a_\pm = \frac{\sqrt{a^2} \pm a}{2}$ are the unique positive elements in $\mathcal{A}$ such that $a = a_+ - a_-$ and $a_+ a_- = 0$. Nothing like this is true for pre–C^*–algebras. Instead, we say a in a pre–C^*–algebra $\mathcal{A}$ is *positive*, if it is positive in the completion of $\mathcal{A}$. As our pre–C^*–algebras are operator algebras, it is important to notice that this definition of positivity is compatible with Definition 1.5.1 for operators.

A C^*–algebra $\mathcal{A}$ is spanned linearly by its unitaries or quasiunitaries. Therefore, any representation (which, once for all, means $*$–representation) of $\mathcal{A}$ by adjointable operators on a pre-Hilbert module maps into the bounded operators. Since homomorphisms (which, once for all, means $*$–homomorphisms) from a C^*–algebra into a pre–C^*–algebra are contractions (by the way, always with norm complete range), so is π. As a corollary, if such a homomorphism is faithful, then it must be isometric. (Otherwise, its left inverse would not be a contraction.) The same statements are true, if $\mathcal{A}$ is a pre–C^*–algebra which is the union or even the closure of its C^*–subalgebras.

In several places in these notes we speak about *essential* objects: essential Hilbert modules (Example 1.1.5), essential conditional expectations (Definition 4.4.1), essential weak

Markov (quasi)flows (Definition 12.4.11). All these notions are directly related to the requirement that certain ideals be essential and, therefore, act faithfully in some sense. We repeat this important notion and extend it to pre–C^*–algebras. A (not necessarily closed) ideal (which, once for all, means $*$–ideal) I in a C^*–algebra $\mathcal{A}$ is ***essential***, if it ***separates*** the points of $\mathcal{A}$, i.e. if $ab = 0$ (or $ba = 0$) for some $a \in \mathcal{A}$ and for all $b \in I$ implies that $a = 0$. Since I is an ideal, elements $a \in \mathcal{A}$ act as (adjointable) operators on the Hilbert I–module I (cf. also the double centralizers in Appendix A.8). If I is essential, then this representation of $\mathcal{A}$ is faithful and, therefore, isometric. It is this last property in which we are interested. Therefore, we define an ideal I in a pre–C^*–algebra to be ***essential***, if it is essential in $\overline{\mathcal{A}}$. In Example 4.4.9 we show that this C^*–algebraic condition is, indeed, indipensable.

A C^*–algebra $\mathcal{A}$ admits an ***approximate unit***, i.e. a net $\big(u_\lambda\big)_{\lambda\in\Lambda}$ in $\mathcal{A}$ such that $\lim\limits_\lambda u_\lambda a = \lim\limits_\lambda a u_\lambda = a$ for all $a \in \mathcal{A}$. The ***standard approximate unit*** is the set of all positive elements u in the open unit-ball of $\mathcal{A}$ which is directed increasingly for the partial order in $\mathcal{A}$. Observe that necessarily $\lim\limits_\lambda \|u_\lambda\| = 1$. If $\mathcal{A}$ is a pre–C^*–algebra, then we may choose an approximate unit $\big(u_\lambda\big)_{\lambda\in\Lambda}$ in the unit-ball of the completion. For each u_λ we choose $v_\lambda \in \mathcal{A}$ such that $\|u_\lambda - v_\lambda\| \leq \frac{1}{2}(1 - \|u_\lambda\|)$. Then $\big(v_\lambda\big)_{\lambda\in\Lambda}$ is an approximate unit in $\mathcal{A}$ even for $\overline{\mathcal{A}}$. Observe that $\|v_\lambda\| < 1$. We may and, usually, will assume that all v_λ are self-adjoint. A standard result, which remains true for pre–C^*–algebras, is that $\lim\limits_{\lambda,\lambda'} \varphi((v_\lambda - v_{\lambda'})^2) = 0$. (For λ, λ' sufficiently big, we may replace v by the standard approximate unit u for $\overline{\mathcal{A}}$ and follow the usual proof.)

We close by collecting some simple, but, useful results.

A.7.2 Proposition. *Let p and q be projections in a pre–C^*–algebra. Then the following properties are equivalent.*

1. $p \geq q$
2. $p - q$ *is a projection.*
3. $pq = q$ *(or, equivalently,* $qp = q$*).*
4. $qpq = q$.
5. $pqp = q$.

Proof. $3 \Rightarrow 4$ and $3 \Rightarrow 5$ are clear. $4 \Rightarrow 3$. We have $0 = q(\mathbf{1} - p)q = q(\mathbf{1} - p)^2 q = ((\mathbf{1} - p))^* q((\mathbf{1} - p)q)$, whence $(\mathbf{1} - p)q = 0$ or $qp = q$. $5 \Rightarrow 3$. We have $0 = pqp(\mathbf{1} - p) = q(\mathbf{1} - p) = q - qp$ or $q = qp$. Therefore, 3, 4, and 5 are equivalent.

$3 \Rightarrow 2$. We have $(p-q)^2 = p - pq - qp + q = p - q$. $2 \Rightarrow 1$ is clear. $1 \Rightarrow 5 (\Rightarrow 3)$. We have $p - q \geq 0$ so that also $qpq - q \geq 0$, and we have $\mathbf{1} - p \geq 0$ so that also $q - qpq \geq 0$. In other words, $q = qpq$. ■

A.7.3 Proposition. *Let a, a', b be elements in a pre–C^*–algebra $\mathcal{A}$.*

1. *$ab = 0$, if and only if $abb^* = 0$.*

2. *Let $a \geq 0$, and $a' \geq 0$. Then $(a + a')b = 0$, if and only if $ab = 0$ and $a'b = 0$.*

PROOF. Of course, $ab = 0 \Rightarrow abb^* = 0$. Conversely, $abb^* = 0 \Rightarrow abb^*a^* = 0 \Rightarrow ab = 0$. This is 1.

Of course, $ab = 0$ and $a'b = 0$ implies $(a+a')b = 0$. So let $a, a' \geq 0$ and, if necessary, pass to $\overline{\mathcal{A}}$ so that a has a positive square root. Then $b^*(a+a')b \geq b^*ab \geq 0$ so that $b^*(a+a')b = 0$ implies $b^*ab = 0 \Rightarrow \sqrt{a}b = 0 \Rightarrow ab = 0$, and, similarly, for a'. This is 2. ■

A.8 Multiplier algebras

Multiplier algebras arise naturally, if we ask, how to embed non-unital C^*–algebras into unital ones. In the context of commutative C^*–algebras (which are isomorphic to some space $\mathcal{C}_0(M)$ of continuous function vanishing at infinity on a locally compact space M, and which are unital if and only if M is compact), the multiplier algebras correspond to the *Stone-Cech compactification*, which is maximal in some sense. The *one-point compactification* correponds to simply adding an artificial unit. We discuss this afterwards. Needless to say that we give pre–C^*–algebraic versions. Besides this exception, we follow Murphy [Mur90].

A ***double centralizer*** on a pre–C^*–algebra $\mathcal{A}$ is a pair (L, R) of mappings in $\mathcal{B}(\mathcal{A})$, fulfilling $aL(b) = R(a)b$. Denote by $M(\mathcal{A})$ the space of double centralizers on $\mathcal{A}$. Let $\big(u_\lambda\big)_{\lambda \in \Lambda}$ be an approximate unit for $\mathcal{A}$. We conclude that

$$L(ab) \;=\; \lim_\lambda u_\lambda L(ab) \;=\; \lim_\lambda R(u_\lambda)ab \;=\; \lim_\lambda u_\lambda L(a)b \;=\; L(a)b \tag{A.8.1}$$

and similarly $R(ab) = aR(b)$ so that $L \in \mathcal{B}^r(\mathcal{A})$ and $R \in \mathcal{B}^l(\mathcal{A})$. Clearly, double centralizers form a vector space. Since $\|L\| = \sup_{\|a\| \leq 1, \|b\| \leq 1} \|aL(b)\| = \sup_{\|a\| \leq 1, \|b\| \leq 1} \|R(a)b\| = \|R\|$ we find that $M(\mathcal{A})$ is a normed space. We may define a product $(L, R)(L', M') = (L \circ L', R' \circ R)$ and an adjoint $(L, R)^* = (R^*, L^*)$ where $L^*(a) = L(a^*)^*$ and $R^*(a) = R(a^*)^*$. Clearly, $M(\mathcal{A})$ with these operations is a normed $*$–algebra. From $\|L(a)\|^2 = \|L(a)^*L(a)\| = \|R(L(a)^*)a\| \leq \|a\|^2 \|R^* \circ L\|$ we conclude that $\|L\|^2 \leq \|R^* \circ L\| \leq \|R^*\| \|L\| = \|L\|^2$ so that $M(\mathcal{A})$ is a pre–C^*–algebra, the ***multiplier algebra*** of $\mathcal{A}$. Clearly, $M(\mathcal{A})$ is unital.

For each $a \in \mathcal{A}$ we may define a double centralizer (L_a, R_a) by setting $L_a(b) = ab$ and $R_a(b) = ab$. In other words, $\mathcal{A}$ is contained (isometrically) as an ideal in $M(\mathcal{A})$. More generally, each pre–C^*–algebra $\mathcal{B}$, containing $\mathcal{A}$ as an ideal, gives rise to a double centralizer. The canonical homomorphism $\mathcal{B} \rightarrow M(\mathcal{A})$ is contractive. It is isometric, if and only is $\mathcal{A}$ is an essential ideal in $\mathcal{B}$. In other words, $M(\mathcal{A})$ is the "biggest" pre–C^*–algebra, containing $\mathcal{A}$ as an essential ideal. If $\mathcal{A}$ is unital, then for each double centralizer (L, R) we may choose $a = L(\mathbf{1}) = R(\mathbf{1})$. Then $(L, R) = (L_a, R_a)$ so that $\mathcal{A}$ coincides with $M(\mathcal{A})$.

The *strict* topology of $M(\mathcal{A})$ is the locally convex Hausdorff topology induced by the families $\|a\bullet\|$ and $\|\bullet a\|$ $(a \in \mathcal{A})$ of seminorms. If $\mathcal{A}$ is a C^*–algebra, then so is $M(\mathcal{A})$. Moreover, $M(\mathcal{A})$ is strictly complete. On the other hand, setting $a_\lambda = u_\lambda L(u_\lambda) = R(u_\lambda)u_\lambda$, computations similar to (A.8.1) show that $(L_{a_\lambda}, R_{a_\lambda})$ converges strictly to (L, R). In other words, the unit-ball of $\mathcal{A}$ is strictly dense in $M(\mathcal{A})$ so that we may identify $M(\mathcal{A})$ with the strict completion of $\mathcal{A}$.

We close with the simplest possible unitization $\widetilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}\widetilde{\mathbf{1}}$. There is one and only one $*$–algebra structure, preserving the multiplication of the subspace (in fact, the ideal) $\mathcal{A}$, for which $\widetilde{\mathbf{1}}$ is a unit. If there is no confusion (for instance, if $\mathcal{A}$ is non-unital), we write $\mathbf{1}$ for $\widetilde{\mathbf{1}}$.

If $\mathcal{A}$ is a non-unital pre–C^*–algebra, then we identify $\widetilde{\mathcal{A}}$ as a pre–C^*–subalgebra of $M(\mathcal{A})$, thus, norming it. Clearly, for all $\widetilde{a} \in \widetilde{\mathcal{A}}$ we find $\|\widetilde{a}\| = \sup\limits_{a \in \mathcal{A}, \|a\| \leq 1} \|\widetilde{a}a\|$. If $\mathcal{A}$ is unital, then $\widetilde{\mathcal{A}}$ is isomorphic to the pre–C^*–algebraic direct sum $\mathcal{A} \oplus \mathbb{C}$, where the copy of $\mathbb{C}$ correponds to multiples of $\widetilde{\mathbf{1}} - \mathbf{1}$. The norm of $(a, \mu) \in \mathcal{A} \oplus \mathbb{C}$ is $\max(\|a\|, |\mu|)$.

A.9 Normal mappings on von Neumann algebras

We repeat some well-known facts on normal mappings which can be found in any text book like [Sak71, Tak79]. We also recommend the almost self-contained appendix in Meyer's book [Mey93].

First of all, recall that a von Neumann algebra is *order complete*, i.e. any bounded increasing net of positive elements in a von Neumann algebra converges in the strong topology to its unique least upper bound. A positive linear mapping T between von Neumann algebras is called *normal*, if it is *order continuous*. In other words, T is normal, if and only if $\limsup\limits_\lambda T(a_\lambda) = T(\limsup\limits_\lambda a_\lambda)$ for each bounded increasing net $\big(a_\lambda\big)$.

Of particular interest is the set of *normal states* on a von Neumann algebra. An increasing net $\big(a_\lambda\big)$ converges *normally* to a, if and only if $\varphi(a_\lambda)$ converges to $\varphi(a)$ for any normal state φ. The linear span of the normal states is a Banach space, the *pre-dual*, and a von Neumann algebra is the dual Banach space of its pre-dual. The σ–weak topology is the topology induced on the von Neumann algebra by this duality.

As normality is a matter of bounded subsets, a positive mapping T is normal, if and only if $\varphi \circ T(a_\lambda)$ converges to $\varphi \circ T(a)$ for all bounded increasing nets $\big(a_\lambda\big)$ and all φ in a subset of normal states which is total in the pre-dual.

If a von Neumann algebra acts on a Hilbert space G, then the functionals of the form $\langle f, \bullet f\rangle$ form a total subset of the pre-dual, whenever f ranges over a dense subset of G. Moreover, using the technique of *cyclic decomposition* (see Section 2.3), it follows that for some von Neumann module E the set of functionals $\langle x \odot g, \bullet x \odot g\rangle$ is total in the pre-dual of $\mathcal{B}(E \bar{\odot} G)$, whenever x ranges over a dense subset of E and g ranges over a dense subset of G.

A.10 Inductive limits

A.10.1 Definition. Let $\mathbb{L}$ be a partially ordered set which is directed increasingly. An *inductive system over* $\mathbb{L}$ is a family $\big(E_t\big)_{t\in\mathbb{L}}$ of vector spaces E_t with a family $\big(\beta_{ts}\big)_{t\geq s}$ of linear mappings $\beta_{ts}\colon E_s \to E_t$ fulfilling

$$\beta_{tr}\beta_{rs} = \beta_{ts}$$

for all $t \geq r \geq s$ and

$$\beta_{tt} = \mathsf{id}_{E_t}\,.$$

The *inductive limit* $E = \liminf_{t\in\mathbb{L}} E_t$ of the family $\big(E_t\big)$ is defined as

$$E = E^{\oplus}/\mathcal{N},$$

where $E^{\oplus} = \bigoplus_{t\in\mathbb{L}} E_t$ and $\mathcal{N}$ denotes the subspace of $E^{\oplus}$ consisting of all those $x = \big(x_t\big)$ for which there exists $s \in \mathbb{L}$ (with $s \geq t$ for all t with $x_t \neq 0$) such that $\sum_{t\in\mathbb{L}} \beta_{st}x_t = 0 \in E_s$. (Clearly, if s fulfills this condition, then so does each $s' \geq s$.)

A.10.2 Proposition. *The family* $\big(i_t\big)_{t\in\mathbb{L}}$ *of canonical mappings* $i_t\colon E_t \to E$ *fulfills*

$$i_t\beta_{ts} = i_s$$

for all $t \geq s$. *Clearly,* $E = \bigcup_{t\in\mathbb{L}} i_tE_t$.

Proof. Let us identify $x_t \in E_t$ with its image in $E^{\oplus}$ under the canonical embedding. We have to check, whether $\beta_{ts}x_s - x_s \in \mathcal{N}(\subset E^{\oplus})$ for all $x_s \in E_s$. But this is clear, because $\beta_{tt}(\beta_{ts}x_s) - \beta_{ts}(x_s) = 0$. ∎

A.10.3 Proposition. *Let F be another vector space and suppose $f\colon E \to F$ is a linear mapping. Then the familiy $\big(f_t\big)_{t\in\mathbb{L}}$ of linear mappings, defined by setting*

$$f_t = f i_t, \tag{A.10.1}$$

fulfills

$$f_t\beta_{ts} = f_s \quad \textit{for all} \quad t \geq s. \tag{A.10.2}$$

Conversely, if $\big(f_t\big)_{t\in\mathbb{L}}$ is a family of linear mappings $f_t\colon E_t \to F$ fulfilling (A.10.2)*, then there exists a unique linear mapping $f\colon E \to F$ fulfilling* (A.10.1).

Proof. Of course, $f = 0$, if and only if $f_t = 0$ for all $t \in \mathbb{L}$, because E is spanned by all i_tE_t. In other words, the correspondence is one-to-one.

Consider a linear mapping $f\colon E \to F$ and set $f_t = fi_t$. Then by Proposition A.10.2 we have $f_t\beta_{ts} = fi_t\beta_{ts} = fi_s = f_s$.

For the converse direction let $\big(f_t\big)$ be a family of linear mappings $f_t\colon E_t \to F$ which satisfies (A.10.2). Define $f^{\oplus} = \bigoplus_{t\in\mathbb{L}} f_t\colon E^{\oplus} \to F$ and let $x = \big(x_t\big) \in \mathcal{N}$ so that for some $s \in \mathbb{L}$ we have $\sum_{t\in\mathbb{L}} \beta_{st}x_t = 0$. Then

$$f^{\oplus}(x) = \sum_{t\in\mathbb{L}} f_tx_t = \sum_{t\in\mathbb{L}} f_s\beta_{st}x_t = f_s \sum_{t\in\mathbb{L}} \beta_{st}x_t = 0,$$

so that $f^{\oplus}$ defines a mapping f on the quotient E fulfilling (A.10.1). ■

A.10.4 Remark. The inductive limit E together with the family $\big(i_t\big)$ is determined by the second part of Proposition A.10.3 up to vector space isomorphism. This is refered to as the *universal property* of E.

If the vector spaces E_t carry additional structures, and if the mediating mappings β_{ts} respect these structures, then simple applications of the universal property show that, usually, also the inductive limit carries the same structures.

A.10.5 Example. If all E_t are right (left) modules and all β_{ts} are right (left) module homomorphisms, then E inherits a right (left) module structure in such a way that all i_t also become right (left) module homomorphisms. A similar statement is true for two-sided modules.

Moreover, if F is another module (right, left, or two-sided) and $\big(f_t\big)$ is a family of homomorphisms of modules (right, left, or two-sided) fulfilling (A.10.2), then also f is homomorphism.

Sometimes it is necessary to work slightly more in order to see that E carries the same structure. Denote by $i\colon E^\oplus \to E$ the canonical mapping.

A.10.6 Proposition. *Let all E_t be pre-Hilbert modules and let all β_{ts} be isometries. Then*

$$\langle x, x'\rangle = \sum_{t,t'} \langle \beta_{st} x_t, \beta_{st'} x'_{t'}\rangle \tag{A.10.3}$$

($x = i((x_t))$, $x' = i((x'_t)) \in E$, and s such that $x_t = x'_t = 0$ whenever $t > s$) defines an inner product on E. Obviously, also the i_t are isometries.

Moreover, if $\bigl(f_t\bigr)_{t\in\mathbb{L}}$ with $f_t\beta_{ts} = f_s$ $(t \geq s)$ is a family of isometries from E_t into a pre-Hilbert module F, then so is f.

Proof. We have to show that (A.10.3) does not depend on the choice of s. So let s_1 and s_2 be different possible choices. Then choose s such that $s \geq s_1$ and $s \geq s_2$ and apply the isometries β_{ss_1} and β_{ss_2} to the elements of E_{s_1} and E_{s_2}, respectively, which appear in (A.10.3).

Since any element of E may be written in the form $i_t x_t$ for suitable $t \in \mathbb{L}$ and $x_t \in E_t$, we see that that the inner product defined by (A.10.3) is, indeed, strictly positive.

The remaining statements are obvious. ■

A.10.7 Remark. Of course, the inductive limit over two-sided pre-Hilbert modules E_t with two-sided β_{ts} is also a two-sided pre-Hilbert module and the canonical mappings i_t respect left multiplication.

A.10.8 Remark. If the mappings β_{ts} are non-isometric, then Equation (A.10.3) does not make sense. However, if $\mathbb{L}$ is a lattice, then we may define an inner product of two elements $i_t x_t$ and $i_{t'} x'_{t'}$ by $\langle \beta_{st} x_t, \beta_{st'} x'_{t'}\rangle$ where s is the unique maximum of t and t'. This idea is the basis for the construction in [Bha99] where also non-conservative CP-semigroups are considered. Cf. also Remark 12.3.6.

Sometimes, however, in topological contexts it will be necessary to enlarge the algebraic inductive limit in order to preserve the structure. For instance, the inductive limit of Hilbert modules will only rarely be complete. If this is the case, we refer to the limit in Definition A.10.1 as the *algebraic* inductive limit.

A.10.9 Definition. By the *inductive limit* of an inductive system of Hilbert modules we understand the norm completion of the algebraic inductive limit.

By the *inductive limit* of an inductive system of von Neumann modules we understand the strong closure of the algebraic inductive limit.

A.10.10 Proposition. *1. Let $\mathcal{A}$ and $\mathcal{B}$ be pre–C^*–algebras. Then the inductive limit of contractive pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–modules is a contractive pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module. Consequently, the inductive limit of contractive Hilbert $\mathcal{A}$–$\mathcal{B}$–modules is a contractive Hilbert $\mathcal{A}$–$\mathcal{B}$–module.*

2. Let $\mathcal{A}$ be a von Neumann algebra and let $\mathcal{B}$ be a von Neumann algebra acting on a Hilbert space G. Then the inductive limit of von Neumann $\mathcal{A}$–$\mathcal{B}$–modules is a von Neumann $\mathcal{A}$–$\mathcal{B}$–module.

PROOF. Any element in the algebraic inductive limit may be written as $i_t x_t$ for suitable $t \in \mathbb{L}$ and $t_t \in E_t$. Therefore, the action of $a \in \mathcal{A}$ is bounded by $\|a\|$ on a dense subset of the inductive limit of Hilbert modules. Moreover, if all E_t are von Neumann modules, then the functionals $\langle i_t x_t \odot g, \bullet i_t x_t \odot g \rangle$ on $\mathcal{A}$ all are normal. ■

Appendix B

Spaces of Banach space valued functions

B.1 Functions on measure spaces

B.1.1 The spaces $\mathfrak{F}$ and $\mathfrak{F}^\infty$ and multiplication operators. Let M be a set (usually $\mathbb{R}$ or $\mathbb{R}_+$) and let V be a Banach space. The space $\mathfrak{F}(M,V)$ of functions $f\colon M \to V$ is a vector space in a natural fashion. By $\mathfrak{F}(M)$ we mean $\mathfrak{F}(M,\mathbb{C})$. We use a similar convention for all other function spaces. Any function $\varphi \in \mathfrak{F}(M)$ gives rise to an operator $\varphi \in \mathcal{L}(\mathfrak{F}(M,V))$ (the *multiplication operator* associated with φ) which acts by *pointwise multiplication*, i.e. $(\varphi f)(x) = \varphi(x)f(x)$. This definition extends in an obvious way to functions $\varphi\colon M \to \mathcal{L}(V)$. For any subset $S \subset M$ we define its *indicator function* $\mathbb{I}_S \in \mathfrak{F}(M)$ by setting $\mathbb{I}_S(x) = 1$ for $x \in S$ and $\mathbb{I}_S(x) = 0$ otherwise.

On $\mathfrak{F}(M,V)$ we define the function $\|\bullet\| : \mathfrak{F}(M,V) \to [0,\infty]$ by setting

$$\|f\| \;=\; \sup_{x \in M} \|f(x)\|\,.$$

We say a sequence $\big(f_n\big)$ in $\mathfrak{F}(M,V)$ converges *uniformly* to $f \in \mathfrak{F}(M,V)$, if $\|f - f_n\| \to 0$. With the exception that possibly $\|f\| = \infty$, the function $\|\bullet\|$ fulfills all properties of a norm (see Definition A.1). Therefore, the space

$$\mathfrak{F}^\infty(M,V) \;=\; \big\{f \in \mathfrak{F}(M,V)\colon\; \|f\| < \infty\big\}$$

of *bounded functions* on M is a normed vector subspace of $\mathfrak{F}(M,V)$. In fact, $\mathfrak{F}^\infty(M,V)$ is a Banach space. (Indeed, if $\big(f_n\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{F}^\infty(M,V)$, then f_n converges pointwise to a function f and $\|f_n\|$ converges to some constant $C \geq 0$. From the inequality $\|f(x)\| \leq \|f(x) - f_n(x)\| + \|f_n\| \to 0 + C$ for fixed x and $n \to \infty$ it follows that $\|f\| \leq C < \infty$. From $\|f(x) - f_m(x)\| \leq \|f(x) - f_n(x)\| + \|f_n - f_m\|$ for fixed x and all n,m sufficiently big it follows after letting $n \to \infty$ that $\|f_m - f\| \leq \sup_{n \geq m} \|f_n - f_m\|$. Letting $m \to \infty$ this expression tends to 0 so that $f_m \to f$.)

If we want to distinguish the norm from other norms, we often write $\|\bullet\|_\infty$. The multiplication operator associated with a function $\varphi \in \mathfrak{F}^\infty(M, \mathcal{B}(V))$ restricts to a bounded operator φ on $\mathfrak{F}^\infty(M, V)$ with norm $\|\varphi\| = \|\varphi\|_\infty$. Therefore, $\mathfrak{F}^\infty(M, \mathcal{B}(V))$ is a (unital) Banach subalgebra of $\mathcal{B}(\mathfrak{F}^\infty(M, V))$.

B.1.2 Simple functions, measurable functions and essentially bounded functions. Let (M, Σ) be a measurable space (usually $\mathbb{R}$ or some subset with the σ–algebra of *Borel sets*). The space of *simple functions* is

$$\mathfrak{E}(M, V) \;=\; \Big\{ f = \sum_{i=1}^{n} v_i \mathbb{I}_{S_i} \;\Big|\; n \in \mathbb{N}, v_i \in V, S_i \in \Sigma \Big\}.$$

Clearly, $\mathfrak{E}(M, V) \subset \mathfrak{F}^\infty(M, V)$. More precisely, if we choose the S_i to be pairwise disjoint (what is always possible), then

$$\|f\| \;=\; \max_{i=1,\ldots,n} \|v_i\|\,. \tag{B.1.1}$$

We consider V always as measurable space equipped with the σ–algebra $\mathfrak{B}(V)$ of *Borel sets*. Clearly, functions in $\mathfrak{E}(M, V)$ are measurable.

By $\mathfrak{L}^\infty(M, V)$ we denote the closure of $\mathfrak{E}(M, V)$ in the Banach space $\mathfrak{F}^\infty(M, V)$ so that $\mathfrak{L}^\infty(M, V)$ is a Banach space, too. Pointwise limits (i.e., in particular, uniform limits) of measurable functions are again measurable. (This can be reduced to the same well-known statement for $\mathbb{R}_+$–valued functions, by the observation that, if $f_n \to f$ pointwise, then $\|f_n - v\| \to \|f - v\|$ pointwise for all $v \in V$. The functions $\|f_n - v\|$ are measurable for all $n \in \mathbb{N}$, because $\|\bullet\|$ is continuous, hence, mesurable and compositions of measurable functions are measurable. Consequently, also the limit function $\|f - v\|$ is measurable. Therefore, the inverse image under f of any ε–neighbourhood of $v \in V$ is measurable, which implies that f is measurable.)

Let (M, Σ, μ) be a measure space (usually, $\mathbb{R}$ with the *Lebesgue measure* λ). Then the set $\mathcal{N}_\mu = \{f \in \mathfrak{L}^\infty(M, V) \mid f(x) = 0 \ \mu\text{–a.e.}\}$ (where *μ–a.e.* means with exception of a set of measure 0) is a Banach subspace of $\mathfrak{L}^\infty(M, V)$. (If $\big(f_n\big)$ is a sequence in $\mathcal{N}_\mu$ converging to f, and if A_n are null-sets such that $f_n \upharpoonright A_n^\complement = 0$, then $A = \bigcup_n A_n$ is a null-set in Σ, such that $f \upharpoonright A^\complement = 0$.) By Remark A.1.6, $\|f\|_{ess} = \inf_{g \in \mathcal{N}_\mu} \|f + g\|$ defines a seminorm on $\mathfrak{L}^\infty(M, V)$ and the quotient $L^\infty(M, V) = \mathfrak{L}^\infty(M, V)/\mathcal{N}_\mu$ is a Banach space with norm $\|f + \mathcal{N}_\mu\|_{ess} = \|f\|_{ess}$. Of course, $\mathfrak{E}(M, V)/(\mathfrak{E}(M, V) \cap \mathcal{N}_\mu)$ is a dense subspace. If it is clear from the context that we speak about $\|\bullet\|_{ess}$ we will not distinguish between the coset $f + \mathcal{N}_\mu \in L^\infty(M, V)$ and its representative $f \in \mathfrak{L}^\infty(M, V)$.

B.1.3 Proposition. *Let $f \in \mathcal{L}^\infty(M,V)$.*

1. $\|f\|_{ess} = \inf\Big\{C > 0 \;\Big|\; \mu\big(\{x \in M\colon\ \|f(x)\| > C\}\big) = 0\Big\}$.

2. *If f^s is a simple function such that $\|f - f^s\|_{ess} \leq \varepsilon$, then there exists a non-null-set $S \in \Sigma$ such that $f^s \restriction S = v \in V$ and $\|v\| \geq \|f\|_{ess} - \varepsilon$.*

Proof. 1. Denote the right-hand side by C_0 and suppose $C \geq C_0$. Then the set S_C of all $x \in M$ for which $\|f(x)\| > C$ is a null-set, so that $f\mathbb{I}_{S_C} \in \mathcal{N}_\mu$. Therefore, $\|f\|_{ess} \leq \|f - f\mathbb{I}_{S_C}\| \leq C$ and, as $C \geq C_0$ was arbitrary, $\|f\|_{ess} \leq C_0$. Now let $C < C_0$ and suppose that $\|f\|_{ess} \leq C$. Then for all $\varepsilon > 0$ there exists a function $f_\varepsilon \in \mathcal{N}_\mu$ such that $\|f + f_\varepsilon\|_\infty \leq C + \varepsilon$. In other words, the set of all $x \in M$ for which $\|f(x)\| > C + \varepsilon$ is a null-set from which $C_0 \leq C + \varepsilon$. As $\varepsilon > 0$ was arbitrary, we find $C_0 \leq C$, contradicting for all $C < C_0$ the assumption $\|f\|_{ess} \leq C$. It follows $\|f\|_{ess} = C_0$.

2. Write f^s in the form $\sum\limits_{i=1}^n v_i \mathbb{I}_{S_i}$ for pairwise disjoint sets $S_i \in \Sigma$. Choosing i such that $\|v_i\|$ is the maximum of all $\|v_j\|$ for which S_j is not a null-set (cf. (B.1.1)), and letting $v = v_i$ and $S = S_i$, we have $f^s\mathbb{I}_S = v$ and $\|f^s\|_{ess} = \|v\|$. It follows that $\|f\|_{ess} - \|v\| = \|f\|_{ess} - \|f^s\|_{ess} \leq \|f - f^s\|_{ess} \leq \varepsilon$. ■

B.1.4 Remark. Allowing the value ∞, the right-hand side of Proposition B.1.3(1) makes sense also for all measurable functions $f\colon M \to V$. This is the way, how $\|\bullet\|_{ess}$ is defined, usually. Here we recover $\|\bullet\|_{ess}$ as a quotient norm, and it is gifted by Remark A.1.6 that the quotient space is a Banach space.

If f is measurable, then $\|f\|_{ess}$ is finite, if and only if there is a measurable function f_0 which is 0 μ–a.e. and $f + f_0$ is bounded. Therefore, we call $L^\infty(M,V)$ the space of *essentially bounded* functions, although it, actually, consists of equivalence classes of functions.

B.1.5 Multiplication operators to essentially bounded functions. If $\varphi\colon M \to \mathcal{B}(V)$ is a bounded measurable function, then the associated multiplication operator on $\mathfrak{F}(M,V)$ leaves invariant the subspaces $\mathcal{L}^\infty(M,V)$ and $\mathcal{N}_\mu$, and, therefore, gives rise to operators on $\mathcal{L}^\infty(M,V)$ and $L^\infty(M,V)$. The norm of the former is $\|\varphi\|_\infty$ and (making use of Proposition B.1.3(2)) the norm of the latter is $\|\varphi\|_{ess}$. We continue denoting both operators by the same symbol φ, but the norms must be distinguished clearly. The canonical mapping from the Banach algebra $\mathcal{L}^\infty(M,\mathcal{B}(V))$ to the Banach algebra $L^\infty(M,\mathcal{B}(V))$ is a contractive surjective homomorphism.

B.1.6 Lemma. *A contractive representation ρ of $L^\infty(M,\mathcal{B}(V))$ by bounded operators on a Banach space W is isometric, if and only if $\|\rho(b\mathbb{I}_S)\| \geq \|b\|$ for all non-null-sets $S \in \Sigma$ and all $b \in \mathcal{B}(V)$.*

Proof. Of course, the condition is necessary. So assume that the condition is satisfied. Let $\varphi^s \in \mathfrak{E}(M, \mathcal{B}(V))$ and choose $b \in \mathcal{B}(V)$ and a non-null-set $S \in \Sigma$ such that $\mathbb{I}_S\varphi = b\mathbb{I}_S$ and $\|\varphi^s\|_{ess} = \|b\|$ like in the proof of Proposition B.1.3(2). Then

$$\|\rho(\varphi^s)\| \;=\; \|\rho(\mathbb{I}_S)\|\,\|\rho(\varphi^s)\| \;\geq\; \|\rho(\mathbb{I}_S\varphi^s)\| \;=\; \|\rho(b\mathbb{I}_S)\| \;\geq\; \|b\| \;=\; \|\varphi^s\|_{ess}\,.$$

As ρ is assumed contractive, we find $\|\rho(\varphi^s)\| = \|\varphi^s\|_{ess}$ for simple functions, which, clearly, extends to all of $L^\infty(M, \mathcal{B}(V))$. ■

B.1.7 Observation. All preceding results about the Banach algebras $\mathcal{L}^\infty(M, \mathcal{B}(V))$ and $L^\infty(M, \mathcal{B}(V))$, first, restrict to the Banach algebras $\mathcal{L}^\infty(M, \mathcal{B})$ and $L^\infty(M, \mathcal{B})$ where $\mathcal{B}$ is a Banach subalgebra of $\mathcal{B}(V)$ and, then, extend to the case where $\mathcal{B}$ is an arbitrary Banach algebra, by considering $\mathcal{B}$ as a subset of $\mathcal{B}(\mathcal{B})$ or $\mathcal{B}(\widetilde{\mathcal{B}})$ (depending on whether $\mathcal{B}$ is unital or not) where an element $b \in \mathcal{B}$ acts on $\mathcal{B}$ or $\widetilde{\mathcal{B}}$ via left multiplication. (Observe that this embedding is an isometry. Here is the only place where we use the unitization $\widetilde{\mathcal{B}} = \mathcal{B} \oplus \mathbb{C}\widetilde{\mathbf{1}}$ of a non-unital Banach algebra which is normed via $\|b \oplus \lambda\mathbf{1}\| = \|b\| + |\lambda|$. This norm does, in general, not coincide with the unique C^*–norm on the unitization of a non-unital C^*–algebra as discussed in Appendix A.8.) If $\mathcal{B}$ is a C^*–algebra, then it is not necessary to consider the unitization, because already the embedding $\mathcal{B} \to \mathcal{B}(\mathcal{B})$ is isometric. Obviously, in this case both $\mathcal{L}^\infty(M, \mathcal{B})$ and $L^\infty(M, \mathcal{B})$ are C^*–algebras with pointwise involution.

B.1.8 Bochner integrable functions. Denote by $\mathfrak{E}_0(M, V)$ the subspace of $\mathfrak{E}(M, V)$ generated linearly by the functions $v\mathbb{I}_S$ where $\mu(S) < \infty$. It is routine to see that

$$\int \;:\; \sum_{i=1}^n v_i \mathbb{I}_{S_i} \;\longrightarrow\; \sum_{i=1}^n v_i \mu(S_i).$$

is a well-defined linear mapping $\mathfrak{E}_0(M, V) \to V$. We also write $\int f = \int f(x)\,\mu(dx)$. Assuming that the S_i are pairwise disjoint, we see that $\int$ is *monotone*, i.e. $f \geq 0$ implies $\int f \geq 0$, for $\mathbb{C}$–valued (or C^*–algebra–valued) functions. Consequently, $\|\bullet\|_1 : f \mapsto \int \|f\|$ (where $\|f\|$ means the function $x \mapsto \|f(x)\|$ in $\mathfrak{E}_0(M)$) defines a seminorm on $\mathfrak{E}_0(M, V)$ and $\|f\|_1 \geq \left\|\int f\right\|$. Obviously, $\|f\|_1 = 0$ if and only if $f \in \mathcal{N}_\mu$. In other words, defining the space $L^1_B(M, V)$ of (equivalence classes of) *Bochner integrable functions* as the completion of $\mathfrak{E}_0(M, V)/(\mathfrak{E}_0(M, V) \cap \mathcal{N}_\mu)$ in $\|\bullet\|_1$, the integral gives rise to a norm–1–mapping $L^1_B(M, V) \to V$, the *Bochner integral.* (One can show that the Bochner integrable functions have measurable representatives in $\mathfrak{F}(M, V)$; see [DU77].) In the cases of $V = \mathbb{C}$ we recover the usual Banach space $L^1_B(M) = L^1(M)$ of integrable functions.

We use the preceding L^1–norm $\|\bullet\|_1$ only in this appendix. In the remainder of these notes $\|\bullet\|_1$ stands for the ℓ^1–norm on the direct sum of Banach spaces as introduced in Appendix A.2. For $M = \mathbb{N}$ the latter is a special case of the former.

$\mathfrak{E}_0(M)$ is a semi-Hilbert space with semiinner product $\langle f, g\rangle = \int(\overline{f}g)$. More generally, let $f \in \mathfrak{E}_0(M,V)$. Applying Cauchy-Schwartz inequality to the semiinner product $\langle \|f\|, \|f\|\rangle$ it follows that $\|\bullet\|_2 : f \mapsto \sqrt{\int \|f\|^2}$ defines a seminorm on $\mathfrak{E}_0(M,V)$. We define the space $L^2_B(M,V)$ of (equivalence classes of) *Bochner square integrable functions* as the completion of $\mathfrak{E}_0(M,V)/(\mathfrak{E}_0(M,V) \cap \mathcal{N}_\mu)$ in $\|\bullet\|_2$. (Also Bochner square integrable functions have measurable representatives in $\mathfrak{F}(M,V)$. Utilizing *Minkowski inequality*, also L^p–versions can be done.) Also here in the case $V = \mathbb{C}$ we recover the usual space $L^2_B(M) = L^2(M)$ of square integrable functions.

B.1.9 Example. If $j\colon C \times D \to V$ is a bounded bilinear mapping, then for all $f \in L^2_B(M,C)$, $g \in L^2_B(M,D)$, the function $x \mapsto j(f(x), g(x))$ is in $L^1_B(M,V)$ and $(f,g) \mapsto \int j(f,g)$ is a bounded bilinear mapping with norm less than or equal $\|j\|$ (actually, equal if Σ contains at least one non-null-set with finite measure).

For instance, if E is a Hilbert $\mathcal{B}$–module than $j(v^*, w) = \langle v, w\rangle$ is a bilinear mapping on $E^* \times E$ and $\|j\| = 1$. Therefore, $\langle f, g\rangle = \int\langle f(x), g(x)\rangle$ defines a $\mathcal{B}$–valued inner product on $L^2_B(M,E)$. However, as we see in Example 4.3.12, $L^2_B(M,E)$ is, usually, not complete in its inner product norm. $L^2_B(M,E)$ is a (dense) pre-Hilbert $\mathcal{B}$–submodule of the exterior tensor product $L^2(M,E)$. Example 4.3.13 shows that $L^2(M,E)$ contains elements which have not a representative in $\mathfrak{F}(M,E)$.

B.1.10 Example. Let $\varphi \in \mathfrak{E}(M, \mathcal{B}(V))$ and $f \in \mathfrak{E}_0(M,V)$. Then φ acts on f by pointwise multiplication from the left and the result is again a function φf in $\mathfrak{E}_0(M,V)$. By the simple estimate

$$\|\varphi f\|_k \;\leq\; \|\varphi\|_{ess} \|f\|_k \quad \text{for} \quad k = 1, 2 \tag{B.1.2}$$

we see that this left action extends to a contractive representation of $L^\infty(M, \mathcal{B}(V))$ (and, of course, of $\mathcal{L}^\infty(M, \mathcal{B}(V))$) on $L^k_B(M,V)$. On the other hand, this representation is not norm decreasing (for $\|\bullet\|_{ess}$) on functions $b\mathbb{I}_S$ (because $\|(b\mathbb{I}_S)(v\mathbb{I}_S)\|_k = \|bv\mathbb{I}_S\|_k = \|bv\|\,\mu(S)^{\frac{1}{k}}$) and, therefore, isometric by Lemma B.1.6. Suppose V is a *Banach $\mathcal{A}$–$\mathcal{B}$–module* for some Banach algebras $\mathcal{A}$ and $\mathcal{B}$ (i.e. $\|avb\| \leq \|a\|\,\|v\|\,\|b\|$). Then analogues of the statements for pointwise left multiplication are true also for pointwise right multiplication $f \mapsto f\varphi$ so that $L^k_B(M,V)$ is a Banach $L^\infty(M,\mathcal{A})$–$L^\infty(M,\mathcal{B})$–module. By Example 4.3.12, this is not true for $L^2(M,E)$ when E is a Hilbert $\mathcal{A}$–$\mathcal{B}$–module.

B.1.11 Strong topology and strongly bounded functions. On $\mathfrak{E}(M, \mathcal{B}(V))$ we can introduce a *strong topology* which is generated by the family $\varphi \mapsto \|\varphi\|_v = \|\varphi v\|$ $(v \in V)$ of seminorms. By $\mathcal{L}^{\infty,s}(M, \mathcal{B}(V))$ we denote the strong closure of $\mathfrak{E}(M, \mathcal{B}(V))$ in $\mathfrak{F}(M, \mathcal{B}(V))$.

By the *principle of uniform boundedness*, any function f in this closure must be bounded, so that $\mathcal{L}^{\infty,s}(M, \mathcal{B}(V))$ is a Banach subspace of $\mathfrak{F}^\infty(M, \mathcal{B}(V))$. We call $\mathcal{L}^{\infty,s}(M, \mathcal{B}(V))$ the space of ***strongly bounded*** functions. As the strong completion of $\mathcal{B}(V)$ is $\mathcal{L}(V)$, the space $\mathcal{L}^{\infty,s}(M, \mathcal{B}(V))$ is strongly complete, if and only if V is finite-dimensional (and M is non-empty).

If $\varphi \in \mathcal{L}^{\infty,s}(M, \mathcal{B}(V))$, then for each $f \in \mathcal{L}^\infty(M, V)$ also the function φf is measurable as limit of measurable functions. Of course, $\|\varphi f\| \leq \|\varphi\| \, \|f\|$. The mapping $f \mapsto \varphi f$ sends elements of $\mathcal{N}_\mu$ to elements of $\mathcal{N}_\mu$ so that

$$\|\varphi f\|_{ess} \;=\; \inf_{g \in \mathcal{N}_\mu} \|\varphi f + g\| \;\leq\; \inf_{g \in \mathcal{N}_\mu} \|\varphi f + \varphi g\| \;\leq\; \|\varphi\| \inf_{g \in \mathcal{N}_\mu} \|f + g\| \;=\; \|\varphi\| \, \|f\|_{ess}.$$

Similarly, we have the inequalities $\|\varphi f\|_k \leq \|\varphi\| \, \|f\|_k$ for $f \in L^k_B(M, V)$ $(k = 1, 2)$. However, φ itself need not be measurable, so if we divide out only those functions which are 0 μ–a.e., then this is not enough to guarantee faithfulness of representations like in Example B.1.10. We could divide out the functions which are strongly 0 μ–a.e., but we content ourselves with the space $\mathcal{L}^{\infty,s}$ and do not define $L^{\infty,s}$.

B.1.12 Observation. $\mathcal{L}^{\infty,s}(M, \mathcal{B}(V))$ is a Banach algebra. Indeed, let φ and ψ be in $\mathcal{L}^{\infty,s}(M, \mathcal{B}(V))$, and let $\big(\varphi_\lambda\big)_{\lambda \in \Lambda}$ and $\big(\psi_\varkappa\big)_{\varkappa \in K}$ be nets in $\mathfrak{E}(M, \mathcal{B}(V))$ approximating φ and ψ, respectively, strongly. For each $\varepsilon > 0$ and $v \in V$ choose $\varkappa_0 \in K$ such that $\|(\psi - \psi_\varkappa)v\| < \frac{\varepsilon}{2\|\varphi\|}$ for all $\varkappa \geq \varkappa_0$. For each $\varkappa$ choose $\lambda(\varkappa)$ such that $\big\|(\varphi - \varphi_{\lambda(\varkappa)})\psi_\varkappa v\big\| < \frac{\varepsilon}{2}$. (This is possible, because the function $\psi_\varkappa v$ takes only finitely many values in V.) It follows that

$$\big\|(\varphi\psi - \varphi_{\lambda(\varkappa)}\psi_\varkappa)v\big\| \;\leq\; \|\varphi(\psi - \psi_\varkappa)v\| + \big\|(\varphi - \varphi_{\lambda(\varkappa)})\psi_\varkappa v\big\| \;<\; \varepsilon$$

for all $\varkappa \geq \varkappa_0$. In other words, the net $\big(\varphi_{\lambda(\varkappa)}\psi_\varkappa\big)_{\varkappa \in K}$ converges strongly to $\varphi\psi$.

B.1.13 Remark. If we consider $\mathfrak{E}(M, \mathcal{B}^a(E))$ for some Hilbert module E, then we may introduce the ***–strong topology**, i.e. the topology generated by the two families $\varphi \mapsto \|\varphi\|_v$ and $\varphi \mapsto \|\varphi^*\|_v$ $(v \in E)$ of seminorms. In this case, the $*$–strong closure of $\mathfrak{E}(M, \mathcal{B}^a(E))$ in $\mathfrak{F}(M, \mathcal{B}^a(E))$ is $*$–strongly complete, because $\mathcal{B}^a(E)$ is $*$–strongly complete. In other words, $\mathcal{L}^{\infty,s}(M, \mathcal{B}^a(E)) \cap \mathcal{L}^{\infty,s}(M, \mathcal{B}^a(E))^*$ is a $*$–strongly complete C^*–algebra.

B.1.14 Finite measures. Suppose M is a ***finite*** measure space (i.e. $\mu(M) < \infty$). Then $\mathfrak{E}_0(M, V) = \mathfrak{E}(M, V)$. Similar to (B.1.2), we find

$$\big\|\textstyle\int f\big\| \;\leq\; \|f\|_{ess}\, \mu(M) \;(\leq\; \|f\|\, \mu(M)).$$

for all $f \in \mathfrak{E}(M, V)$. Therefore, $L^\infty(M, V) \subset L^k_B(M, V)$ and $\|f\|_k \leq \|f\|_{ess}\, \mu(M)^{\frac{1}{k}}$ $(k = 1, 2)$. Moreover, if $S = \{x \in M \colon \|f(x)\| \leq 1\}$, then $\|f\|_1 \leq \|\mathbb{I}_S f\|_1 + \|\mathbb{I}_{S^\complement} f\|_1 \leq \mu(M) +$

$\|f\|_2$ so that $L^2_B(M,V) \subset L^1_B(M,V)$. In particular, $\int f$ is a well-defined bounded mapping on all these spaces.

If a net $\big(\varphi_\lambda\big)_{\lambda\in\Lambda}$ of functions in $\mathfrak{E}(M,\mathcal{B}(V))$ converges strongly to $\varphi \in \mathcal{L}^{\infty,s}(M,\mathcal{B}(V))$, then for each $v \in V$ the net $\int(\varphi_\lambda v)$ converges to $\int(\varphi v)$. In other words, $\int \varphi_\lambda$ converges in the strong topology of $\mathcal{B}(V)$ to the operator $\int \varphi\colon v \mapsto \int(\varphi v)$.

B.1.15 Local variants. Let K be a subset of M (usually, compact intervals in $\mathbb{R}$). Then the multiplication operator $\mathbb{I}_K$ defines a projection onto a subspace of $\mathfrak{F}(M,V)$. Identifying $\mathbb{I}_K f$ and $f \upharpoonright K$, we may consider $\mathfrak{F}(K,V)$ as the subspace $\mathbb{I}_K\mathfrak{F}(M,V)$ of $\mathfrak{F}(M,V)$. This identification turns over isometrically to the several (semi)normed subspaces of $\mathfrak{F}(M,V)$ and the projection $\mathbb{I}_K$ always has norm 1. (This implies that the norm on these spaces is admissible in the sense of Definition A.2.1, when we decompose them according to a decomposition of M into families of pairwise disjoint subspaces; cf. B.1.17.) By $\|f\|^K = \|\mathbb{I}_K f\|$ we denote the restriction of the supremum norm to the subset K. We use similar notations for all other (semi)norms. We also denote $\int_K f = \int(\mathbb{I}_K f)$.

Suppose M is a locally compact space (usually, $\mathbb{R}$ or $\mathbb{R}^d$). The space $\mathfrak{F}^\infty_{loc}(M,V)$ of *locally bounded functions* consists of all function $f \in \mathfrak{F}(M,V)$ such that $\mathbb{I}_K f \in \mathfrak{F}^\infty(K,V)$ for all compact subsets K of M. We may consider $\mathfrak{F}^\infty_{loc}(M,V)$ as the projective limit of the family $\big(\mathfrak{F}^\infty(K,V)\big)$. Similar considerations lead to local variants of the other spaces. We may equip these spaces with the respective projective limit topology. In other words, a net of functions converges, if the net of restrictions to K converges for all compact subsets K of M.

B.1.16 Example. If μ is a *Borel measure* (i.e. any point in M has an open neighbourhood U such that $\mu(U) < \infty$ so that, in particular, $\mu(K) < \infty$ for all compact subsets K of M), then by B.1.14

$$L^\infty_{loc}(M,V) \;\subset\; L^2_{B,loc}(M,V) \;\subset\; L^1_{B,loc}(M,V).$$

In particular, $\int_K$ is a linear mapping on all three spaces and bounded in the respective norm. Recall that in all spaces $\mathfrak{E}(M,V)$ is a dense subspace in the respective norm. Therefore, each space is dense in each space in which it is included. Moreover, $\int_K$ extends in the respective strong topologies to $\mathcal{L}^{\infty,s}_{loc}(M,\mathcal{B}(V))$.

B.1.17 Decomposition. Let (M,Σ,μ) be a measure space, and let $\mathfrak{C}$ be a *disjoint covering* of M (i.e. a family of disjoint subsets of M such that $\bigcup\limits_{C\in\mathfrak{C}} C = M$). Then the linear span of all $L^1_B(C,V)$ ($C \in \mathfrak{C}$) may be identified isometrically with their ℓ^1–direct sum (see Appendix A.2), and its closure consists of all families $\big(f^{(C)}\big)_{C\in\mathfrak{C}}$ of vectors $f_C \in L^1_B(C,V)$ for which

$\|f^{(C)}\|$ is absolutely summable over $\mathfrak{C}$ and the net $(\mathfrak{p}_{\mathfrak{C}_0})_{\mathfrak{C}_0 \subset \mathfrak{C}, \#\mathfrak{C}_0 < \infty}$ converges strongly to a norm one projection onto this closure. We show that the range of this projection is all of $L^1_B(M,V)$. Since the net $\mathfrak{p}_{\mathfrak{C}_0}$ is bounded, it is sufficient to show that $\lim_{\mathfrak{C}_0} \mathfrak{p}_{\mathfrak{C}_0} f = f$ for all $f = v \mathbb{I}_S$ $(v \in V, S \in \Sigma, \mu(S) < \infty)$, because these form a total subset. By the usual counting argument ($\|\mathbb{I}_C f\| \geq \frac{1}{n}$ for at most finitely many $C \in \mathfrak{C}$), $\|\mathbb{I}_C f\|$ can be different from 0 for at most countably many $C \in \mathfrak{C}$. Now convergence follows by σ–additivity of μ. A similar argument gives the same result for $L^2_B(M,V)$.

B.2 Continuous functions

B.2.1 General properties. If $(M, \mathfrak{O})$ is a topological space and V a Banach space, then we may consider the subspace $\mathcal{C}(M,V) \subset \mathfrak{F}(M,V)$ of *continuous functions*, and the space $\mathcal{C}_b(M,V) = \mathcal{C}(M,V) \cap \mathfrak{F}^\infty(M,V)$ of *bounded* continuous functions. Suppose that for a sequence (f_n) in $\mathcal{C}(M,V)$ we have $\|f_n - f\| \to 0$ for a (unique) function $f \in \mathfrak{F}(M,V)$. Then f is continuous. (Indeed, let $x \in M$ and for $\varepsilon > 0$ choose $N \in \mathbb{N}$ such that $\|f_n - f\| \leq \frac{\varepsilon}{3}$. As f_n is continuous, the inverse image $O = f_n^{-1}\big(U_{\frac{\varepsilon}{3}}(f_n(x))\big)$ of the $\frac{\varepsilon}{3}$–neighbourhood of $f(x)$ is an open set in M containing x. It follows that $\|f(x) - f(y)\| \leq \|f(x) - f_n(x)\| + \|f_n(x) - f_n(y)\| + \|f_n(y) - f(y)\| < \varepsilon$ for all $y \in O$. In other words, $f^{-1}\big(U_\varepsilon(f(x))\big)$ contains the open set O. As this is true for all points $f(x) \in V$, the inverse image of any open subset of V is open, i.e. f is continuous.) Therefore, $\mathcal{C}_b(M,V)$ is a Banach space. If M comes along with the σ–algebra $\mathfrak{B}(M)$ of Borel sets, then all continuous functions are measurable.

Let B be a Banach algebra. Then the product of two functions in $\mathcal{C}(M,B)$ is again in $\mathcal{C}(M,B)$. In particular, $\mathcal{C}_b(M,B)$ is a Banach subalgebra of $\mathfrak{F}^\infty(M,B)$. Clearly, we have $\mathcal{C}_b(M,\mathcal{B}(V))\mathcal{C}_b(M,V) = \mathcal{C}_b(M,V)$ so that $\mathcal{C}_b(M,\mathcal{B}(V)) \subset \mathcal{B}(\mathcal{C}_b(M,V))$, isometrically.

B.2.2 Locally compact spaces. If M is compact (and Hausdorff), then $\mathcal{C}(M,V) = \mathcal{C}_b(M,V)$, because the numerical continuous function $x \mapsto \|f(x)\|$ has a maximum on the compact set M. If M is locally compact, then it is the projective limit of its compact subspaces. It follows that $\mathcal{C}(M,V) = \mathcal{C}_{b,loc}(M,V)$. By $\mathcal{C}_c(M,V)$ we denote the space of continuous functions f with *compact support*, i.e. there exists a compact subset K of M such that $\mathbb{I}_K f = f$. By $\mathcal{C}_0(M,V)$ we denote the space of continuous functions f *vanishing at infinity*, i.e. for all $\varepsilon > 0$ there exists a compact subset K of M such that $\|f\|^{K^\complement} < \varepsilon$. Of course, $\mathcal{C}_c(M,V) \subset \mathcal{C}_0(M,V) \subset \mathcal{C}_b(M,V)$. Clearly, $\mathcal{C}_0(M,V)$ is a Banach space. (For $\varepsilon > 0$ approximate the limit $f \in \mathfrak{F}^\infty(M,V)$ of a Cauchy sequence $f_n \in \mathcal{C}_0(M,V)$ by some f_n up to $\frac{\varepsilon}{2}$. Then choose a compact subset $K \subset M$ such that $\|f_n\|^{K^\complement} < \frac{\varepsilon}{2}$. It follows that $\|f\|^{K^\complement} < \varepsilon$.) And $\mathcal{C}_c(M,V)$ is dense in $\mathcal{C}_0(M,V)$. (Making use of the *one-point compactification* and *Uryson's lemma*, on shows that there exists a net of norm-one functions

$\chi_\lambda \in \mathcal{C}_c(M)$, converging on any compact subset K to 1. Therefore, for any $f \in \mathcal{C}_0(M,V)$ the net $\chi_\lambda f \in \mathcal{C}_c(M,V)$ converges in norm to f.)

If B is a (unital) Banach algebra, then $\mathcal{C}_0(M,B)$ is a Banach subalgebra of $\mathcal{C}_b(M,B)$. We have the invariances

$$\begin{aligned}
\mathcal{C}_b(M,B)\mathcal{C}_0(M,B) &= \mathcal{C}_0(M,B)\mathcal{C}_b(M,B) = \mathcal{C}_0(M,B)\\
\mathcal{C}(M,B)\mathcal{C}_c(M,B) &= \mathcal{C}_c(M,B)\mathcal{C}(M,B) = \mathcal{C}_c(M,B)\\
\mathcal{C}_0(M,\mathcal{B}(V))\mathcal{C}_b(M,V) &\subset \mathcal{C}_b(M,\mathcal{B}(V))\mathcal{C}_0(M,V) = \mathcal{C}_0(M,V)\\
\mathcal{C}_c(M,\mathcal{B}(V))\mathcal{C}(M,V) &\subset \mathcal{C}(M,\mathcal{B}(V))\mathcal{C}_c(M,V) = \mathcal{C}_c(M,V).
\end{aligned}$$

However, observe that an element $\varphi \in \mathcal{C}_b(M,\mathcal{B}(V))$ acts as operator with norm $\|\varphi\|$ on both Banach spaces $\mathcal{C}_0(M,V)$ and $\mathcal{C}_b(M,V)$, whereas, an element $\varphi \in \mathcal{C}(M,\mathcal{B}(V))$ which is unbounded also acts as an unbounded operator on the normed space $\mathcal{C}_c(M,V)$.

B.2.3 Strong variants. By $\mathcal{C}^s(M,\mathcal{B}(V))$ we denote the space of *strongly continuous* $\mathcal{B}(V)$–valued functions on M. The bounded portion $\mathcal{C}^s_b(M,\mathcal{B}(V))$ is a strongly closed Banach subalgebra of $\mathfrak{F}^\infty(M,\mathcal{B}(V))$. (From $\|(\varphi(x)\psi(x) - \varphi(y)\psi(y))v\| \leq \|\varphi\|\;\|(\psi(x) - \psi(y))v\| + \|(\varphi(x) - \varphi(y))\psi(y)v\| \to 0$ for $x \to y$ we conclude that $\mathcal{C}^s_b(M,\mathcal{B}(V))$ is an algebra. For any net $\varphi_n \in \mathcal{C}_b(M,\mathcal{B}(V))$ converging strongly to some $\varphi \in \mathfrak{F}^\infty(M,\mathcal{B}(V))$ the limit φv of the continuous functions $\varphi_n v$ is again continuous, i.e. φ is strongly continuous.) If M is locally compact, then already $\mathcal{C}^s(M,\mathcal{B}(V))$ is an algebra. Moreover, $\mathcal{C}^s_c(M,\mathcal{B}(V))$ and $\mathcal{C}^s_0(M,\mathcal{B}(V))$, i.e. the function $\varphi \in \mathcal{C}^s(M,\mathcal{B}(V))$ for which $\varphi v \in \mathcal{C}_c(M,V)$ and $\varphi v \in \mathcal{C}_0(M,V)$, are an ideal and a strongly closed ideal, respectively, in $\mathcal{C}^s_b(M,\mathcal{B}(V))$. We have

$$\begin{aligned}
\mathcal{C}^s_0(M,\mathcal{B}(V))\mathcal{C}_b(M,V) &\subset \mathcal{C}^s_b(M,\mathcal{B}(V))\mathcal{C}_0(M,V) = \mathcal{C}_0(M,V)\\
\mathcal{C}^s_c(M,\mathcal{B}(V))\mathcal{C}(M,V) &\subset \mathcal{C}^s(M,\mathcal{B}(V))\mathcal{C}_c(M,V) = \mathcal{C}_c(M,V).
\end{aligned}$$

Also here the representations of $\mathcal{C}^s_b(M,\mathcal{B}(V))$ on $\mathcal{C}_b(M,V)$, $\mathcal{C}_0(M,V)$, and $\mathcal{C}_c(M,V)$ are isometric.

B.2.4 Example. If $V = E$ is a Hilbert $\mathcal{B}$–module, then $\mathcal{C}_b(M,E)$ is a Hilbert $\mathcal{C}_c(M,\mathcal{B})$–module with inner product $\langle f,g\rangle(t) = \langle f(t),g(t)\rangle$ as in Section 12.5. (It is complete, because the identification is isometric.) The canonical action of $\mathcal{C}^s_b(M,\mathcal{B}^r(E))$ defines a faithful representation by operators in $\mathcal{B}^r(\mathcal{C}_b(M,E))$. It restricts to a faithful representation of $\mathcal{C}^s_b(M,\mathcal{B}^a(E)) \cap \mathcal{C}^s_b(M,\mathcal{B}^a(E))^*$ by operators in $\mathcal{B}^a(\mathcal{C}_b(M,E))$. It is easy to see that this representation is onto, and that the property of $\mathcal{B}^a(E)$ to be $*$–strongly complete turns over to the $*$–strong topology of $\mathcal{B}^a(\mathcal{C}_b(M,E))$ considered as $\mathcal{C}^s_b(M,\mathcal{B}^a(E)) \cap \mathcal{C}^s_b(M,\mathcal{B}^a(E))^*$.

B.3 Some lattices

In the sequel, we are interested in showing that certain subsets of $\mathfrak{E}(M, V)$, related to indicator functions $I\!I_K$ where K stems from some sublattice of subsets of the *power set* $\mathcal{P}(M)$, are dense in the spaces introduced so far. Here regularity of the measure plays a crucial role. An exposition in full generality would require deeper theorems from the theory of *polish* or locally compact measure spaces. As we need only the case $M = \mathbb{R}$ with the Lebesgue measure (or products $\mathbb{R}^d$), we restrict ourselves to that case, and introduce the lattice of interval partitions together with some related lattices, which play a role in Part III.

B.3.1 Definition. By $\mathbb{P} = \big\{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^n \colon n \in \mathbb{N}, t_0 < \ldots < t_n\big\}$ we denote set of *interval partitions of* $\mathbb{R}$. For each $P \in \mathbb{P}$ we define the *norm* $\|P\| = \max_{i=1,\ldots,n}(t_i - t_{i-1})$. For $K = [\tau, \mathfrak{T}]$ $(\tau < \mathfrak{T})$ we denote by $\mathbb{P}_K = \big\{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^n \colon n \in \mathbb{N}, \tau = t_0 < \ldots < t_n = \mathfrak{T}\big\}$ the *interval partitions of* K.

Each $P \in \mathbb{P}$ gives rise to a *partition* of $\mathbb{R}$ into finitely many non-empty intervals $(-\infty, t_0), [t_0, t_1), \ldots, [t_{n-1}, t_n), [t_n, \infty)$, thus, justifying the name interval partitions. The elements of $\mathbb{P}_K$ correspond to partitions $[t_0, t_1), \ldots, [t_{n-1}, t_n]$ of the interval K. (Of course, also other choices of for in- or excluding the end points are possible, but the given choice is most convenient for our purposes.) Thinking only of the tuples P, clearly, $\mathbb{P}_K$ is a subset of $\mathbb{P}$. We have a natural notion of *inclusion*, *union*, and *intersection* of tuples. By inclusion we define a *partial order*. Clearly, $\mathbb{P}$ is a latice with intersection of two tuples being their unique minimum and union of two tuples being their unique maximum. Of course, $\mathbb{P}_K$ is a sublattice of $\mathbb{P}$. Notice that $\mathbb{P}_K$ has a global minimum $(\tau, \mathfrak{T})$, whereas $\mathbb{P}$ has not. None of the two has a global maximum. For each $\varepsilon > 0$ and each partition P there exists $P' \geq P$ such that $\|P'\| < \varepsilon$. In particular, for two (or finitely many) partitions P_i there exists $P' \geq \max_j P_j \geq P_i$ for all i such that $\|P'\| < \varepsilon$. Often, this enables us transform limits over $\mathbb{P}_K$ into limits $\|P\| \to 0$. Such limits will carry the prefix *equi*.

There exist, essentially, two ways of looking at an interval partition. Firstly, as above, with emphasis on the end points of each subinterval. Secondly, with emphasis on the length of each subinterval. The different pictures are useful for different purposes. In the first picture we have seen easily that the interval partitions form a lattice. The second point of view is more useful in the context of product systems in Part III. We comment on this point in Observation B.3.4. We concentrate on the interval $[0, t]$, and it is convenient to reverse the order of tuples. Finally, we require that the involved *time points* come from $\mathbb{T}$ which can be $\mathbb{R}_+$ or $\mathbb{N}_0$.

Let $t > 0$ in $\mathbb{T}$. We define $\mathbb{I}_t$ as the set of all tuples $\{(t_n, \dots, t_1) \in \mathbb{T}^n : n \in \mathbb{N}, t = t_n > \dots > t_1 > 0\}$. Obviously, $\mathbb{I}_t$ is isomorphic to $\mathbb{P}_{[0,t]}$ if $\mathbb{T} = \mathbb{R}_+$ and to a sublattice of $\mathbb{P}_{[0,t]}$ if $\mathbb{T} = \mathbb{N}_0$.

We define $\mathbb{J}_t$ to be the set of all tuples $\mathfrak{t} = (t_n, \dots, t_1) \in \mathbb{T}^n$ ($n \in \mathbb{N}, t_i > 0$) having *length*

$$|\mathfrak{t}| := \sum_{i=1}^{n} t_i = t.$$

For two tuples $\mathfrak{s} = (s_m, \dots, s_1) \in \mathbb{J}_s$ and $\mathfrak{t} = (t_n, \dots, t_1) \in \mathbb{J}_t$ we define the *joint tuple* $\mathfrak{s} \smile \mathfrak{t} \in \mathbb{J}_{s+t}$ by

$$\mathfrak{s} \smile \mathfrak{t} = ((s_m, \dots, s_1), (t_n, \dots, t_1)) = (s_m, \dots, s_1, t_n, \dots, t_1).$$

We equip $\mathbb{J}_t$ with a partial order by saying $\mathfrak{t} \geq \mathfrak{s} = (s_m, \dots, s_1)$, if for each j $(1 \leq j \leq m)$ there are (unique) $\mathfrak{s}_j \in \mathbb{J}_{s_j}$ such that $\mathfrak{t} = \mathfrak{s}_m \smile \dots \smile \mathfrak{s}_1$.

We extend the definitions of $\mathbb{I}_t$ and $\mathbb{J}_t$ to $t = 0$, by setting $\mathbb{I}_0 = \mathbb{J}_0 = \{()\}$, where $()$ is the *empty tuple*. For $\mathfrak{t} \in \mathbb{J}_t$ we put $\mathfrak{t} \smile () = \mathfrak{t} = () \smile \mathfrak{t}$.

B.3.2 Proposition. *The mapping* $\mathfrak{o} \colon (t_n, \dots, t_1) \mapsto \left(\sum_{i=1}^{n} t_i, \dots, \sum_{i=1}^{1} t_i\right)$ *is an order isomorphism* $\mathbb{J}_t \to \mathbb{I}_t$.

Proof. Of course, $\mathfrak{o}$ is bijective. Obviously, the image in $\mathbb{I}_t$ of a tuple $(|\mathfrak{s}_m|, \dots, |\mathfrak{s}_1|)$ in $\mathbb{J}_t$ is *contained* in the image of $\mathfrak{s}_m \smile \dots \smile \mathfrak{s}_1$. Conversely, let $(s_m, \dots, s_1)$ be a tuple in $\mathbb{I}_t$ and $(t_n, \dots, t_1) \geq (s_m, \dots, s_1)$. Define a function $\mathfrak{n} \colon \{0, \dots, m\} \to \{0, \dots, n\}$ by requiring $t_{\mathfrak{n}(j)} = s_j$ $(j \geq 1)$ and $\mathfrak{n}(0) = 0$. Set $\mathfrak{t} = \mathfrak{o}^{-1}(t_n, \dots, t_1)$ and $\mathfrak{s} = \mathfrak{o}^{-1}(s_m, \dots, s_1)$. Furthermore, define $\mathfrak{s}_j = \mathfrak{o}^{-1}(t_{\mathfrak{n}(j)}, \dots, t_{\mathfrak{n}(j-1)+1})$ $(j \geq 1)$. Then $\mathfrak{t} = \mathfrak{s}_m \smile \dots \smile \mathfrak{s}_1 \geq (|\mathfrak{s}_m|, \dots, |\mathfrak{s}_1|) = \mathfrak{s}$. ∎

B.3.3 Remark. For ordered sets it is sufficient to check only, whether the set isomorphism $\mathfrak{o}$ preserves order. Nothing like this is true for partial orders. Consider, for instance, $\mathbb{N}$ equipped with its natural order and $\bar{\mathbb{N}} = \mathbb{N}$ as set, but equipped with the partial order $n \leq m$ for $n = 1$ or $n = m$ (and all other pairs of different elements are not comparable). Clearly, the identification mapping is not an order isomorphism.

B.3.4 Observation. $\mathbb{I}_t$ is a lattice with the union of two tuples being their unique least upper bound and the intersection of two tuples being their unique greatest lower bound. In particular, $\mathbb{I}_t$ is directed increasingly. (t) is the unique minimum of $\mathbb{I}_t$ $(t > 0)$. By Proposition B.3.2 all these assertions are true also for $\mathbb{J}_t$.

The reason why we use the lattice $\mathbb{J}_t$ instead of $\mathbb{I}_t$ is the importance of the operation $\smile$. Notice that $\smile$ is an operation not on $\mathbb{J}_t$, but rather an operation $\mathbb{J}_s \times \mathbb{J}_t \to \mathbb{J}_{s+t}$. We can say two tuples $\mathfrak{s} \in \mathbb{J}_s$ and $\mathfrak{t} \in \mathbb{J}_t$ are just *glued together* to a tuple $\mathfrak{s} \smile \mathfrak{t} \in \mathbb{J}_{s+t}$. Before we can *glue together* the corresponding tuples $\mathfrak{o}(\mathfrak{s}) \in \mathbb{I}_s$ and $\mathfrak{o}(\mathfrak{t}) \in \mathbb{I}_t$, we first must *shift* all points in $\mathfrak{o}(\mathfrak{s})$ by the time t. (This behaviour is not surprising. Recall that the t_i in a tuple in $\mathbb{J}_t$ stand for time differences. These do not change under time shift. Whereas the t_i in a tuple in $\mathbb{I}_t$ stand for time points, which, of course, change under time shift.) Hence, in the description by $\mathbb{I}_t$ the time shift must be acted out explicitly, whereas in the description by $\mathbb{J}_t$ the time shift is intrinsic and works automatically. Our decision to use $\mathbb{J}_t$ instead of the more common $\mathbb{I}_t$ is the reason why in many formulae where one intuitively would expect a time shift, no explicit time shift appears. It is, however, always encoded in our notation. (Cf., for instance, Equations (7.1.1), (12.1.1), and (12.3.1).)

B.4 Bilinear mappings on spaces of Banach space valued functions

In Part IV we are dealing with integrals $\int_\tau^{\mathfrak{T}} A_t\, dI_t\, B_t$ where the integrands A and B are processes of operators on a Fock module and $dI_t = I_{t+dt} - I_t$ are differentials constructed from certain basic integrator processes I. It is typical for calculus on modules that the integrands do not commute with the differentials. Therefore, we have to work with two integrands A and B. In other words, we investigate bilinear mappings $(A, B) \mapsto \int_\tau^{\mathfrak{T}} A_t\, dI_t\, B_t$.

In this appendix we collect some general results concerning conditions under which such an integral exists. The basic spaces for the integrands are continuous functions or strongly continuous functions. These may be approximated on compact intervals by *right continuous step functions*, whose closure consists right continuous functions with left limits (càdlàg functions). Therefore, we provide the basics about such functions.

B.4.1 Definition. Let $K = [\tau, \mathfrak{T}]$ $(\tau < \mathfrak{T})$ be a compact interval. Let V be a Banach space. For any function $A\colon K \to V$ and any partition $P \in \mathbb{P}_K$ we set

$$A_t^P = \sum_{i=1}^n A_{t_{k-1}} I\!I_{[t_{k-1}, t_k)}(t).$$

By $\mathfrak{S}^r(K, V) \subset \mathfrak{E}(K, V)$ we denote the space of *right continuous V–valued step functions on* $[\tau, \mathfrak{T}]$. In other words, for each $A \in \mathfrak{S}^r(K, V)$ there exists a partition $P \in \mathbb{P}_K$ such that $A = A^P + A_{\mathfrak{T}} I\!I_{[\mathfrak{T},\mathfrak{T}]}$ (and, of course, $A = A^{P'} + A_{\mathfrak{T}} I\!I_{[\mathfrak{T},\mathfrak{T}]}$ for all $P' \in \mathbb{P}_K$ with $P' \geq P$).

By $\mathfrak{R}(K, V)$ we denote the space of *bounded right continuous V–valued functions with left limit* (or short *càdlàg functions*)on $[\tau, \mathfrak{T}]$.

Let $\mathcal{B}$ be a Banach subalgebra of $\mathcal{B}(V)$. By $\mathfrak{R}^s(K, \mathcal{B})$ we denote the space of *bounded strongly right continuous $\mathcal{B}$–valued functions with strong left limit* (or short *strong càdlàg* *functions*) on $[\tau, \mathfrak{T}]$.

B.4.2 Proposition. *1. $\mathfrak{R}(K,V)$ is a Banach subspace of $\mathfrak{F}^\infty(K,V)$, and $\mathcal{C}(K,V)$ is a Banach subspace of $\mathfrak{R}(K,V)$. For all $A \in \mathfrak{R}(K,V)$ we have $A^P + A_{\mathfrak{T}}\mathit{I\!I}_{[\mathfrak{T},\mathfrak{T}]} \to A$. In other words, the step functions $\mathfrak{S}^r(K,V)$ form a dense subset of $\mathfrak{R}(K,V)$. Moreover, for each $A \in \mathcal{C}(K,V)$, and each $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$\left\|A - A^P\right\|^{K'} < \epsilon$$

for all compact intervals $K' \subset K$ with $\mathfrak{T} \notin K$ and all $P \in \mathbb{P}_K$ with $\|P\| < \delta$. We say the continuous functions can be approximated by step functions ***equiuniformly****.*

2. Also $\mathfrak{R}^s(K,\mathcal{B})$ is a Banach subspace of $\mathfrak{F}^\infty(K,V)$, and $\mathcal{C}^s(K,\mathcal{B})$ is a Banach subspace of $\mathfrak{R}^s(K,\mathcal{B})$. Moreover, $\mathcal{C}^s(K,\mathcal{B})$ is a strongly closed subset of $\mathfrak{R}^s(K,\mathcal{B})$, and each strongly closed subset of $\mathfrak{R}^s(K,\mathcal{B})$ is also norm closed. For all $A \in \mathfrak{R}^s(K,\mathcal{B})$ we have $A^P + A_{\mathfrak{T}}\mathit{I\!I}_{[\mathfrak{T},\mathfrak{T}]} \to A$ in the strong topology. In other words, the step functions $\mathfrak{S}^r(K,V)$ form a strongly dense subset of $\mathfrak{R}^s(K,\mathcal{B})$. Moreover, for each $A \in \mathcal{C}^s(K,\mathcal{B})$, each $v \in V$, and each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left\|A - A^P\right\|_v^{K'} < \epsilon$$

for all compact intervals $K' \subset K$ with $\mathfrak{T} \notin K$ and all $P \in \mathbb{P}_K$ with $\|P\| < \delta$. We say the strongly continuous functions can be approximated by step functions ***equistrongly****.*

PROOF. $\mathcal{C}(K,V)$ is a Banach space and a subspace of $\mathfrak{R}(K,V)$. A usual $\frac{\varepsilon}{3}$–argument show that also $\mathfrak{R}(K,V)$ is a Banach space. The corresponding statements for $\mathfrak{R}^s(K,\mathcal{B})$ and $\mathcal{C}^s(K,\mathcal{B})$ follow by an application of the *principle of uniform boundedness*, and from the observation that the strong topology is weaker than the norm topology.

Density of the step functions in $\mathfrak{R}(K,V)$ follows by the usual compactness arguments for the intervall $[\tau,\mathfrak{T}]$ (see e.g. [Die85, Section 7.6] for limits of arbitrary step functions), and equiuniform approximation of continuous functions uses standard arguments well-known from Riemann integral building on uniform continuity of continuous functions on compact sets. For the strong versions we just apply these arguments to functions of the form Av in $\mathfrak{R}(K,V)$ and in $\mathcal{C}(K,V)$, respectively. Of course, the statements for K' are just restrictions of the statement for K and thanks to $\mathfrak{T} \notin K'$ we can forget about the appendix $A_{\mathfrak{T}}\mathit{I\!I}_{[\mathfrak{T},\mathfrak{T}]}$. ∎

We want to define an integral

$$\int_\tau^{\mathfrak{T}} A_t \, dI_t \, B_t := \lim_{P \in \mathbb{P}_K} \sum_{k=1}^N A_{t_{k-1}} \, dI_{t_k} \, B_{t_{k-1}} \tag{B.4.1}$$

where $A, B \in \mathfrak{R}^s(K, \mathcal{B})$ and I is some function $K \to \mathcal{B}$ and $dI_{t_k} = I_{t_k} - I_{t_{k-1}}$. Suppose A and B are step functions, i.e. $A = A^{P_A}$ and $B = B^{P_B}$ for suitable $P_A, P_B \in \mathbb{P}_K$. Then

$$\int_\tau^{\mathcal{T}} A_t \, dI_t \, B_t = \sum_{k=1}^N A_{t_{k-1}} \, dI_{t_k} \, B_{t_{k-1}}$$

for all partitions $P \geq \max(P_A, P_B)$. The following proposition is a simple consequence of Proposition B.4.2.

B.4.3 Proposition. *Suppose*

$$(A, B) \longmapsto \int_\tau^{\mathcal{T}} A_t \, dI_t \, B_t$$

is bounded on $\mathfrak{S}^r(K, \mathcal{B}) \times \mathfrak{S}^r(K, \mathcal{B})$. *Then* (B.4.1) *exists*

1. *as equiuniform limit on* $\mathcal{C}(K, \mathcal{B}) \times \mathcal{C}(K, \mathcal{B})$.

2. *as equistrong limit on* $\mathcal{C}(K, \mathcal{B}) \times \mathcal{C}^s(K, \mathcal{B})$.

3. *as uniform limit on* $\mathfrak{R}(K, \mathcal{B}) \times \mathfrak{R}(K, \mathcal{B})$.

4. *as strong limit on* $\mathfrak{R}(K, \mathcal{B}) \times \mathfrak{R}^s(K, \mathcal{B})$.

In Proposition B.4.3 we needed boundedness on step function of a bilinear mapping, but we did not specify further properties of the integrator I_t. Now we introduce a generalized integral under the condition that the *measure* $I_{t+dt} - I_t$ has something like a density.

By B.1.14 for all functions $I' \in \mathcal{L}^\infty_{loc}(\mathbb{R}, \mathcal{B})$ ($I' \in \mathcal{L}^{\infty,s}_{loc}(\mathbb{R}, \mathcal{B})$) we may define $\int_\tau^{\mathcal{T}} I'_s \, ds = \int_K I'$ by (strong) extension from simple functions.

B.4.4 Definition. We say an integrator function $I \colon \mathbb{R} \to \mathcal{B}$ has a locally bounded and a locally strongly bounded ***density*** $I' \in \mathcal{L}^\infty_{loc}(\mathcal{B})$ and $I' \in \mathcal{L}^{\infty,s}_{loc}(\mathcal{B})$, respectively, if

$$I_t - I_0 = \begin{cases} \int_0^t I'_s \, ds & \text{for } t \geq 0 \\ -\int_t^0 I'_s \, ds & \text{otherwise.} \end{cases}$$

B.4.5 Proposition. *Suppose I has a locally bounded **density** I'. Then* (B.4.1) *exists*

1. *as equiuniform limit on* $\mathcal{C}(K, \mathcal{B}) \times \mathcal{C}(K, \mathcal{B})$.

2. *as equistrong limit on* $\mathcal{C}^s(K, \mathcal{B}) \times \mathcal{C}^s(K, \mathcal{B})$.

3. *as uniform limit on* $\mathfrak{R}(K, \mathcal{B}) \times \mathfrak{R}(K, \mathcal{B})$.

4. as strong limit on $\mathfrak{R}^s(K,\mathcal{B}) \times \mathfrak{R}^s(K,\mathcal{B})$.

In all cases we have

$$\int_\tau^{\mathcal{T}} A_t \, dI_t \, B_t = \int_\tau^{\mathcal{T}} A_t I'_t B_t \, dt. \tag{B.4.2}$$

where, obviously, $AI'B \in \mathcal{L}^\infty_{loc}(\mathcal{B})$ *and* $AI'B \in \mathcal{L}^{\infty,s}_{loc}(\mathcal{B})$*, respectively. Formally, we write* $dI_t = I'_t \, dt$*.*

B.4.6 Remark. We call $\int_\tau^{\mathcal{T}} A_t \, dI_t \, B_t$ a ***generalized time integral***. The difference between $\int_\tau^{\mathcal{T}} A_t \, dI_t \, B_t$ and $\int_\tau^{\mathcal{T}} A_t I'_t B_t \, dt$ lies in the Riemann-Stieltjes sums which are suggested by the respective integral. Proposition B.4.5 tells us that $\int_\tau^{\mathcal{T}} A_t \, dI_t \, B_t$ is a limit of the associated Riemann-Stieltjes sums. For $\int_\tau^{\mathcal{T}} A_t I'_t B_t \, dt$, in general, this is not true, already in the scalar case.

PROOF OF PROPOSITION B.4.5. The proof is based on the rude estimate

$$\Big\| \sum_{k=1}^N A_{t_{k-1}} \, dI_{t_k} \, B_{t_{k-1}} \Big\| \le (\mathcal{T} - \tau) \, \|I'\|^K \, \|A\|^K \, \|B\|^K \tag{B.4.3}$$

which holds for arbitrary functions $A, B \colon \mathbb{R} \to \mathcal{B}$ and all partitions $P \in \mathbb{P}_K$. By (B.4.3) we may assume that I' is simple. (Otherwise, replace it by a simple function sufficiently close to I' in $\|\bullet\|^K$.) As simple functions are finite sums over functions of the form $\mathbb{I}_S a$, we even may assume $I' = \mathbb{I}_S a$. Thus, we are reduced to the case

$$\sum_{k=1}^N A_{t_{k-1}} \, dI_{t_k} \, B_{t_{k-1}} = \sum_{k=1}^N d\widetilde{I}_{t_k} \, A_{t_{k-1}} a B_{t_{k-1}}$$

with $\widetilde{I}' = \mathbb{I}_S$. It remains to mention that $AaB \in \mathcal{C}^s(K,\mathcal{B})$, whenever $A, B \in \mathcal{C}^s(K,\mathcal{B})$, and (by similar arguments) that $AaB \in \mathfrak{R}^s(K,\mathcal{B})$, whenever $A, B \in \mathfrak{R}^s(K,\mathcal{B})$. Then the desired convergences follow by Proposition B.4.2 as for the usual Riemann integral. (B.4.2) follows from the observation that the integral on the right-hand side is approximated by simple functions of the form $A^P I' B^P$ (I' simple) in the respective topologies. ■

B.4.7 Example. As mentioned in the preceding proof, the case $I' = 1$ (i.e. $I_t = t$) corresponds to the usual (strong) Riemann integral over the (strongly) continuous function AB. We may use this to define an inner product on $\mathcal{C}_c(\mathbb{R}, E)$ for some Hilbert $\mathcal{B}$–module E, by setting $\langle f, g \rangle = \int \langle f(t), g(t) \rangle \, dt$. If $\mathcal{B}$ is a von Neumann algebra on some Hilbert space G, then E is represented by the Stinespring representation in $\mathcal{B}(G, H)$ where $H = E \,\bar{\odot}\, G$. If we extend the inner product to $\mathcal{C}^s_c(\mathbb{R}, \mathcal{B}(G,H))$, then the integral converges at least weakly.

B.4.8 Remark. It is well-known that the step functions $\mathfrak{S}(\mathbb{R})$ (i.e. linear combinations of indicator functions to bounded intervals) are dense in $L^k(\mathbb{R})$ $(k = 1, 2)$. This is true already for left or right continuous step functions alone. Since the L^k–closure of $\mathcal{C}_c(\mathbb{R})$ contains the step functions, $\mathcal{C}_c(\mathbb{R})$ is also dense $L^k(\mathbb{R})$. Similar statements are true for V–valued functions (including strong versions for $\mathcal{B}$–valued functions).

Appendix C

Hilbert modules over P^*–algebras

In the main part of these notes the semiinner product of a semi-Hilbert $\mathcal{B}$–module takes values in a pre–C^*–algebra $\mathcal{B}$. In a pre–C^*–algebra the positive elements and the positive functionals can be characterized in many equivalent ways. For instance, we can say an element $b \in \mathcal{B}$ is positive, if it can be written in the form b'^*b' for a suitable $b' \in \overline{\mathcal{B}}$. We can also give a weak definition and say that b is positive, if $\varphi(b) \geq 0$ for all positive functionals φ. Here we can say a (bounded) functional on $\mathcal{B}$ is positive, if $\varphi(b^*b) \geq 0$ for all $b \in \mathcal{B}$, but also if $\|\varphi\| = \varphi(\mathbf{1})$. Also if we want to divide out the length-zero elements in a semi-Hilbert module in order to have a pre-Hilbert module, we can either use Cauchy-Schwarz inequality (1.2.1) or we can use a weak Cauchy-Schwarz inequality

$$\varphi(\langle x, y\rangle)\varphi(\langle y, x\rangle) \leq \varphi(\langle y, y\rangle)\varphi(\langle x, x\rangle)$$

(φ running over all positive functionals), because the positive functionals separate the points in a pre–C^*–algebra. It is easy to equip the tensor product of Hilbert modules with an inner product and to show that it is positive; cf. Section 4.2.

In some of our applications we have to consider Hilbert modules over more general $*$–algebras, where the preceding characterizations of positive elements and positive functionals lead to different notions of positivity. The algebraic definition, where only elements of the form b^*b or sums of such are positive, excludes many good candidates for positive elements and, in fact, is too restrictive to include our applications. The weak definition, where $b \geq 0$, if $\varphi(b) \geq 0$ for all positive functionals φ, allows for many positive elements. However, in many cases, for instance, if we want to show positivity of the inner product on the tensor product, this condition is uncontrollable. Here we give an extended algebraic definition as proposed in Accardi and Skeide [AS98], where we put by hand some distinguished elements to be positive and consider a certain convex cone which is generated by these elements. Of course, a suitable choice of these distinguished elements depends highly on the concrete application.

If we want to divide out the length-zero elements, we should require that the positive functionals separate the points of the $*$–algebra. However, also here it turns out that we cannot consider all positive functionals (i.e. functionals φ on $\mathcal{B}$ for which $\varphi(b^*b) \geq 0$ for all $b \in \mathcal{B}$), because we cannot guarantee that these functionals send our distinguished positive cone into the positive reals (see Remark C.1.2).

The connection of our abstract notion of positivity with concrete positivity of certain complex numbers is done by representations. Also representations π have to respect the positivity in the sense that they shoud send elements of the positive cone to operators on a pre-Hilbert space G which are positive in the usual sense (i.e. $\langle g, \pi(b)g\rangle \geq 0$ for all $g \in G$).

Also a left multiplication on a pre-Hilbert module is a representation and, thus, should be compatible with the notion of positivity. Consequently, our definition of positivity of the inner product involves left multiplication so that a two-sided module is no longer a right module with some positive inner product and an additional structure. We have to give an integrated definition of two-sided pre-Hilbert module from the beginning.

C.1 P^*–Algebras

C.1.1 Definition. Let $\mathcal{B}$ be a unital $*$–algebra. We say a subset P of $\mathcal{B}$ is a *$\mathcal{B}$–cone*, if $b \in P$ implies $b'^*bb' \in P$ for all $b' \in \mathcal{B}$. A *convex $\mathcal{B}$–cone* is a $\mathcal{B}$–cone P which is stable under sums (i.e. $b, b' \in P$ implies $b + b' \in P$).

Let S be a distinguished subset of $\mathcal{B}$. Then by $P(S)$ we denote the convex $\mathcal{B}$–cone *generated* by S (i.e. the set of all sums of elements of the form b'^*bb' with $b \in S, b' \in \mathcal{B}$). If S contains $\mathbf{1}$ and consists entirely of self-adjoint elements, then we say the elements of $P(S)$ are *S–positive*. We say the pair $(\mathcal{B}, S)$ is a *P_0^*–algebra*.

A *P_0^*–(quasi)homomorphism* between P_0^*–algebras $(\mathcal{A}, S_\mathcal{A})$ and $(\mathcal{B}, S_\mathcal{B})$ is a homomorphism which sends $S_\mathcal{A}$ into $S_\mathcal{B}$ ($P(S_\mathcal{A})$ into $P(S_\mathcal{B})$). A *P_0^*–(quasi)isomorphism* is a P_0^*–(quasi)homomorphism with an inverse which is also a P_0^*–(quasi)homomorphism.

Of course, in a reasonable choice for a set of positive elements, S should contain only self-adjoint elements. If $S = \{\mathbf{1}\}$, then $P(S)$ contains all sums over elements of the form b^*b, i.e. we recover the usual algebraic definition of positivity. In a reasonable choice for S, at least, these elements should be positive. Therefore, we require $\mathbf{1} \in S$.

C.1.2 Remark. Notice that even contradictory choices of S are possible. Consider, for instance, the $*$–algebra $\mathbb{C}\langle x\rangle$ generated by one self-adjoint indeterminate x. Then both $S_+ = \{\mathbf{1}, x\}$ and $S_- = \{\mathbf{1}, -x\}$ are possible choices. Indeed, there exist faithful representations of this algebra which send either x or $-x$ to a positive operator on a Hilbert space which, of course, cannot be done simultaneously. Notice that $x \mapsto -x$ extends to an isomorphism which does not preserve either of the notions of positivity. It is, however, an isomorphism of the pairs $(\mathbb{C}\langle x\rangle, S_+)$ and $(\mathbb{C}\langle x\rangle, S_-)$.

C.1.3 Definition. Let G be a pre-Hilbert space and denote by S_G the set of all $b \in \mathcal{L}^a(G)$ for which $\langle g, bg\rangle \geq 0$ $(g \in G)$. Then $(\mathcal{L}^a(G), S_G)$ is a P_0^*–algebra. If not stated otherwise, explicitly, then we speak of the P^*–algebra $\mathcal{L}^a(G)$ with this positivity structure.

Let $(\mathcal{B}, S)$ be a P_0^*–algebra. An *S–representation* (π, G) on a pre-Hilbert space G is a P_0^*–homomorphism $\pi\colon \mathcal{B} \to \mathcal{L}^a(G)$; cf. Proposition C.2.2.

A *P^*–algebra* is a P_0^*–algebra $(\mathcal{B}, S)$ which admits a faithful S–representation.

C.1.4 Remark. In our applications we will identify $\mathcal{B} \subset \mathcal{L}^a(G)$ as a $*$–algebra of operators on a pre-Hilbert space G. In order to have reasonable results, the defining representation should be an S–representation. This is, for instance, the case, if S is a set of elements $b \in \mathcal{B}$ which can be written as a sum over B^*B where $B \in \mathcal{L}^a(G)$ but not necessarily $B \in \mathcal{B}$. In other words, we have algebraic positivity in a bigger algebra.

C.1.5 Definition. Let $(\mathcal{B}, S)$ be a P_0^*–algebra. We say a functional φ on $\mathcal{B}$ is *S–positive*, if $\varphi(b) \geq 0$ for all $b \in P(S)$. Let S^* be a set of S–positive functionals. We say S^* *separates the points* (or is *separating*), if $\varphi(b) = 0$ for all $\varphi \in S^*$ implies $b = 0$ $(b \in \mathcal{B})$.

C.1.6 Observation. By definition, each functional $\varphi_g = \langle g, \pi(\bullet)g\rangle$, where (π, G) is an S–representation and $g \in G$, is S–positive. Conversely, if φ is an S–positive functional, then its GNS-representation is an S–representation. (To see this, observe that any vector in the representation space can be written as $g = \pi(b_g)g_0$ for suitable $b_g \in \mathcal{B}$, and $b \mapsto b_g^* b b_g$ maps $P(S)$ into itself. Therefore, $\varphi_g = \varphi(b_g^* b b_g)$ is S–positive.)

C.1.7 Observation. A P_0^*–algebra is a P^*–algebra, if and only if it admits a separating set of S–positive functionals. Indeed, if $\mathcal{B}$ is a P^*–algebra, then the states $\langle g, \bullet g\rangle$, where g ranges over the unit vectors in the representations space G of a faithful representation, separate the points. On the other hand, if the S–positive functionals separate the points, then the direct sum over the GNS-representations of all S–positive functionals is a faithful S–representation.

C.1.8 Example and convention. If $\mathcal{B}$ is a unital pre–C^*–algebra, then we assume that $S = P(S)$ consists of all elements which are positive in the usual sense (see Appendix A.7), if not stated otherwise explicitly. We could also set $S = \{\mathbf{1}\}$, and still end up with the same positive $\mathcal{B}$–cone $P(S)$. Clearly, $\mathcal{B}$ is a P^*–algebra. In particular, we consider $\mathbb{C}$ always as the P^*–algebra $(\mathbb{C}, \mathbb{R}_+)$.

C.1.9 Example. Let $\mathcal{B}$ be a unital C^*–algebra. We ask, what happens, if there is a (non-zero) self-adjoint element $b \in S$ which is not among the usual positive elements. We can write b as $b_+ - b_-$ $(b_- \neq 0)$ where b_+, b_- are unique positive (in the usual sense) elements fulfilling $b_+ b_- = 0$. It follows that $b_- b b_- = -b_-^3$ is in $P(S)$, but also b_-^3 is in $P(S)$. Therefore, there do not exist S–positive functionals nor S–representations which send b_-^3 to a non-zero element. Consequently, the P_0^*–algebra $(\mathcal{B}, S)$ has neither a faithful S–representation nor the S–positive functionals separate the points. It is not a P^*–algebra.

C.1.10 Example. Let us consider the complex numbers $\mathbb{C}$ as a two-dimensional real algebra with basis $\{1, \imath\}$. The complexification of $\mathbb{C}$ (i.e. $\{\mu 1 + \nu \imath \ (\mu, \nu \in \mathbb{C})\}$) becomes a (complex) $*$–algebra, if we define 1 and $\imath$ to be self-adjoint. In this $*$–algebra the element $-1 = -(1^2) = \imath^2$ is negative *and* positive, so that $\varphi(-1) \leq 0$ and $\varphi(-1) \geq 0$, i.e $\varphi(-1) = 0$ for all states φ. Of course, $-1 \neq 0$, so that in this $*$–algebra the states do not separate the points.

Another example is the $*$–algebra of differentials of complex polynomials in a real indeterminate t. Here we have $dt^2 = 0$. Since dt is self-adjoint, we conclude that $\varphi(dt) = 0$ for all states φ. The next example is the Ito algebra of differentials of stochastic processes; see [Bel92].

C.2 Hilbert P^*–modules

C.2.1 Definition. Let $(\mathcal{A}, S_\mathcal{A})$ and $(\mathcal{B}, S_\mathcal{B})$ be P_0^*–algebras. A *pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module* is an $\mathcal{A}$–$\mathcal{B}$–module E with a sesquilinear *inner product* $\langle \bullet, \bullet \rangle \colon E \times E \to \mathcal{B}$, fulfilling the following requirements

$$\begin{aligned} \langle x, x \rangle = 0 \Rightarrow x = 0 &\qquad \textit{(definiteness)},\\ \langle x, yb \rangle = \langle x, y \rangle b &\qquad \textit{(right } \mathcal{B}\textit{–linearity)},\\ \langle x, ay \rangle = \langle a^* x, y \rangle &\qquad \textit{(}*\textit{–property)}, \end{aligned}$$

and the positivity condition that for all choices of $a \in S_\mathcal{A}$ and of finitely many $x_i \in E$ there exist finitely many $b_k \in S_\mathcal{B}$ and $b_{ki} \in \mathcal{B}$, such that

$$\langle x_i, a x_j \rangle = \sum_k b_{ki}^* b_k b_{kj}.$$

If definiteness is missing, then we speak of a *semiinner product* and a *semi-Hilbert module*.

C.2.2 Proposition. *Let π be a representation of a P_0^*–algebra $(\mathcal{B}, S)$ on a pre-Hilbert space G. Then π is an S–representation, if and only if G is a pre-Hilbert $\mathcal{B}$–$\mathbb{C}$–module.*

Proof. Let π be an S–representation. For $g_i \in G$ $(i = 1, \ldots, n)$ we find

$$\sum_{i,j} \overline{c}_i \langle g_i, \pi(z) g_j \rangle c_j \;=\; \sum_{i,j} \langle c_i g_i, \pi(z) g_j c_j \rangle \;\geq\; 0$$

for all $\bigl(c_i\bigr) \in \mathbb{C}^n$. Therefore, the matrix $\bigl(\langle g_i, \pi(z) g_j \rangle\bigr)$ is positive in M_n and, henceforth, can be written in the form $\sum\limits_k \overline{d}_{ki} d_{kj}$ for suitable $\bigl(d_{ki}\bigr) \in M_n$, i.e. G is a pre-Hilbert $\mathcal{B}$–$\mathbb{C}$–module. The converse direction is obvious. ■

Let us return to Definition C.2.1. Since $\mathbf{1}_{\mathcal{A}} \in S_{\mathcal{A}}$, the inner product is S–*positive* (i.e. $\langle x, x \rangle \in P(S_{\mathcal{B}})$), and since $S_{\mathcal{B}}$ consists only of self-adjoint elements, the inner product is *symmetric* (i.e. $\langle x, y \rangle = \langle y, x \rangle^*$) and *left anti-linear* (i.e. $\langle xb, y \rangle = b^* \langle x, y \rangle$).

C.2.3 Observation. It is sufficient to check positivity on a subset $E_{\mathfrak{g}}$ of E which generates E as a right module. Indeed, for finitely many $x_i \in E$ there exist finitely many $y_\ell \in E_{\mathfrak{g}}$ and $b_{\ell i} \in \mathcal{B}$ such that $x_i = \sum\limits_\ell y_\ell b_{\ell i}$ for all i. It follows that for $a \in S_{\mathcal{A}}$

$$\langle x_i, a x_j \rangle \;=\; \sum_{\ell,m} b_{\ell i}^* \langle y_\ell, a y_m \rangle b_{mj} \;=\; \sum_{\ell,m,k} b_{\ell i}^* {b'_{k\ell}}^* b_k b'_{km} b_{mj} \;=\; \sum_k c_{ki}^* b_k c_{kj},$$

where $c_{ki} = \sum\limits_\ell b'_{k\ell} b_{\ell i}$.

C.2.4 Proposition. *Let $(\mathcal{A}, S_{\mathcal{A}})$, $(\mathcal{B}, S_{\mathcal{B}})$, and $(\mathcal{C}, S_{\mathcal{C}})$ be P_0^*–algebras. Let E be a semi-Hilbert $\mathcal{A}$–$\mathcal{B}$–module and let F be a semi-Hilbert $\mathcal{B}$–$\mathcal{C}$–module. Then their tensor product $E \underline{\odot} F$ over $\mathcal{B}$ is turned into a semi-Hilbert $\mathcal{A}$–$\mathcal{C}$–module by setting*

$$\langle x \odot y, x' \odot y' \rangle = \langle y, \langle x, x' \rangle y' \rangle. \tag{C.2.1}$$

Proof. We only check that the positivity condition is fulfilled, because the remaining conditions are clear. (See Appendix C.4 for the tensor product over $\mathcal{B}$ and well-definedness of mappings on it.) By Observation C.2.3 it is sufficient to check positivity for elementary tensors $x \odot y$. So let $x_i \odot y_i$ be finitely many elementary tensors in $E \underline{\odot} F$, and let $a \in S_{\mathcal{A}}$. Then

$$\langle x_i \odot y_i, a x_j \odot y_j \rangle \;=\; \langle y_i, \langle x_i, a x_j \rangle y_j \rangle \;=\; \sum_k \langle b_{ki} y_i, b'_k b_{kj} y_j \rangle \;=\; \sum_{k,\ell} c_{\ell(ki)}^* c'_\ell c_{\ell(kj)},$$

where for each k we have finitely many elements $c_{\ell(ki)} \in \mathcal{B}$ corresponding to the finitely many elements $b_{ki} y_i$ in F. ■

C.2.5 Observation. Also here it is sufficient to consider elementary tensors $x_i \odot y_i$ where x_i and y_i come from (right) generating subsets of E and F, respectively. This follows, because any element in F, in particular, an element of the form by, can be written as sum over $y_i c_i$, and elements of the form $x_i \odot by = x_i b \odot y$, clearly, span $E \underline{\odot} F$.

So far we were concerned with semi-Hilbert modules. For several reasons it is desirable to have a strictly positive inner product. For instance, contrary to a semiinner product, an inner product guarantees for uniqueness of adjoints. We provide a quotienting procedure which allows to construct a pre-Hilbert module out of a given semi-Hilbert module, at least, in the case of P^*–algebras.

C.2.6 Definition. A *(semi-) pre-Hilbert P^*–module* is a (semi-) pre-Hilbert P_0^*–module where the algebra to the right is a P^*–algebra.

C.2.7 Proposition. *Let E be a semi-Hilbert P^*–module. Then the set*

$$\mathcal{N}_E \;=\; \bigl\{x \in E \colon \langle x, x\rangle = 0\bigr\}$$

is a two-sided submodule of E. Moreover, the quotient module $E_0 = E/\mathcal{N}_E$ inherits a pre-Hilbert P^*–module structure by setting $\langle x + \mathcal{N}_E, y + \mathcal{N}_E\rangle = \langle x, y\rangle$.

Proof. Let E be a semi-Hilbert $\mathcal{A}$–$\mathcal{B}$–module, and let S^* be a separating set of S–positive functionals on $\mathcal{B}$. We have $x \in \mathcal{N}_E$, if and only if $\varphi(\langle x, x\rangle) = 0$ for all $\varphi \in S^*$. Let $\varphi \in S^*$. Then the sesquilinear form $\langle x, y\rangle_\varphi = \varphi(\langle x, y\rangle)$ on E is positive. By Cauchy-Schwarz inequality we find that $\langle x, x\rangle_\varphi = 0$ implies $\langle y, x\rangle_\varphi = 0$ for all $y \in E$. Consequently, $x, y \in \mathcal{N}_E$ implies $x + y \in \mathcal{N}_E$. Obviously, $x \in \mathcal{N}_E$ implies $xb \in \mathcal{N}_E$ for all $b \in \mathcal{B}$. And by the $*$–property and Cauchy-Schwarz inequality we find $x \in \mathcal{N}_E$ implies $ax \in \mathcal{N}_E$ for all $a \in \mathcal{A}$. Therefore, $\mathcal{N}_E$ is a two-sided submodule of E so that $E/\mathcal{N}_E$ is a two-sided $\mathcal{A}$–$\mathcal{B}$–module. Once again, by Cauchy-Schwarz inequality we see that $\langle x + \mathcal{N}_E, y + \mathcal{N}_E\rangle$ is a well-defined element of $\mathcal{B}$. ■

C.2.8 Observation. Notice that an operator $a \in \mathcal{L}^a(E)$ respects $\mathcal{N}_E$, automatically. Like in Corollary 1.4.3 any adjoint $a^* \in \mathcal{L}^a(E)$ gives rise to a unique adjoint of a in $\mathcal{L}^a(E_0)$.

C.2.9 Definition. Let $\mathcal{A}$, $\mathcal{B}$ be P_0^*–algebras, and let $\mathcal{C}$ be a P^*–algebra. The tensor product of a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module and a pre-Hilbert $\mathcal{B}$–$\mathcal{C}$–module is the pre-Hilbert $\mathcal{A}$–$\mathcal{C}$–module $E \odot F = E\underline{\odot}F/\mathcal{N}_{E\underline{\odot}F}$.

Now we can construct, in particular, the tensor product of E and a representation space G of an S–representation of $\mathcal{B}$. Also here we use the whole terminology concerning the

Stinespring representation as introduced in Section 2.3. The crucial property of E, being a functor which sends representations of $\mathcal{B}$ to representations of $\mathcal{A}$ as explained in Remark 4.2.8, remains true in the framework of P_0^*–algebras, if we restrict to S–representations. In this context, also the notion of complete positivity and GNS-representation generalize due to the fact that all P_0^*–algebras are assumed unital.

C.2.10 Remark. Also the exterior tensor product fits well into the framework of P^*–modules, if we consider the tensor product $(\mathcal{B}_1 \otimes \mathcal{B}_2, S_1 \otimes S_2)$ of P^*–algebras $(\mathcal{B}_i, S_i)$ which is again a P^*–algebra. Proposition 4.3.1 and many other algebraic results from Section 4.3 remain true.

C.3 Full Fock modules over pre-Hilbert P^*–modules

C.3.1 Definition. Let $\mathcal{B}$ be a unital $*$–algebra with a subset S of self-adjoint elements containing $\mathbf{1}$, and let S^* be a separating set of S–positive functionals on $\mathcal{B}$. Let E be pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module . The *full Fock module* over E is the pre-Hilbert $\mathcal{B}$–$\mathcal{B}$–module

$$\underline{\mathcal{F}}(E) = \bigoplus_{n \in \mathbb{N}_0} E^{\odot n}$$

where we set $E^{\odot 0} = \mathcal{B}$ and $E^{\odot 1} = E$. Also creators and annihilators are defined in the usual way. By $\mathcal{A}(\underline{\mathcal{F}}(E))$ we denote the unital $*$–subalgebra of $\mathcal{L}^a(\underline{\mathcal{F}}(E))$ generated by $\ell^*(E)$ and $\mathcal{B}$, where $\mathcal{B}$ acts canonically on $\underline{\mathcal{F}}(E)$ by left multiplication.

C.3.2 Remark. Like in the case of the usual full Fock space, the $*$–algebra $\mathcal{A}(\underline{\mathcal{F}}(E))$ is determined by the generalized Cuntz relations $\ell(x)\ell^*(y) = \langle x, y \rangle$; see [Pim97].

For examples we refer to Section 9.2 and Chapter 8 starting from Section 8.2.

C.4 Appendix: Tensor products over $\mathcal{B}$

We repeat some basics about the tensor product of modules over an algebra. In the main part of these notes we did not need this tensor product, because we always were able to divide out the length-zero elements. In Appendix C this is not possible for P_0^*–algebras. In Proposition C.4.2 we provide the universal property which shows that in the definition of the the semiiner product in (C.2.1), at least, relations like $xb \odot y - x \odot by$ may be divided out. Finally, we show the result from [Ske98a] that centered modules behave nice under tensor product also in this algebraic framework.

C.4.1 Definition. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be algebras, let E be an $\mathcal{A}$–$\mathcal{B}$–module and let F a $\mathcal{B}$–$\mathcal{C}$–module. Define the $\mathcal{A}$–$\mathcal{C}$–submodule

$$\mathfrak{N}_{\mathcal{B}} \;=\; \{xb \otimes y - x \otimes by \; (x \in E, y \in F, b \in \mathcal{B})\}$$

of the $\mathcal{A}$–$\mathcal{C}$–module $E \otimes F$. The ***tensor product*** $E \odot F$ ***over*** $\mathcal{B}$ is the $\mathcal{A}$–$\mathcal{C}$–module $(E \otimes F)/\mathfrak{N}_{\mathcal{B}}$. We set $x \odot y = x \otimes y + \mathfrak{N}_{\mathcal{B}}$.

We say a mapping $j \colon E \times F \to G$ into an $\mathcal{A}$–$\mathcal{C}$–module G is ***$\mathcal{A}$–$\mathcal{C}$–bilinear***, if it is left $\mathcal{A}$–linear in the first and right $\mathcal{C}$–linear in the second argument. We say j is ***balanced***, if $j(xb, y) = j(x, by)$ for all $x \in E, b \in \mathcal{B}, y \in F$. Obviously, the mapping $i \colon (x, y) \mapsto x \odot y$ is balanced and $\mathcal{A}$–$\mathcal{C}$–bilinear.

C.4.2 Proposition. *The pair $(E \odot F, i)$ is determined uniquely up to $\mathcal{A}$–$\mathcal{C}$–module isomorphism by the universal property: For an arbitrary balanced $\mathcal{A}$–$\mathcal{C}$–bilinear mapping $j \colon E \times F \to G$ there exists a unique $\mathcal{A}$–$\mathcal{C}$–linear mapping $\widehat{j} \colon E \odot F \to G$ fulfilling $j = \widehat{j} \circ i$.*

Proof. Uniqueness follows in the usual way. We sketch this once for all other cases in these notes. Assume we have two pairs (G, i) and (G', i') which have the universal property. Denote by $\widehat{i'} \colon G \to G'$ and $\widehat{i} \colon G' \to G$ the unique mappings determined by the universal property of G applied to i' and conversely. We have $\widehat{i} \circ \widehat{i'} \circ i = \widehat{i} \circ i' = i$. By the universal property of G there is precisely one $\mathcal{A}$–$\mathcal{C}$–linear mapping $j \colon G \to G$ fulfilling $j \circ i = i$, namely $j = \mathsf{id}_G$. We conclude $\widehat{i} \circ \widehat{i'} = \mathsf{id}_G$ and, similarly, $\widehat{i'} \circ \widehat{i} = \mathsf{id}_{G'}$. This means that G and G' are isomorphic $\mathcal{A}$–$\mathcal{C}$–modules. ∎

C.4.3 Corollary. *Let $j \colon E \to E'$ be an $\mathcal{A}$–$\mathcal{B}$–linear mapping and let $k \colon F \to F'$ be a $\mathcal{B}$–$\mathcal{C}$–linear mapping. There exists a unique a $\mathcal{A}$–$\mathcal{C}$–linear mapping $j \odot k \colon E \odot F \to E' \odot F'$, fulfilling $(j \odot k)(x \odot y) = j(x) \odot k(y)$.*

C.4.4 Remark. If E and F are submodules of E' and F' and j and k are the canonical embeddings, respectively, then $j \odot k$ defines a canonical *embedding* of $E \odot F$ into $E' \odot F'$. However, unlike the vector space case, this *embedding* is, in general, not injective. This may happen, because the number of relations to be divided out in the definition of $E \odot F$ is, usually, much smaller than the corresponding number for $E' \odot F'$.

C.4.5 Theorem. *Let E, F be two centered $\mathcal{B}$–$\mathcal{B}$–modules. There is a unique $\mathcal{B}$–$\mathcal{B}$–module isomorphism $\digamma \colon E \odot F \to F \odot E$, called* flip *isomorphism, fulfilling*

$$\digamma(x \odot y) \;=\; y \odot x \tag{C.4.1}$$

for all $x \in C_{\mathcal{B}}(E)$ and $y \in C_{\mathcal{B}}(F)$.

Proof. Let (x, y) $(x \in E, y \in F)$ denote an arbitrary element of $E \times F$. Since E and F are centered, we have $x = \sum_i a_i x_i$ and $y = \sum_j y_j b_j$ for suitable $x_i \in C_{\mathcal{B}}(E); y_j \in C_{\mathcal{B}}(F); a_i, b_j \in \mathcal{B}$. Let $x'_i \in C_{\mathcal{B}}(E); y'_j \in C_{\mathcal{B}}(F); a'_i, b'_j \in \mathcal{B}$ denote another suitable choice. We find

$$\begin{aligned}\sum_{ij} a_i y_j \odot x_i b_j &= \sum_{ij} y_j \odot a_i x_i b_j = \sum_{ij} y_j \odot a'_i x'_i b_j \\ &= \sum_{ij} a'_i y_j \odot x'_i b_j = \sum_{ij} a'_i y_j b_j \odot x'_i \\ &= \sum_{ij} a'_i y'_j b'_j \odot x'_i = \sum_{ij} a'_i y'_j \odot x'_i b'_j.\end{aligned}$$

Therefore,

$$\mathcal{F}^{\times} \colon (x, y) \longmapsto \sum_{ij} a_i y_j \odot x_i b_j$$

is a well-defined mapping $E \times F \to F \underline{\odot} E$.

Obviously, $\mathcal{F}^{\times}$ is $\mathcal{B}$–$\mathcal{B}$–bilinear. We show that it is balanced. Indeed, for an arbitrary $a \in \mathcal{B}$ we find

$$\mathcal{F}^{\times}(xa, y) = \sum_{ij} a_i a y_j \odot x_i b_j = \sum_{ij} a_i y_j \odot x_i a b_j = \mathcal{F}^{\times}(x, ay).$$

Thus, by the universal property of the $\mathcal{B}$–tensor product there exists a unique $\mathcal{B}$–$\mathcal{B}$–linear mapping $\mathcal{F} \colon E \underline{\odot} F \to F \underline{\odot} E$ fulfilling

$$\mathcal{F}(x \odot y) = \mathcal{F}^{\times}(x, y).$$

Of course, $\mathcal{F}$ fulfills (C.4.1).

By applying $\mathcal{F}$ a second time (now to $F \underline{\odot} E$), we find $\mathcal{F} \circ \mathcal{F} = \mathsf{id}$. Combining this with surjectivity, we conclude that $\mathcal{F}$ is an isomorphism. ■

C.4.6 Remark. It is noteworthy that in Appendix C and in the applications in Part II of these notes we do not distinguish very carefully between $x \otimes y + \mathcal{N}_{\mathcal{B}}$ and $x \otimes y + \mathcal{N}_{E \otimes F}$, when E and F are pre-Hilbert modules, and we denote both by $x \odot y$. It should be clear from the context which quotient we have in mind.

Appendix D

The stochastic limit of the QED-Hamiltonian

In this appendix we present our results from Skeide [Ske98a] which show how Hilbert modules can help to understand the *stochastic limit* for an electron in the vacuum electric field as computed by Accardi and Lu [AL96, AL93]. For a detailed account on the stochastic limit we recommend the monograph Accardi, Lu and Volovich [ALV01]. In particular, we show (with the help of Lemma D.3.6) that the inner product computed in [AL96] (Proposition D.3.5) determines an element in a pre–C^*–algebra $\mathcal{B}$, and that the limit module over $\mathcal{B}$ is a full Fock module over a suitable one-particle module (Theorem D.4.3). The decisive step for this identification consists in finding the correct left multiplication (D.3.6) on the one-particle module. Without left multiplication there would be no full Fock module.

In Section D.1 we discuss the physical model. The most important objects are the collective operators defined by (D.1.3). In Section D.2 we translate (basically, as in Example 4.4.12) the description in terms of Hilbert spaces into the language of modules. After that the collective operators appear just as creators and annihilators on a symmetric Fock module.

In Section D.3 we compute the (two-sided) limit module of the one-particle sector. In Section D.4 we show that the limit of the symmetric Fock module is a full Fock module. More precisely, we show in a central limit theorem that the moments of collective operators in the vaccum conditional expectation on the symmteric Fock module converge to moments of free creators and annihilators on a full Fock module. All technical difficulties already arise in the limit of the one-particle sector. The extension from the one particle sector is based on the fact that the algebra of operators on the full Fock module is determined by the generalized free commutation relations (6.1.1) (see Pimsner [Pim97]). In a more probabilistic interpretation we may say that the inner product on the full Fock module is determined by the inner product of the one-particle module (i.e. the two-point function), a behaviour which is typical for central limit distributions and which is refered to as *gaußianity* in Accardi, Lu and Volovich [ALV99]. It throws a bridge from the stochastic limit of elementary particle physics to Voiculescu's *operator-valued free probability* (see also the introduction to Part II and Section 17.2) with examples by Speicher [Spe98] and Shlyakhtenko [Shl96], and further to the construction of new C^*–algebras generalizing Cuntz-Krieger algebras and crossed products; see [Pim97].

D.1 The physical model

The *stochastic limit* is a general procedure to separate in the dynamical evolution of a physical system *slowly moving* degrees of freedom (typically, those of one or some particles) from *quickly moving* or *noise* degrees of freedom (typically, a field in an equilibrium state). A careful rescaling of parameters describing the compound system (in our example these are time and coupling constant, but there can be involved also other parameters), avoiding divergences on the one hand, and trivialities on the other hand (in this respect, indeed, very similar to a central limit theorem), provides a new description of the dynamics of the original system. (In our case, for instance, Gough [Gou96] shows that the description obtained is equivalent to 2nd-order perturbation theory.) However, under such a limit the character of the equations which govern dynamics changes. The original unitary evolution fufills a so-called *hamiltonian equation* (the *Schrödinger equation*), whereas the evolution after the limit is the solution of a quantum stochastic differential equation driven by a *quantum white noise* (in our case operator-valued free white noise in the sense of Speicher [Spe98] which can be resolved using our calculus from Part IV). While the original hamiltonian equation can be treated only rarely, the quantum stochastic differential equations have already been dealt with successfully in many cases. For a comprehensive account on the stochastic limit (with many examples) we refer the reader to Accardi and Kozyrev [AK00].

In the sequel, we describe the stochastic limit for the non-relativistic QED–Hamiltonian in $d \in \mathbb{N}$ dimensions of a single free electron coupled to the photon field without dipole approximation. Originally, the photon field has d components. However, since we neglect the possibility of polarization, we may restrict to a single component. From the mathematical point of view this is not a serious simplification. Our results can be generalized easily to d components. In addition, we forget about the fact that the electron couples to the field via the component of p into the direction of the field. The p may be reinserted after the computations easily, because we work in Coulomb gauge. Throughout this appendix we assume $d \geq 3$. In the sequel, we describe our simplified set-up and refer to [AL96, Gou96] for a detailed description.

The Hilbert space R of the field is the symmetric Fock space $\Gamma(L^2(\mathbb{R}^d))$ over $L^2(\mathbb{R}^d)$

with the Hamiltonian $H_R = \int dk\ |k|\, a_k^* a_k$. (By a_k^* and a_k we denote the usual creator and annihilator densities which fulfill $a_k a_{k'}^* - a_{k'}^* a_k = \delta(k-k')$.) The particle space S is the representation space $L^2(\mathbb{R}^d)$ of the d–dimensional *Weyl algebra* $\mathfrak{W}$ in *momentum representation* with the usual free Hamiltonian $H_S = \frac{p^2}{2}$.

The interaction is described on the compound system $S \bar{\otimes} R$ by the interaction Hamiltonian

$$H_I = \lambda \int dk\, a_k^* \otimes e^{ik\cdot q} c(k) + \mathsf{h.c.}\,.$$

λ is a (positive) coupling constant. In the original physical model the function c is given by $c(k) = \frac{1}{\sqrt{|k|}}$; see [AL96]. As in [AL96] we replace it by a suitable cut-off function $c \in \mathcal{C}_c(\mathbb{R}^d)$. Since we will identify operators on R and S, respectively, with their ampliations to $S \bar{\otimes} R$, we omit in the sequel the $\otimes$–sign in between such operators. The time-dependent interaction Hamiltonian in the interaction picture, defined by $H_I(t) = e^{it(H_R+H_S)} H_I e^{-it(H_R+H_S)}$, takes the form

$$H_I(t) = \lambda \int dk\, a_k^* e^{ik\cdot q} e^{itk\cdot p} e^{it(|k|+\frac{1}{2}|k|^2)} c(k) + \mathsf{h.c.}\,.$$

This follows directly from the commutation relations fulfilled by a_k^* and a_k and from the basic relation $f(p)e^{ik\cdot q} = e^{ik\cdot q} f(p+k)$ for all $f \in L^\infty(\mathbb{R}^d)$. In the sequel, the special case

$$e^{ik\cdot p} e^{ik'\cdot q} = e^{ik'\cdot q} e^{ik\cdot p} e^{ik\cdot k'} \tag{D.1.1}$$

is of particular interest.

The *wave operators* $U(t)$ defined by $U(t) = e^{it(H_R+H_S)} e^{-it(H_R+H_S+H_I)}$ are the objects of main physical interest. They fulfill the differential equation

$$\frac{dU}{dt}(t) = -iH_I(t)U(t) \quad \text{and} \quad U(0) = \mathbf{1}. \tag{D.1.2}$$

For the stochastic limit the time t is replaced by $\frac{t}{\lambda^2}$ and one considers the limit $\lambda \to 0$. So we define the rescaled wave operators $U_\lambda(t) = U(\frac{t}{\lambda^2})$. The problem is to give sense to $U_0 = \lim\limits_{\lambda\to 0} U_\lambda$. For this aim [AL96] proceed in the following way. Let V denote the vector space which is linearly spanned by all functions $f\colon \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$ of the form $f(\tau,k) = I\!I_{[t,T]}(\tau)\widetilde{f}(k)$ ($t < T, \widetilde{f} \in \mathcal{C}_c(\mathbb{R}^d)$). Obviously, we have $V = \mathfrak{S}(\mathbb{R}) \otimes \mathcal{C}_c(\mathbb{R}^d)$. For $f \in V$ define the *collective creators*

$$A_\lambda^*(f) = \int d\tau \int dk\, a_k^* \gamma_\lambda(\tau,k) f(\tau,k) \tag{D.1.3}$$

and their adjoints $A_\lambda(f)$, the *collective annihilators*. Here we set

$$\gamma_\lambda(\tau,k) = \frac{1}{\lambda} e^{ik\cdot q} e^{i\frac{t}{\lambda^2}k\cdot p} e^{i\frac{t}{\lambda^2}\omega(|k|)} \quad \text{and} \quad \omega(r) = r + \frac{1}{2}r^2.$$

In view of Remark D.2.2 and Lemma D.3.6 we keep in mind that in some respects also more general choices for γ_λ and ω are possible.

Obviously, we have

$$A^*_\lambda(\mathbb{I}_{[t,T]}c) + A_\lambda(\mathbb{I}_{[t,T]}c) = \int_{\frac{t}{\lambda^2}}^{\frac{T}{\lambda^2}} d\tau\, H_I(\tau). \tag{D.1.4}$$

With the definition $A^*_\lambda(t) = A^*_\lambda(\mathbb{I}_{[0,t]}c)$ Equation (D.1.2) transforms into

$$\frac{dU_\lambda}{dt}(t) = -i\Big(\frac{dA^*_\lambda}{dt}(t) + \frac{dA_\lambda}{dt}(t)\Big)U_\lambda(t) \quad \text{and} \quad U_\lambda(0) = \mathbf{1}. \tag{D.1.5}$$

Henceforth, if we are able to give sense to the limit not only of U_λ but also of $A^*_\lambda(f)$ for fixed $f \in V$, we may expect this differential equation to hold also in the limit $\lambda \to 0$. On the other hand, if we find a limit of the $A^*_\lambda(f)$ and a quantum stochastic calculus in which (D.1.5) makes sense and has a solution U_0, we may hope that U_0 is the limit of the U_λ. Here we will consider exclusively the limit of the collective operators.

Let Ω denote the vacuum in R. Then the vacuum conditional expectation $\mathbb{E}_0 \colon \mathcal{B}(S \bar{\otimes} R) \to \mathcal{B}(S)$ is

$$\mathbb{E}_0(a) = (\mathsf{id} \otimes \Omega^*)a(\mathsf{id} \otimes \Omega). \tag{D.1.6}$$

In [AL96] the limit $\lim\limits_{\lambda\to 0}\langle \xi, \mathbb{E}_0(M_\lambda)\zeta\rangle$, for M_λ being an arbitrary monomial in collective operators and ξ, ζ being Schwartz functions, was calculated. In the sequel, we repeat the major ideas of the proof in a new formulation. Moreover, we show that the limit considered as a sesquilinear form in ξ and ζ, indeed, determines an element of $\mathcal{B}(S)$.

D.2 The module for finite λ

For the time being we fix on the algebra $\mathcal{B}(S)$ which contains the Weyl algebra $\mathfrak{W}$ as a strongly dense subalgebra. In the limit $\lambda \to 0$ the algebra $\mathcal{B}(S)$ turns out to be too big.

We know from Example 3.3.4 that the algebra $\mathcal{B}(S \bar{\otimes} R)$ coincides with the algebra $\mathcal{B}^a(\mathcal{B}(S, S \bar{\otimes} R))$ of adjointable operators on the von Neumann $\mathcal{B}(S)$–$\mathcal{B}(S)$–module $\mathcal{B}(S, S \bar{\otimes} R)$. By Example 4.4.12 $\mathcal{B}(S, S \bar{\otimes} R)$ is the GNS-module of the vacuum conditional expectation (D.1.6). As in Example 6.1.6 we see that $\mathcal{B}(S, S \bar{\otimes} R) = \mathcal{B}(S) \bar{\otimes}^s \Gamma(L^2(\mathbb{R}^d))$ is nothing but the strong closure of the symmetric Fock module $\Gamma(L^2(\mathbb{R}^d, \mathcal{B}(S))$ over the one-particle sector $L^2(\mathbb{R}^d, \mathcal{B}(S))$ as investigated in Section 8.1.

If we want to represent the $*$–algebra $\mathcal{A}_\lambda$ of collective operators, then we must be slightly more careful, because these operators are unbounded. We consider the algebraic symmetric Fock module $\Gamma_c = \underline{\Gamma}(\mathcal{C}^s_c(\mathbb{R}^d, \mathcal{B}(S))$ over the strongly continuous functions with compact support (see Appendix B). Then the creators and annihilators defined as in Section 8.1 leave invariant Γ_c, and it is clear that the conditional expectation extends to the $*$–algebra generated by them.

Let f be an element in V (see Section D.1). We define a mapping $\varphi_\lambda\colon V \to \mathcal{C}_c^s(\mathbb{R}^d, \mathcal{B}(S))$, by setting

$$[\varphi_\lambda(f)](k) = \int d\tau\, \gamma_\lambda(\tau, k) f(\tau, k).$$

(Since γ_λ is only strongly continuous, φ_λ maps, indeed, into $\mathcal{C}_c^s(\mathbb{R}^d, \mathcal{B}(S))$ and not into $\mathcal{C}_c(\mathbb{R}^d, \mathcal{B}(S))$.) Having a look at (D.1.3), we get the impression, as if $A_\lambda^*(f)$ "wants to create" the function $\varphi_\lambda(f)$. This impression is fully reconfirmed in the module picture. It is rather obvious that the equation

$$\alpha(A_\lambda^*(f)) = a^*(\varphi_\lambda(f))$$

defines a $*$–algebra monomorphism $\alpha\colon \mathcal{A}_\lambda \to \mathcal{L}^a(\Gamma_c)$. This follows more or less by checking

$$\mathbb{E}_0(M_\lambda) = \langle \omega, \alpha(M_\lambda)\omega \rangle \tag{D.2.1}$$

for any monomial M_λ in collective operators where ω is the vacuum in Γ_c. (For a detailed proof see [Ske98a]. To make a precise statement it is necessary to find a common dense invariant domain of all elements in $\mathcal{A}_\lambda$.)

ω is not yet necessarily a cyclic vector for the range of α. However, if we denote by $V(\mathcal{B}(S))$ the module spanned by functions $f\colon \mathbb{R} \times \mathbb{R}^d \to \mathcal{B}(S)$ of the form $f(\tau, k) = I\!I_{[t,T]}(\tau)\breve{f}(\tau, k)$ $\big(\breve{f} \in \mathcal{C}_c(\mathbb{R} \times \mathbb{R}^d, \mathcal{B}(S))^s\big)$, then it is possible to extend the definitions of the collective operators and of φ_λ to $V(\mathcal{B}(S))$. Also α extends to the bigger $*$–algebra generated by $A_\lambda^*\big(V(\mathcal{B}(S))\big)$. We will see in Remark D.2.2 that now ω is at least topologically cyclic. Notice that φ_λ is right linear automatically.

We turn $V(\mathcal{B}(S))$ into a semi-Hilbert $\mathcal{B}(S)$–module, by defining the semi-inner product $\langle f, g\rangle_\lambda = \langle \varphi_\lambda(f), \varphi_\lambda(g)\rangle$. By defining the left multiplication

$$[b.f](t, k) = \gamma_\lambda^{-1}(t, k) b \gamma_\lambda(t, k) f(t, k) \tag{D.2.2}$$

$V(\mathcal{B}(S))$ becomes a semi-Hilbert $\mathcal{B}(S)$–$\mathcal{B}(S)$–module and φ_λ a ($\mathcal{B}(S)$–$\mathcal{B}(S)$–linear) isometry.

D.2.1 Proposition. *φ_λ extends to an isomorphism between the von Neumann $\mathcal{B}(S)$–$\mathcal{B}(S)$–modules $\overline{V(\mathcal{B}(S))}^s$ and $L^{2,s}(\mathbb{R}^d, \mathcal{B}(S))$. A fortiori, all $\overline{V(\mathcal{B}(S))}^s$ for different $\lambda > 0$ are isomorphic.*

PROOF. Observe that $\gamma_\lambda^{-1}V(\mathcal{B}(S)) \subset V(\mathcal{B}(S))$. Let $b \in \mathcal{B}(S)$ and $f \in \mathcal{C}_c(\mathbb{R}^d)$. We have $\varphi_\lambda(\gamma_\lambda^{-1}\mathbb{I}_{[0,1]}fb) = fb$, so that $\varphi_\lambda\big(V(\mathcal{B}(S))\big) \supset \mathcal{C}_c(\mathbb{R}^d) \otimes \mathcal{B}(S)$. We conclude that $\overline{\varphi_\lambda\big(V(\mathcal{B}(S))\big)}^s = L^{2,s}(\mathbb{R}^d, \mathcal{B}(S))$.

φ_λ is an isometry and extends as a surjective isometry from $\overline{V(\mathcal{B}(S))}^s$ to $L^2(\mathbb{R}^d, \mathcal{B}(S))^s$. Clearly, this extension is an isomorphism. ■

D.2.2 Remark. The preceeding proof shows that the operators $A_\lambda^*\big(V(\mathcal{B}(S))\big)$ applied successively to $\mathbf{1}$ generate a strongly dense subspace of $\overline{\Gamma_c}^s$. Therefore, ω is topologically cyclic. Notice that all results obtained so far remain valid, if we choose for γ_λ an arbitrary invertible element of $\mathcal{C}_b^s(\mathbb{R} \times \mathbb{R}^d, \mathcal{B}(S))$ (the bounded strongly continuous functions).

D.2.3 Remark. The two pictures $L^{2,s}(\mathbb{R}^d, \mathcal{B}(S))$ and $\overline{V(\mathcal{B}(S))}^s$ of the same Hilbert module are useful for two different purposes. $L^{2,s}(\mathbb{R}^d, \mathcal{B}(S))$ shows more explicitly the algebraic structure which appears simply as the pointwise operations on a two-sided module of functions with values in an algebra. The property that the module is centered can be seen clearly only in this picture. For the limit $\lambda \to 0$, however, we concentrate on the elements of the generating subset $V \subset \overline{V(\mathcal{B}(S))}^s$. (The image of $f \in V$ in $L^2(\mathbb{R}^d, \mathcal{B}(S))^s$ under φ_λ does not converge to anything.)

D.3 The limit of the one-particle sector

This section is the analytical heart of this appendix. We compute the *limit* of the module $V(\mathcal{B}(S))$. In Section D.4 we point out how the results of this section can be generalized to the full system.

Let $V^f = \mathcal{B} \otimes V \otimes \mathcal{B}$ denote the free $\mathcal{B}$–$\mathcal{B}$–module generated by the vector space V. Motivated by Remark D.2.3 we give the definition of what we understand by a limit of Hilbert modules.

D.3.1 Definition. Let V denote a vector space. A family of semi-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules $\big(E_\lambda\big)_{\lambda\in\Lambda}$ with linear embeddings $i_\lambda \colon V \to E_\lambda$ is called V*–related*, if the $\mathcal{B}$–$\mathcal{B}$–submodule generated by $i_\lambda(V)$ is E_λ. In this case i_λ extends to a $\mathcal{B}$–$\mathcal{B}$–linear mapping from V^f onto E_λ. We turn V^f into a semi-Hilbert $\mathcal{B}$–$\mathcal{B}$–module V_λ, by defining the semi-inner product

$$\langle f, g\rangle^\lambda = \langle i_\lambda(f), i_\lambda(g)\rangle$$

for $f, g \in V^f$.

Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be locally convex Hausdorff topologies on $\mathcal{B}$. A semi-Hilbert $\mathcal{B}$–$\mathcal{B}$–module E is called *sequentially* $\mathfrak{T}_1$*–*$\mathfrak{T}_2$*–continuous*, if for all $f, g \in E$ any of the four functions $b \mapsto \langle f, gb\rangle$, $b \mapsto \langle f, bg\rangle$, $b \mapsto \langle fb, g\rangle$ and $b \mapsto \langle bf, g\rangle$ on $\mathcal{B}$ is sequentially $\mathfrak{T}_1$–$\mathfrak{T}_2$–continuous.

Let Λ be a net converging to $\lambda_0 \in \Lambda$ and $\mathcal{B}_0$ a $*$–subalgebra of $\mathcal{B}$ which is sequentially $\mathfrak{T}_1$–dense. We say a V–related family of sequentially $\mathfrak{T}_1$–$\mathfrak{T}_2$–continuous semi-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules $\bigl(E_\lambda\bigr)_{\lambda\in\Lambda}$ *converges* to E_{λ_0}, if

$$\lim_\lambda \langle f,g\rangle^\lambda = \langle f,g\rangle^{\lambda_0} \tag{D.3.1}$$

in the topology $\mathfrak{T}_2$ for all $f,g \in V^f_{\mathcal{B}_0}$. We write $\lim\limits_\lambda E_\lambda = E_{\lambda_0}$.

D.3.2 Remark. Some comments on this definition are in place. We are interested in the limits of the semi-inner products of elements of V^f. However, it turns out that the limit may be calculated only on the submodule $V^f_{\mathcal{B}_0}$, where $\mathcal{B}_0$ is a sufficiently small subalgebra of $\mathcal{B}$, and in a sufficiently weak topology $\mathfrak{T}_2$. (If this limit took values also in $\mathcal{B}_0$, we could stay with $V^f_{\mathcal{B}_0}$ and forget about $\mathcal{B}$. Unfortunately, this will not be the case.) By the requirement that $\mathcal{B}_0$ is a sequentially dense subalgebra of $\mathcal{B}$ in a sufficiently strong topology $\mathfrak{T}_1$ and by the $\mathfrak{T}_1$–$\mathfrak{T}_2$–continuity conditions we assure that the semi-inner product on $V^f_{\mathcal{B}_0}$ (with values in $\mathcal{B}$) already determines the semi-inner product on V^f.

Suppose that $\bigl(E_\lambda\bigr)_{\lambda\in\Lambda\setminus\{\lambda_0\}}$ is V–related and sequentially $\mathfrak{T}_1$–$\mathfrak{T}_2$–continuous and that Equation (D.3.1) holds. Furthermore, suppose that also the limit semi-inner product fulfills the continuity conditions and its extension to elements of V^f still takes values in $\mathcal{B}$. Then V^f with extension of the semi-inner product (D.3.1) by $\mathfrak{T}_1$–$\mathfrak{T}_2$–continuity is a sequentially $\mathfrak{T}_1$–$\mathfrak{T}_2$–continuous semi-Hilbert $\mathcal{B}$–$\mathcal{B}$–module. Letting $E_{\lambda_0} = V^f$, the family $\bigl(E_\lambda\bigr)_{\lambda\in\Lambda}$ is V–related, sequentially $\mathfrak{T}_1$–$\mathfrak{T}_2$–continuous and we have $\lim\limits_\lambda E_\lambda = E_{\lambda_0}$.

Obviously, after dividing out all null-spaces, Definition D.3.1 may be restricted to the case of pre-Hilbert modules. If $\mathcal{B}$ is a pre-C^*–algebra and left multiplication is norm continuous on all E_λ, we may perform a completion. Convergence of a family of Hilbert modules means that there is a familiy of dense submodules for which Definition D.3.1 applies.

D.3.3 Remark. If the $*$ is continuous in both topologies, then it is sufficient to check the $\mathfrak{T}_1$–$\mathfrak{T}_2$–continuity conditions only for either the left or the right argument of the semi-inner product.

Furthermore, if the multiplication in $\mathcal{B}_0$ is separately $\mathfrak{T}_2$–continuous, then it is sufficient to compute (D.3.1) on elements of the left generate of V in $V^f_{\mathcal{B}_0}$. However, there is no way out of the necessity to compute the limit on any single element in the left generate. This had been avoided in [AL96], so that the convergence used therein is at most a convergence of right Hilbert modules. However, notice that, in particular, the left multiplication will cause later on a big growth of the limit module. The algebraic operations in the construction of a full Fock module cannot even be formulated without the left multiplication.

Now we start choosing the ingredients of Definition D.3.1 for our problem. For $\mathcal{B}_0$ we choose the $*$–algebra $\mathfrak{W}_0 = \operatorname{span}\{e^{i\varkappa\cdot p}e^{i\rho\cdot q} \colon \varkappa, \rho \in \mathbb{R}^d\}$ of Weyl operators.

In order to proceed, we have to recall some basic facts about the Weyl algebra. As a reference see e.g. the book [Pet90] of Petz. The Weyl algebra $\mathfrak{W}$ is the C^*–algebra generated by unitary groups of elements of a C^*–algebra subject to Relations (D.1.1). By *Slawny's theorem* this C^*–algebra is unique, so that the definition makes sense. A representation of $\mathfrak{W}$ on a Hilbert space induces a weak topology on $\mathfrak{W}$. However, this topology depends highly on the representation under consideration. For instance, we identify elements $b \in \mathfrak{W}$ always as operators in $\mathcal{B}(S)$. In this representation the operators depend strongly continuous on the parameters $\varkappa$ and ρ. (Such a representation is called *regular*. An irreducible regular representation of $\mathfrak{W}$ is determined up to unitary equivalence.)

Denote by $\mathfrak{W}_p$ and $\mathfrak{W}_q$ the $*$–subalgebras of $\mathfrak{W}_0$ spanned by all $e^{i\varkappa\cdot p}$ and spanned by all $e^{i\rho\cdot q}$, respectively. The Weyl operators are linearly independent, i.e. as a vector space we may identify $\mathfrak{W}_0$ with $\mathfrak{W}_p \otimes \mathfrak{W}_q$ via $e^{i\varkappa\cdot p}e^{i\rho\cdot q} \equiv e^{i\varkappa\cdot p} \otimes e^{i\rho\cdot q}$. Since $\{e^{i\rho\cdot q}\}_{\rho\in\mathbb{R}^d}$ is a basis for $\mathfrak{W}_q$, we may identify $\mathcal{B}_0$ with $\bigoplus_{\rho\in\mathbb{R}^d} \mathfrak{W}_p$. We identify $\mathfrak{W}_p$ as a subalgebra of $L^\infty(\mathbb{R}^d) \subset \mathcal{B}(S)$. By the *momentum algebra* $\mathcal{P}$ we mean the $*$–subalgebra $\mathcal{C}_b(\mathbb{R}^d)$ of $L^\infty(\mathbb{R}^d)$. Notice that the C^*–algebra $\mathcal{P}$ contains $\mathfrak{W}_p$. For $\mathcal{B}$ we choose $\bigoplus_{\rho\in\mathbb{R}^d} \mathcal{P}$. We have $\mathcal{B} \subset \bigoplus_{\rho\in\mathbb{R}^d} L^\infty(\mathbb{R}^d) \subset \mathcal{B}(S)$.

In order to define the topology $\mathfrak{T}_1$, we need the weak topology arising from a different representation. We define a representation π of $\mathfrak{W}$ on $\overline{\bigoplus_{\rho\in\mathbb{R}^d} L^2(\mathbb{R}^d)}$ (consisting of families $\big(f_\rho\big)_{\rho\in\mathbb{R}^d}$ where $f_\rho \in L^2(\mathbb{R}^d)$) by setting

$$\pi(e^{i\varkappa\cdot p}e^{i\rho'\cdot q})\big(f_\rho\big)_{\rho\in\mathbb{R}^d} = \big(e^{i\varkappa\cdot p}e^{i\rho'\cdot q}f_{\rho-\rho'}\big)_{\rho\in\mathbb{R}^d}.$$

This representation extends to elements $b \in \bigoplus_{\rho\in\mathbb{R}^d} L^\infty(\mathbb{R}^d)$. It is, roughly speaking, regular with respect to $\varkappa$, however, 'discrete' with respect to ρ'.

Let $\mathcal{I}$ denote a finite subset of $\mathbb{R}^d$. We equip $\bigoplus_{\rho\in\mathcal{I}} L^\infty(\mathbb{R}^d)$ with the restriction of the weak topology on $\bigoplus_{\rho\in\mathbb{R}^d} L^\infty(\mathbb{R}^d)$ induced by the representation π. We equip $\bigoplus_{\rho\in\mathbb{R}^d} L^\infty(\mathbb{R}^d)$ with a different topology by considering it as the *strict inductive limit* of $\Big(\bigoplus_{\rho\in\mathcal{I}} L^\infty(\mathbb{R}^d)\Big)_{\mathcal{I}\subset\mathbb{R}^d}$; see e.g. Yosida [Yos80, Definition I.1.6]. Clearly, a sequence $\Big(\sum_{\rho\in\mathbb{R}^d} e^{i\rho\cdot q}h^n_\rho\Big)_{n\in\mathbb{N}} = \big(\big(h^n_\rho\big)_{\rho\in\mathbb{R}^d}\big)_{n\in\mathbb{N}}$ in $\bigoplus_{\rho\in\mathbb{R}^d} L^\infty(\mathbb{R}^d)$, where $h^n_\rho \in L^\infty(\mathbb{R}^d)$, converges, if and only if the h^n_ρ are different from zero only for a finite number of $\rho \in \mathbb{R}^d$ and if any of the sequences $\big(h^n_\rho\big)_{n\in\mathbb{N}}$ ($\rho \in \mathbb{R}^d$) converges in the weak topology of $L^\infty(\mathbb{R}^d)$. Notice that $\bigoplus_{\rho\in\mathbb{R}^d} L^\infty(\mathbb{R}^d)$ is sequentially complete and that $\mathcal{B}_0$ is sequentially dense in this topology. By restriction to $\mathcal{B}$, we obtain the topology $\mathfrak{T}_1$. Notice that convergence of a sequence in the topology $\mathfrak{T}_1$ also implies convergence in the weak topology of $\mathcal{B}(S)$.

The topology $\mathfrak{T}_2$ is the topology induced by matrix elements with respect to the Schwartz functions $\mathcal{S}(\mathbb{R}^d)$. Thus, $\langle f, g\rangle_\lambda$ converges to $b \in \mathcal{B}(S)$, if and only if $\langle \xi, \langle f, g\rangle_\lambda \zeta\rangle$ converges to $\langle \xi, b\zeta\rangle$ for all $\xi, \zeta \in \mathcal{S}(\mathbb{R}^d)$. Since an element in $\mathcal{B}_0$ leaves invariant the domain of Schwartz functions, the multiplication with elements of $\mathcal{B}_0$ is a $\mathfrak{T}_2$–continuous operation. Also the $*$ is continuous in both topologies, i.e. Remark D.3.3 applies.

Of course, we choose $\Lambda = [0, \infty)$, ordered decreasingly, and $\lambda_0 = 0$. We return to $V = \mathfrak{S}(\mathbb{R}) \otimes \mathcal{C}_c(\mathbb{R}^d)$. Fix $\lambda > 0$ and consider $V(\mathcal{B}(S))$ equipped with its semi-inner product $\langle \bullet, \bullet\rangle_\lambda$, the left multiplication (D.2.2) and the embedding i_λ being the extension of the canonical embedding $i \colon V \to V(\mathcal{B}(S))$. Then our E_λ are $i_\lambda(V^f)$.

D.3.4 Proposition. *The $\big(E_\lambda\big)_{\lambda>0}$ form a V–related, sequentially $\mathfrak{T}_1$–$\mathfrak{T}_2$–continuous family of semi-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules.*

Proof. First, we show $\mathfrak{T}_1$–$\mathfrak{T}_2$–continuity. Notice that for sequences convergence in $\mathfrak{T}_1$ implies convergence in the weak topology and that convergence in the weak topology implies convergence in $\mathfrak{T}_2$. Therefore, it suffices to show that for all $f, g \in V^f_{\mathcal{B}}$ the mappings $b \mapsto \langle f, gb\rangle^\lambda$ and $b \mapsto \langle f, b.g\rangle^\lambda$ are sequentially weakly continuous. However, by right $\mathcal{B}$–linearity, continuity of the first mapping is a triviality. The second mapping, actually, is an inner product of elements of $\mathcal{C}^s_c(\mathbb{R}^d, \mathcal{B}(S))$. The mapping depends weakly continuous on b on bounded subsets. In particular, it is sequentially weakly continuous.

It remains to show that the inner product maps into $\mathcal{B}$. For $f = I\!I_{[t,T]}\widetilde{f}$, $g = I\!I_{[s,S]}\widetilde{g} \in V$ we have

$$\begin{aligned}\langle f, g\rangle^\lambda &= \int dk \int_t^T d\tau \int_s^S d\sigma\, \overline{\widetilde{f}(k)}\widetilde{g}(k)\gamma^*_\lambda(\tau, k)\gamma_\lambda(\sigma, k)\\ &= \frac{1}{\lambda^2}\int dk \int_t^T d\tau \int_s^S d\sigma\, \overline{\widetilde{f}(k)}\widetilde{g}(k)e^{i\frac{\sigma-\tau}{\lambda^2}(p\cdot k+\omega(|k|))}\\ &= \int dk \int_t^T d\tau \int_{\frac{s-\tau}{\lambda^2}}^{\frac{S-\tau}{\lambda^2}} du\, \overline{\widetilde{f}(k)}\widetilde{g}(k)e^{iu(p\cdot k+\omega(|k|))}.\end{aligned} \tag{D.3.2}$$

This is the weak limit of elements in $\mathfrak{W}_p$ and, therefore, an element of $\mathcal{P} \subset \mathcal{B}$. Automatically, we have $\langle f, gb\rangle^\lambda \in \mathcal{B}$ for $b \in \mathcal{B}$.

Now consider $b = h(p)e^{i\rho\cdot q} \in \mathcal{B}$ ($h \in \mathcal{P}$). By Equation (D.1.1) and manipulations similar to (D.3.2) we find

$$\langle f, b.g\rangle^\lambda = \int dk \int_t^T d\tau \int_{\frac{s-\tau}{\lambda^2}}^{\frac{S-\tau}{\lambda^2}} du\, \overline{\widetilde{f}(k)}\widetilde{g}(k)e^{-i\frac{\tau}{\lambda^2}\rho\cdot k}e^{iu((p-\rho)\cdot k+\omega(|k|))}h(p+k)e^{i\rho\cdot q}. \tag{D.3.3}$$

The integral without the factor $e^{i\rho\cdot q}$ is a continuous bounded function of p, i.e. an element of the momentum algebra $\mathcal{P} \subset \mathcal{B}$. It follows that also $\langle f, b.g\rangle^\lambda \in \mathcal{B}$. ■

Next we evaluate the limit in (D.3.1). The following proposition is just a repetition of a result in [AL96]. However, notice that the integrations have to be performed precisely in the order indicated (i.e. the p–integration first).

D.3.5 Proposition [AL96]. *Let $f, g \in V$ be given as for Equation* (D.3.2) *and $\xi, \zeta \in \mathcal{S}(\mathbb{R}^d)$. Then*

$$\lim_{\lambda\to 0}\big\langle \xi, \langle f, g\rangle^\lambda \zeta\big\rangle = \langle \mathbb{I}_{[t,T]}, \mathbb{I}_{[s,S]}\rangle \int dk\, \overline{\widetilde{f}(k)}\widetilde{g}(k) \int du \int dp\, \overline{\xi(p)}\zeta(p) e^{iu(p\cdot k+\omega(|k|))}. \tag{D.3.4}$$

The factor $\langle \mathbb{I}_{[t,T]}, \mathbb{I}_{[s,S]}\rangle$ is the inner product of elements of $L^2(\mathbb{R})$.

Proof. The matrix element of Equation (D.3.2) is

$$\begin{aligned}\big\langle \xi, \langle f, g\rangle^\lambda \zeta\big\rangle &= \int dk\, \overline{\widetilde{f}(k)}\widetilde{g}(k) \int_t^T d\tau \int_{\frac{s-\tau}{\lambda^2}}^{\frac{S-\tau}{\lambda^2}} du\, e^{iu\omega(|k|)} \int dp\, \overline{\xi(p)}\zeta(p) e^{iup\cdot k} \\ &= \int dk\, \overline{\widetilde{f}(k)}\widetilde{g}(k) \int_t^T d\tau \int_{\frac{s-\tau}{\lambda^2}}^{\frac{S-\tau}{\lambda^2}} du\, e^{iu\omega(|k|)}\, \widehat{\overline{\xi}\zeta}(uk).\end{aligned}$$

For $\lambda > 0$ the order of integrations does not matter, so we may, indeed, decide to perform the p–integration first. $\widehat{\overline{\xi}\zeta}$, the Fourier transform of $\overline{\xi}\zeta$, is a rapidly decreasing function. Therefore, the λ–limit in the bounds of the u–integral may by performed for almost all k (namely $k \neq 0$) and all τ. Depending on the sign of $s-\tau$ and $S-\tau$, respectively, the bounds converge to $\pm\infty$. A careful analysis, involving the *theorem of dominated convergence*, yields the scalar product of the indicator functions in front of (D.3.4). The resulting function of k is bounded by a positive multiple of the function $\left|\frac{f(k)}{k}\right|$ which is integrable for $d \geq 2$. By another application of the *theorem of dominated convergence* and a resubstitution of $\widehat{\overline{\xi}\zeta}$ the formula follows. ■

Now we will show as one of our main results that the sesquilinear form on $\mathcal{S}(\mathbb{R}^d)$ given by (D.3.4) indeed defines an element of $\mathcal{B}$. In [AL96] it was not clear, if (D.3.4) defines any operator on S. Denote by e_k the unit vector in the direction of $k \neq 0$ and by $\int de_k$ the angular part of an integration over k in polar coordinates.

D.3.6 Lemma. *Let f be an element of $\mathcal{C}_c(\mathbb{R}^d)$, ξ be an element of $\mathcal{S}(\mathbb{R}^d)$ and $d \geq 3$. Furthermore, let ω be a $\mathcal{C}^1$–function $\mathbb{R}_+ \to \mathbb{R}_+$ of the form $\omega(r) = r\omega_0(r)$, where $0 \leq \omega_0(0) < \infty$ and ω_0' bounded below by a constant $c > 0$. Denote by ω_0^{-1} the inverse function of ω_0 extended by zero to arguments less than $\omega_0(0)$. Then*

$$\int dk\, f(k) \int du \int dp\, \xi(p) e^{iu(p\cdot k+\omega(|k|))} = 2\pi \int dp\, \xi(p) \int de_k \frac{\omega_0^{-1}(-p\cdot e_k)^{d-2}}{\omega_0'(\omega_0^{-1}(-p\cdot e_k))} f(\omega_0^{-1}(-p\cdot e_k)e_k).$$

Moreover,

$$\int de_k \frac{\omega_0^{-1}(-p\cdot e_k)^{d-2}}{\omega_0'(\omega_0^{-1}(-p\cdot e_k))} f(\omega_0^{-1}(-p\cdot e_k)e_k)$$

as a function of p *is an element of* $\mathcal{C}_b(\mathbb{R}^d)$.

D.3.7 Remark. Formally, we can perform the u–integration and obtain

$$2\pi \int dk\, f(k) \int dp\, \xi(p)\delta\bigl(p\cdot k + \omega(|k|)\bigr)$$

where the δ–distribution is one-dimensinal (not d–dimensional). The statement of the lemma arises by performing the integration over $|k|$ first and use of the formal rules for δ–functions. However, f is in general not a test function and the domain of the $|k|$–integration is $\mathbb{R}_+$, not $\mathbb{R}$. Therefore, some attention has to be paid. We will use this formal δ–notation whenever it is justified by Lemma D.3.6.

Proof of Lemma D.3.6. Let us write k in polar coordinates, i.e. $k = re_k$. For fixed $k \neq 0$ we write the p–integral in cartesian coordinates with the first coordinate p_0 being the component of p along e_k. Then p has the form $p = p_0e_k + p_\perp$ with $p_\perp$ the unique component of p perpendicular to e_k. In this representation the exponent has the form $iur(p_0 + \omega_0(r))$ and we may apply the inversion formula of the theory of Fourier transforms to the p_0–integration followed by the u–integration. The result may be described formally by the δ–function $2\pi\delta\bigl(r(p_0 + \omega_0(r))\bigr)$ for the p_0–integration. We obtain

$$\begin{aligned}\int du \int dp\, \xi(p) e^{iu(p\cdot k+\omega(|k|))} &= 2\pi \int dp\, \xi(p)\delta\bigl(r(p_0 + \omega_0(r))\bigr)\\ &= \lim_{\varepsilon\to 0} \frac{2\pi}{\varepsilon} \int dp\, \xi(p) I\!I_{[0,\varepsilon]}\bigl(p\cdot k + \omega_0(|k|)\bigr).\end{aligned}$$

It is routine to check that the right-hand side is bounded uniformly in $\varepsilon \in (0, 1]$ by a positive multiple of $\frac{1}{|k|}$. Therefore, again by the *theorem of dominated convergence* we may postpone the ε–limit also for the k–integration and obtain

$$\int dk\, f(k) \int du \int dp\, \xi(p) e^{iu(p\cdot k+\omega(|k|))} = \lim_{\varepsilon\to 0} \frac{2\pi}{\varepsilon} \int dk\, f(k) \int dp\, \xi(p) I\!I_{[0,\varepsilon]}\bigl(p\cdot k + \omega_0(|k|)\bigr).$$

Now the order of integrations no longer matters.

We choose polar coordinates for the k–integration and perform first the integral over $r = |k|$. The above formula for finite ε becomes

$$\frac{2\pi}{\varepsilon} \int dp\, \xi(p) \int de_k \int dr\, r^{d-1} f(re_k) I\!I_{[0,\varepsilon]}\bigl(r(p\cdot e_k + \omega_0(r))\bigr).$$

Consider the function $F(r) = r(p \cdot e_k + \omega_0(r))$. From the properties of ω_0 it follows that $\omega_0(r) \geq \omega_0(0) + cr$. Consequently, $F(r) \geq r(p \cdot e_k + \omega_0(0)) + cr^2$. If $p \cdot e_k + \omega_0(0) \geq 0$, then $F(r) \geq cr^2$ and, because $d \geq 3$, the integral $\frac{1}{\varepsilon} \int dr\, r^{d-1} f(re_k) I\!I_{[0,\varepsilon]}\bigl(r(p \cdot e_k + \omega_0(r))\bigr)$ converges to 0 for $\varepsilon \to 0$ uniformly in $p \cdot e_k \geq -\omega_0(0)$.

On the other hand, if $p \cdot e_k + \omega_0(0) < 0$, then $F(r)$ starts with 0 at $r = 0$, is negative until the second zero $r_0 = \omega_0^{-1}(-p \cdot e_k))$ and increases monotonically faster than cr^2. We make the substitution $\mu = F(r)$ and obtain

$$\frac{1}{\varepsilon} \int dr\, r^{d-1} f(re_k) I\!I_{[0,\varepsilon]}\bigl(r(p \cdot e_k + \omega_0(r))\bigr) = \frac{1}{\varepsilon} \int_0^\varepsilon d\mu\, \frac{r(\mu)^{d-1}}{p \cdot e_k + \omega_0(r(\mu)) + r(\mu)\omega_0'(r(\mu))} f(r(\mu)e_k).$$

The integrand is bounded by $\frac{r(\mu)^{d-2}}{\omega_0'(r(\mu))} \sup\limits_{k \in \mathbb{R}^d} |f(k)|$. Therefore, the integral converges uniformly in $p \cdot e_k < -\omega_0(0)$ to the limit

$$\frac{r_0^{d-2}}{\omega_0'(r_0)} f(r_0 e_k).$$

Substituting the concrete form of r_0 and extending ω_0^{-1} by $\omega_0^{-1}(F) = 0$ for $F \leq \omega_0(0)$, we obtain the claimed formula.

The last statement of the lemma follows from the observation that ω_0^{-1} is a continuous function and that if $\omega_0^{-1}(-p \cdot e_k)$ is big, then $f(\omega_0^{-1}(-p \cdot e_k)) = 0$. ■

D.3.8 Corollary. *The sesquilinear form on $\mathcal{S}(\mathbb{R}^d)$ given by* (D.3.4) *defines an element of $\mathcal{B}$.*

Formally, we denote this element by $\langle f, g \rangle^0 = 2\pi \langle I\!I_{[t,T]}, I\!I_{[s,S]} \rangle \int dk\, \overline{\widetilde{f}(k)} \widetilde{g}(k) \delta\bigl(p \cdot k + \omega(|k|)\bigr)$. Notice also the commutation relations

$$2\pi \int dk\, \overline{\widetilde{f}(k)} \widetilde{g}(k) \delta\bigl((p - \rho) \cdot k + \omega(|k|)\bigr) = e^{i\rho \cdot q} 2\pi \int dk\, \overline{\widetilde{f}(k)} \widetilde{g}(k) \delta\bigl(p \cdot k + \omega(|k|)\bigr) e^{-i\rho \cdot q}.$$

Again it is clear that the limit extends to the right $\mathcal{B}_0$–generate of V and that the function $b \mapsto \langle f, gb \rangle^0$ extends weakly continuous, i.e. a fortiori $\mathfrak{T}_1$–$\mathfrak{T}_2$–continuous, from $\mathcal{B}_0$ to $\mathcal{B}$. It remains to show this also for the left $\mathcal{B}_0$–generate.

D.3.9 Proposition. *Let again $f, g \in V$ be given as for Equation* (D.3.2) *and $\xi, \zeta \in \mathcal{S}(\mathbb{R}^d)$. Furthermore, let $b = e^{i\varkappa \cdot p} e^{i\rho \cdot q} \in \mathcal{B}_0$. Then*

$$\lim_{\lambda \to 0} \bigl\langle \xi, \langle f, b.g \rangle^\lambda \zeta \bigr\rangle = \delta_{\rho 0} \langle I\!I_{[t,T]}, I\!I_{[s,S]} \rangle \int dk\, \overline{\widetilde{f}(k)} \widetilde{g}(k) e^{i\varkappa \cdot k} \int du \int dp\, \overline{\xi(p)} e^{iu((p-\rho)\cdot k + \omega(|k|))} e^{i\varkappa \cdot p} \zeta(p). \quad \text{(D.3.5)}$$

D.3.10 Remark. $\delta_{\rho 0}$ is, indeed, the Kronecker δ. So the left multiplication in the limit is no longer weakly continuous. This is the reason for our rather complicated choice of the topology $\mathfrak{T}_1$.

PROOF OF PROPOSITION D.3.9. Our starting point is equation (D.3.3). If $\rho = 0$ the statement follows precisely as in the proof of Proposition D.3.5.

If $\rho \neq 0$, the expression is similar to the case $\rho = 0$ (where ζ is replaced by $e^{i\rho\cdot q}\zeta$). The only difference is the *oscillating factor* $e^{-i\frac{\tau}{\lambda^2}\rho\cdot k}$. Similarly, one argues that the λ–limit in the bounds of the u–integral may be performed first. By an application of the Riemann-Lebesgue *lemma* the resulting integral over the oscillating factor converges to 0. ∎

D.3.11 Remark. The proposition shows in a particularly simple example how the *Riemann-Lebesgue lemma* makes a lot of matrix elements disappear in the limit. This fundamental idea is due to [AL96]. However, in [AL96] the idea was not applied to the left multiplication.

D.3.12 Corollary. *The sesquilinear form on $\mathcal{S}(\mathbb{R}^d)$, defined by* (D.3.5)*, determines the element*

$$\langle f, b.g\rangle^0 = \delta_{\rho 0}\langle f, (e^{i\varkappa\cdot k}g)\rangle^0 e^{i\varkappa\cdot p}$$

in $\mathcal{P}$. Moreover, the mapping $b \mapsto \langle f, b.g\rangle^0$ extends sequentially $\mathfrak{T}_1$–$\mathfrak{T}_2$–continuously from $\mathcal{B}_0$ to $\mathcal{B}$.

PROOF. Like for $\rho = 0$, it follows that (D.3.5), indeed, defines an element of $\mathcal{P}$. Now we observe that a matrix element $\langle f, h.g\rangle^0$, written in the form according to Lemma D.3.6, may be extended from elements in $\mathfrak{W}_p$ to all elements $h \in \mathcal{P}$. It suffices to show that the mapping $h \mapsto \langle f, h.g\rangle^0$ is sequentially weakly continuous on $\mathcal{P}$. To see this we perform first the p–integral and obtain a bounded function on e_k. Inserting a sequence $\big(h_n\big)_{n\in\mathbb{N}}$, the resulting sequence of functions on e_k is uniformly bounded. By the *theorem of dominated convergence* we may exchange limit and e_k–integration. ∎

The following theorem is proved just by collecting all the results.

D.3.13 Theorem. *The $\big(E_\lambda\big)_{\lambda\geq 0}$ form a V–related, sequentially $\mathfrak{T}_1$–$\mathfrak{T}_2$–continuous family of semi-Hilbert $\mathcal{B}$–$\mathcal{B}$–modules and*

$$\lim_{\lambda\to 0} E_\lambda = E_0.$$

Now we are going to understand the structure of E_0 better. Consider $E_0 = \mathcal{B} \otimes V \otimes \mathcal{B} = \bigoplus_{\rho \in \mathbb{R}^d} \mathcal{P} \otimes V \otimes \mathcal{B}$. Any of the summands $\mathcal{P} \otimes V \otimes \mathcal{B}$ inherits a semi-Hilbert $\mathcal{P}$–$\mathcal{B}$–module structure just by restriction of the operations of E_0. Notice that the inner products differ for different indices ρ. However, the left multiplications by elements $h \in \mathcal{P}$ coincide. Of course, multiplication of an element in the ρ–th summand by $e^{i\rho' \cdot q}$ from the left, is not only pointwise multiplication, but *shifts* this element into the $(\rho + \rho')$–th summand.

Next we recall that $V = \mathfrak{S}(\mathbb{R}) \otimes \mathcal{C}_c(\mathbb{R})$. The factor $\langle I\!I_{[t,T]}, I\!I_{[s,S]} \rangle$ tells us that E_0 is the exterior tensor product of the pre-Hilbert $\mathbb{C}$–$\mathbb{C}$–module $\mathfrak{S}(\mathbb{R})$ and $\mathcal{B} \otimes \mathcal{C}_c(\mathbb{R}) \otimes \mathcal{B}$ with a suitable semi-Hilbert $\mathcal{B}$–$\mathcal{B}$–module structure.

In order to combine both observations we make the following definition. Fix $\rho \in \mathbb{R}^d$. We turn $\mathcal{C}_c(\mathbb{R}^d, \mathcal{B})^s$ into a $\mathcal{P}$–$\mathcal{B}$–module by pointwise multiplication by elements of $\mathcal{B}$ from the right and the left multiplication defined by setting $[h.f](k) = h(p + k) f(k)$. Denote by V^r_ρ the $\mathcal{P}$–$\mathcal{B}$–subsubmodule of $\mathcal{C}_c(\mathbb{R}^d, \mathcal{B})^s$ generated by $\mathcal{C}_c(\mathbb{R}^d)$. We turn V^r_ρ into a semi-Hilbert $\mathcal{P}$–$\mathcal{B}$–module by setting

$$\langle f, g \rangle^{(\rho)} = 2\pi \int dk\, f(k)^* \delta\big((p - \rho) \cdot k + \omega(|k|)\big) g(k).$$

Set $\mathcal{E} = \bigoplus_{\rho \in \mathbb{R}^d} V^r_\rho$. For an element $\big(f_\rho\big)_{\rho \in \mathbb{R}^d} \in \mathcal{E}$ we define the left action of $e^{i\rho' \cdot q}$ by $e^{i\rho' \cdot q}.\big(f_\rho\big)_{\rho \in \mathbb{R}^d} = \big(e^{i\rho' \cdot q} f_{\rho - \rho'}\big)_{\rho \in \mathbb{R}^d}$. The following theorem may be checked simply by inspection.

D.3.14 Theorem. *The mapping*

$$\sum_{\rho \in \mathbb{R}^d} (h_\rho e^{i\rho \cdot q}) \otimes \big(I\!I_{[t,T]} f_\rho\big) \otimes b_\rho \longmapsto I\!I_{[t,T]} \otimes \big(h_\rho.(e^{i\rho \cdot q} f_\rho) b_\rho\big)_{\rho \in \mathbb{R}^d},$$

where $I\!I_{[t,T]} \in \mathfrak{S}(\mathbb{R}), f_\rho \in \mathcal{C}_c(\mathbb{R}^d) \subset V^r_\rho$, $h_\rho \in \mathcal{P}$, *and* $b \in \mathcal{B}$, *(all different from 0 only for finitely many* $\rho \in \mathbb{R}^d$*) defines a surjective* $\mathcal{B}$–$\mathcal{B}$*–linear isometry*

$$E_0 = \mathcal{B} \otimes V \otimes \mathcal{B} \longrightarrow \mathfrak{S}(\mathbb{R}) \otimes \mathcal{E}.$$

The $\otimes$*–sign on the right-hand side is that of the exterior tensor product.*

D.3.15 Remark. $\mathcal{C}^s_c(\mathbb{R}^d, \mathcal{B})$ may be considered as a completion of $\mathcal{C}_c(\mathbb{R}^d) \otimes \mathcal{B}$. The left multiplication by elements of $\mathcal{P}$ leaves invariant $\mathcal{C}^s_c(\mathbb{R}^d, \mathcal{B})$ and the inner product of E_0, first restricted to $\mathcal{P} \otimes V \otimes \mathcal{B} = \mathcal{P} \otimes V^r$ and then extended to $\mathcal{P} \otimes \overline{V^r}^s$, does not distinguish between elements $h \otimes f$ and $\mathbf{1} \otimes (h.f)$. Therefore, already the comparably small spaces V^r_ρ are sufficient to obtain an isometry.

We find the commutation relations

$$\Big[(e^{i\varkappa\cdot p}e^{i\rho'\cdot q}).\big(f_\rho\big)_{\rho\in\mathbb{R}^d}\Big](t,k) = \big(e^{i\varkappa\cdot k}f_{\rho-\rho'}(t,k)\big)_{\rho\in\mathbb{R}^d}(e^{i\varkappa\cdot p}e^{i\rho'\cdot q}) \tag{D.3.6}$$

for elements $f_\rho \in V$. This means for an arbitrary element in $f \in \mathfrak{S}(\mathbb{R}) \otimes \mathcal{E}$ and $\rho, \varkappa \in \mathbb{R}^d$ there exists $f' \in \mathfrak{S}(\mathbb{R}) \otimes \mathcal{E}$ such that $(e^{i\varkappa\cdot p}e^{i\rho'\cdot q}).f = f'e^{i\varkappa\cdot p}e^{i\rho'\cdot q}$ and conversely. (The same is true already for $\mathcal{E}$.) Such a possibility is not unexpected, because it already occurs for finite λ.

D.3.16 Remark. Of course, it is true that $\mathcal{E}$ is non-separable. This is not remarkable due to the non-separability of $\mathfrak{W}_0$. However, the separabilty condition usually imposed on Hilbert modules is that of being countably generated. Clearly, $\mathcal{E}$ fulfills this condition, because V is separable.

A much more remarkable feature is that the left multiplication is no longer weakly continuous. However, also this behaviour is not completely unexpected. It often happens in certain limits of representations of algebras that certain elements in the representation space, fixed for the limit, become orthogonal. Consider, for instance, the limit $\hbar \to 0$ for the canonical commutation relations or the limits $q \to \pm 1$ for the quantum group $SU_q(2)$; see [Ske99b]. In both examples the limits of suitably normalized coherent vectors become orthogonal in the limit.

D.3.17 Remark. Since $\mathcal{B} \subset \mathcal{B}(S)$ is a pre-C^*–algebra, $\mathcal{E}$ has a semi-norm and right multiplication fulfills $\|fb\| \leq \|f\|\,\|b\|$. We show that left multiplication by an element of $\mathcal{B}$ acts at least boundedly on $\mathcal{E}$. Indeed, we have $\|f\|^2 = \sup\limits_{\xi\in\mathcal{S}(\mathbb{R}^d),\|\xi\|=1} \big\langle\xi, \langle f,f\rangle\xi\big\rangle$. Any element in $b = \big(h_\rho\big)_{\rho\in\mathbb{R}^d} \in \mathcal{B}$ may be $\mathfrak{T}_1$–approximated by a sequence $\big(b_n\big)_{n\in\mathbb{N}}$ of elements in $\mathcal{B}_0$ where $b_n = \big(h^n_\rho\big)_{\rho\in\mathbb{R}^d}$. By the *Kaplansky density theorem* and weak separabilty of the unit-ball of $L^\infty(\mathbb{R}^d)$, we may assume that $\big\|h^n_\rho\big\| = \|h_\rho\|$ $(\rho \in \mathbb{R}^d, n \in \mathbb{N})$. We have

$$\big\langle\xi, \langle b_n\varphi_\lambda(f), b_n\varphi_\lambda(f)\rangle\xi\big\rangle \leq \|b_n\|^2 \big\langle\xi, \langle\varphi_\lambda(f), \varphi_\lambda(f)\rangle\xi\big\rangle \leq \big\langle\xi, \langle\varphi_\lambda(f), \varphi_\lambda(f)\rangle\xi\big\rangle\Big(\sum_{\rho\in\mathbb{R}^d}\|h_\rho\|\Big)^2.$$

The number of ρ's for which $h^n_\rho \neq 0$ for at least one $n \in \mathbb{N}$ is finite. Our claim follows, performing the limits first $\lambda \to 0$ and then $n \to \infty$. Therefore, if necessary, we may change to the Hilbert $\mathcal{B}$–$\mathcal{B}$–module $\overline{\mathcal{E}}$ where, however, $\mathcal{B}$ is only a pre-C^*–algebra.

D.3.18 Remark. It is not difficult to see that the left and right multiplication, actually, are sequentially $\mathfrak{T}_1$–continuous. Therefore, all our results in this section and in Section D.4 may be extended to the sequentially $\mathfrak{T}_1$–complete algebra $L^\infty(\mathbb{R}^d) \otimes \mathfrak{W}_q$.

D.4 The central limit theorem

In this section we prove in a central limit theorem that the moments of the collective creators and annihilators in the vacuum conditional expectation, represented in Section D.2 by symmetric creators and annihilators on the symmetric Fock module Γ_c, converge to the moments of the corresponding free creators and annihilators on the full Fock module $\underline{\mathcal{F}}(\mathfrak{S}(\mathbb{R}) \otimes \mathcal{E})$ over the limit of the one-particle sector computed in Section D.3.

In a first step we show that in a pyramidally ordered product (i.e., so to speak, an anti-normally ordered product) the moments of the free operators for finite λ converge to the moments of the free operators for $\lambda = 0$. In the next step we show that nothing changes, if we replace for finite λ the free operators by symmetric operators. For this step an explicit knowledge of the embedding of the symmetric Fock module into the full Fock module is indispensable. The final step consists in showing that the limits for arbitrary monomials respect the free commutation relations (6.1.1).

In the course of this section we compute a couple of $\mathfrak{T}_2$–limits of elements of $\mathcal{B}(S)$. For the sesquilinear forms on $\mathcal{S}(\mathbb{R}^d)$, defined by these algebra elements, all the limits already have been calculated by Accardi and Lu in [AL96]. Since the combinatorical problems of, for instance, how to write down an arbitrary monomial in creators and annihilators and so on, have been treated in [AL96] very carefully, we keep short in the proofs. Sometimes, we give only the main idea of a proof in a typical example.

New is that the limit sesquilinear forms define operators. This means that the limit conditional expectation, indeed, takes values in $\mathcal{B}(S)$. Also new is the interpretation of the limit of the moments of the collective operators as moments of free operators on a full Fock module in the vacuum expectation. The idea to see this, roughly speaking, by checking Relations (6.1.1) (see proof of Theorem D.4.3), has its drawback also in the computation of the limit of the sesquilinear forms. The structure of the proof is simplified considerably.

D.4.1 Theorem. *Let $f_i = \mathbb{I}_{[t_i,T_i]}\widetilde{f}_i, g_i = \mathbb{I}_{[s_i,S_i]}\widetilde{g}_i$ be in V $(i = 1, \ldots, n; n \in \mathbb{N})$. Then*

$$\lim_{\lambda\to 0}\langle \mathbf{1}, \ell(f_1)\cdots\ell(f_n)\ell^*(g_n)\cdots\ell^*(g_1)\mathbf{1}\rangle_\lambda = \langle \mathbf{1}, \ell(f_1)\cdots\ell(f_n)\ell^*(g_n)\cdots\ell^*(g_1)\mathbf{1}\rangle_0.$$

PROOF. First, we show that

$$\langle f_n \odot \ldots \odot f_1, g_n \odot \ldots \odot g_1\rangle_\lambda = \int dk_n \ldots \int dk_1\, \overline{\widetilde{f}_n(k_n)}\widetilde{g}_n(k_n)\cdots\overline{\widetilde{f}_1(k_1)}\widetilde{g}_1(k_1)$$
$$\int_{t_n}^{T_n} d\tau_n \int_{s_n}^{S_n} d\sigma_n \ldots \int_{t_1}^{T_1} d\tau_1 \int_{s_1}^{S_1} d\sigma_1\, \gamma^*_\lambda(\tau_1, k_1)\cdots\gamma^*_\lambda(\tau_n, k_n)\gamma_\lambda(\sigma_n, k_n)\cdots\gamma_\lambda(\sigma_1, k_1) \tag{D.4.1}$$

converges to the inner product on $\underline{\mathcal{F}}_0$ in $\mathfrak{T}_2$. We proceed precisely as in the proof of Proposition D.3.5. Here we are not very explicit, because we have been explicit there. We consider matrix elements of (D.4.1) with Schwartz functions ξ, ζ. The q's in the γ_λ's dissappear by extensive use of Relations (D.1.1), however, cause some shifts to the p's. We make the substitutions $u_i = \frac{\sigma_i - \tau_i}{\lambda^2}$ and after performing the p–integral we obtain the function $\widehat{\overline{\xi}\zeta}(u_n k_n + \ldots + u_1 k_1)$. Its modulo is for almost all $k_n, \ldots, k_1$ and all $\tau_n, \ldots, \tau_1$ a rappidly decreasing upper bound for the u_i–integrations. Similarly, as in the proof of Proposition D.3.5 one checks that the λ–limits for the u_i–integrals may be performed first. We obtain the result

$$\lim_{\lambda \to 0} \langle \xi, \langle f_n \odot \ldots \odot f_1, g_n \odot \ldots \odot g_1 \rangle_\lambda \zeta \rangle = \langle \mathit{I\!I}_{[t_n,T_n]}, \mathit{I\!I}_{[s_n,S_n]} \rangle \cdots \langle \mathit{I\!I}_{[t_1,T_1]}, \mathit{I\!I}_{[s_1,S_1]} \rangle$$
$$\cdot \int dk_1\, \overline{\widetilde{f}_1(k_1)} \widetilde{g}_1(k_1) \int du_1 \cdots \int dk_n\, \overline{\widetilde{f}_n(k_n)} \widetilde{g}_n(k_n) \int du_n$$
$$\cdot \int dp\, \overline{\xi(p)} \zeta(p) e^{iu_n((p+k_{n-1}+\ldots+k_1)\cdot k_n + \omega(|k_n|))} \cdots e^{iu_1(p\cdot k_1 + \omega(|k_1|))}.$$

of [AL96].

Now we proceed as in Lemma D.3.6 and bring the p–integration step by step to the outer position. (Take into account that after performing the integrals over p and over u_i, k_i $(i = m+1, \ldots, n)$ the result is still a rapidly decreasing function on u_i $(i = 1, \ldots, m)$ for almost all k_i $(i = 1, \ldots, m)$. Therfore, *Fubini's theorem* applies.) We obtain (by the same notational use of the δ–functions) that (D.4.1) converges to

$$\langle f_n \odot \ldots \odot f_1, g_n \odot \ldots \odot g_1 \rangle_0$$
$$= (2\pi)^n \langle \mathit{I\!I}_{[t_n,T_n]}, \mathit{I\!I}_{[s_n,S_n]} \rangle \cdots \langle \mathit{I\!I}_{[t_1,T_1]}, \mathit{I\!I}_{[s_1,S_1]} \rangle \int dk_n \ldots \int dk_1\, \overline{\widetilde{f}_n(k_n)} \widetilde{g}_n(k_n) \cdots \overline{\widetilde{f}_1(k_1)} \widetilde{g}_1(k_1)$$
$$\delta\big((p + k_{n-1} + \ldots + k_1) \cdot k_n + \omega(|k_n|)\big) \cdots \delta\big(p \cdot k_1 + \omega(|k_1|)\big). \blacksquare$$

D.4.2 Theorem. *Theorem D.4.1 remains true, if we replace on the left-hand side the free creators and annihilators by the symmetric creators and annihilators, i.e.*

$$\lim_{\lambda \to 0} \mathbb{E}_0(A_\lambda(f_1) \cdots A_\lambda(f_n) A_\lambda^*(g_n) \cdots A_\lambda^*(g_1))$$
$$= \lim_{\lambda \to 0} \langle \mathbf{1}, a(\varphi_\lambda(f_1)) \cdots a(\varphi_\lambda(f_n)) a^*(\varphi_\lambda(g_n)) \cdots a^*(\varphi_\lambda(g_1)) \mathbf{1} \rangle$$
$$= \langle \mathbf{1}, \ell(f_1) \cdots \ell(f_n) \ell^*(g_n) \cdots \ell^*(g_1) \mathbf{1} \rangle_0.$$

PROOF. Notice that

$$a^*(\varphi_\lambda(g_n)) \cdots a^*(\varphi_\lambda(g_1)) \mathbf{1} = \sqrt{n!} P \ell^*(\varphi_\lambda(g_n)) \cdots \ell^*(\varphi_\lambda(g_1)) \mathbf{1}.$$

Therefore, we are ready, if we show that in the sum over the permutations only the identity permutation contributes to the limit of the inner product.

Applying the flip to two neighbouring elements $\varphi_\lambda(g_{i+1}) \odot \varphi_\lambda(g_i)$ means exchanging the arguments $k_{i+1} \leftrightarrow k_i$. (The σ_i are dummies and may be labeled arbitrarily.) We find

$$\begin{aligned}&\big[\mathcal{F}(\varphi_\lambda(g_{i+1}) \odot \varphi_\lambda(g_i))\big](k_{i+1}, k_i)\\ &\qquad = \int_{s_i}^{S_i} d\sigma_{i+1} \int_{s_{i+1}}^{S_{i+1}} d\sigma_i \, e^{i\frac{\sigma_i - \sigma_{i+1}}{\lambda^2} k_{i+1}\cdot k_i} \gamma_\lambda(\sigma_{i+1}, k_{i+1}) \gamma_\lambda(\sigma_i, k_i) \widetilde{g}_i(k_{i+1}) \widetilde{g}_{i+1}(k_i).\end{aligned}$$

This differs only by the oscillating factor $e^{i\frac{\sigma_i - \sigma_{i+1}}{\lambda^2} k_{i+1}\cdot k_i}$ from the expression

$$\varphi_\lambda(\mathbb{I}_{[s_{i+1}, S_{i+1}]}\widetilde{g}_i) \odot \varphi_\lambda(\mathbb{I}_{[s_i, S_i]}\widetilde{g}_{i+1})$$

whose inner products are known to have finite limits. This oscillating factor cannot be neutralized by any other flip operation on a different pair of neighbours. Assume, for instance, for a certain permutation π that i is the first position, counting from the right, which is changed by π. Then π may be written in the form $\pi' \mathcal{F}_{(i,i+1)} \pi''$ where $\mathcal{F}_{(i,i+1)}$ is the flip of positions i and $i+1$ and π', π'' are permutations involving only the positions $i+1, \ldots, n$. A look at the concrete form of the exponents in the oscillating factors tells us that the oscillating factor arising from $\mathcal{F}_{(i,i+1)}$ will be neutralized at most on a null-set for the k_j–σ_j–integrations ($j = i+1, \ldots, n$). Therefore, any non-identical permutation does not contribute to the sum of all permutations. (Notice that also here for a proper argument the *theorem of dominated convergence* is involved.) ■

D.4.3 Central limit theorem. *Theorem D.4.2 remains true, if we replace on the left-hand side $A_\lambda(f_1) \cdots A_\lambda(f_n) A^*_\lambda(g_n) \cdots A^*_\lambda(g_1)$ by an arbitrary monomial in collective creators and annihilators and on the right-hand side $\ell(f_1) \cdots \ell(f_n) \ell^*(g_n) \cdots \ell^*(g_1)$ by the corresponding monomial in the free creators and annihilators. In other words, we expressed the limit of arbitrary moments of collective operators in the vacuum conditional expectation $\mathbb{E}_0$ as the moments of the corresponding free operators in the vacuum expectation on the limit full Fock module.*

Proof. We will show that, in a certain sense, $A_\lambda(f) A^*_\lambda(g) \to \langle f, g \rangle_0$ for $\lambda \to 0$; cf. Relations (6.1.1). Indeed, one easily checks that

$$\begin{aligned}&[a(f) a^*(g) F](k_n, \ldots, k_1)\\ &\qquad = \langle f, g \rangle F(k_n, \ldots, k_1) + \int dk \sum_{i=1}^{n} f^*(k) g(k_i) F(k, k_n, \ldots, \widehat{k_i}, \ldots, k_1)\end{aligned}$$

for $f, g \in \mathcal{C}^s_c(\mathbb{R}^d, \mathcal{B})$.

Replacing f, g by $\varphi_\lambda(f), \varphi_\lambda(g)$ $(f, g \in V)$, the first summand converges precisely to what we want, namely, $\langle f, g\rangle_0 F$. In the remaing sum we may, like in the proof of Theorem D.4.2, exchange the position of f^* and g. This produces an oscillating factor which makes the k–integral dissappear in the limit.

Finally, we must show that in a concrete expression, e.g. like

$$\begin{aligned}\mathbb{E}_0\big(A_\lambda(f_1)\cdots A_\lambda(f_n)A^*_\lambda(g_n)\cdots A^*_\lambda(g_{m+1})A_\lambda(f)A^*_\lambda(g)A^*_\lambda(g_m)\cdots A^*_\lambda(g_1)\big) \\ = \big\langle \mathbf{1}, a(\varphi_\lambda(f_1))\cdots a(\varphi_\lambda(f_n))a^*(\varphi_\lambda(g_n))\cdots a^*(\varphi_\lambda(g_{m+1})) \\ a(\varphi_\lambda(f))a^*(\varphi_\lambda(g))a^*(\varphi_\lambda(g_m))\cdots a^*(\varphi_\lambda(g_1))\mathbf{1}\big\rangle,\end{aligned}$$

the limit $a(\varphi_\lambda(f))a^*(\varphi_\lambda(g)) \to \langle f, g\rangle_0$ for the inner pairing may be computed first. But this follows in the usual way using arguments involving the *theorem of dominated convergence* and the *Riemann-Lebesgue lemma.* ■

D.4.4 Remark. It is possible to extend the preceeding results in an obvious manner to elements f in the $\mathcal{B}_0$–generate of V. This means that the moments of both $A^*(f)$ and $\ell^*(f)$ for finite λ converge to the moments of $\ell^*(f)$ on $\underline{\mathcal{F}}(\mathfrak{S}(\mathbb{R}) \otimes \mathcal{E})$. By a slight weakening of Definition D.3.1 in the sense that the generating set needs only to be topologically generating, one can show that $\lim_{\lambda\to 0} \underline{\mathcal{F}}(E_\lambda) = \underline{\mathcal{F}}(\mathfrak{S}(\mathbb{R}) \otimes \mathcal{E})$ and more or less also $\lim_{\lambda\to 0} \Gamma(\mathfrak{C}^s_c(\mathbb{R}^d, \mathcal{B})) = \underline{\mathcal{F}}(\mathfrak{S}(\mathbb{R}) \otimes \mathcal{E})$. However, since the notational effort and a precise reasoning would take a lot of time, we content ourselves with the central limit theorem. Since the moments of all creators and, henceforth, the inner products on the full Fock module are already determined by Relations (6.1.1), we do not really loose information on the limit module.

Appendix E

A summary on Hilbert modules

In this appendix we recall briefly for a quick reference the notions used in these notes which may be not so standard in Hilbert module theory. In any case we assume that the basic notations from Chapter 1 are known. This includes, in particular, Section 1.4 about operators on Hilbert modules and pre-Hilbert modules sometimes only over pre–C^*–algebras. Most important are Corollaries 1.4.3 and 1.4.4 which describe circumstances under which an operator on a semi-Hilbert module E respects the kernel of the semiinner product (i.e. the submodule of $\mathcal{N}_E = \{x \in E \colon \langle x, x \rangle = 0\}$). In particular, Corollary 1.4.3 guaranties that a mapping which is formally adjointable on a subset which generates E (algebraically) as a right module, gives rise to a mapping on the quotient $E/\mathcal{N}_E$.

In Section E.1 we review quickly the notion of von Neumann modules and their basic properties, and we recall the notions related to $\mathcal{B}(G)$–modules introduced in Examples 3.1.2, 3.3.4, 4.1.15, 4.2.13 and 6.1.6. In Section E.2 we discuss matrices of Hilbert modules as introduced in Examples 1.4.10, 1.6.6, 1.7.6, 1.7.7, 4.2.12 and 4.3.8. The organiziation of the few arguments we give there, differs considerably form the presentation in Part I. For instance, in Part I we use matrices to show positivity of tensor product (often refered to as the interior tensor product; see Section 4.2) and exterior tensor product (Section 4.3), whereas here we assume these notions as well-known and derive from them the properties of matrices. For the extensions of the exterior tensor product to the framework of von Neumann algebras see Remark 4.3.4.

Recall also that a two-sided module is a right module with a left action by another algebra as right module homomorphisms, and that we always assume that this left action is *non-degenerate*. Here *non-degenerate* is meant in the algebraic sense, whereas, *total* refers to the more common topological sense; see Definition 1.6.1. For unital algebras all notions coincide and mean that the unit acts as unit. If a left action is degenerate, instead, then we speak only of a representation. Notice that for the right multiplication on a (pre-)Hilbert module totality (and non-degeneracy in the unital case) is automatic.

E.1 Von Neumann modules

Let $\mathcal{B}$ be a pre–C^*–algebra acting non-degenerately on a pre-Hilbert space G (in other words, G is a pre-Hilbert $\mathcal{B}$–$\mathbb{C}$–module), and let E be a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module. Then the tensor product $H = E \odot G$ is a pre-Hilbert $\mathcal{A}$–$\mathbb{C}$–module, i.e. a pre-Hilbert space with a representation ρ of $\mathcal{A}$ by (adjointable) operators on H. We refer to ρ as the *Stinespring representation* of $\mathcal{A}$ (associated with E and G); cf. Remark 4.1.9.

To each $x \in E$ we associate an operator $L_x \colon G \to H, g \mapsto x \odot g$ in $\mathcal{B}^a(G,H)$. We refer to the mapping $\eta \colon x \mapsto L_x$ as the *Stinespring representation* of E (associated with G). If the representation of $\mathcal{B}$ on G is faithful (whence, isometric), then so is η. More precisely, we find $L_x^* L_y = \langle x, y \rangle \in \mathcal{B} \subset \mathcal{B}^a(G)$. We also have $L_{axb} = \rho(a) L_x b$ so that we may identify E as a *concrete* $\mathcal{A}$–$\mathcal{B}$–submodule of $\mathcal{B}^a(G,H)$.

In particular, if $\mathcal{B}$ is a von Neumann algebra on a Hilbert space G, then we consider E always as a concrete subset of $\mathcal{B}(G, E \bar{\odot} G)$. We say E is a *von Neumann $\mathcal{B}$–module*, if it is strongly closed in $\mathcal{B}(G, E \bar{\odot} G)$. If also $\mathcal{A}$ is a von Neumann algebra, then a *von Neumann $\mathcal{A}$–$\mathcal{B}$–module* E is a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module and a von Neumann $\mathcal{B}$–module such that the Stinespring representation ρ of $\mathcal{A}$ on $E \bar{\odot} G$ is normal.

The (strong closure of the) tensor product of von Neumann modules is again a von Neumann module (Proposition 4.2.24). Left multiplication by any element in $\mathcal{B}^a(E)$ (in particular, those coming from elements of $\mathcal{A}$) is a strongly continuous operation on E (Proposition 3.1.5). The $*$–algebra $\mathcal{B}^a(E)$ is a von Neumann subalgebra of $\mathcal{B}(E \bar{\odot} G)$ (Proposition 3.1.3).

One may easily show that if $\mathcal{B} = \mathcal{B}(G)$, then $E = \mathcal{B}(G,H)$ and $\mathcal{B}^a(E) = \mathcal{B}(H)$ (Example 3.1.2). If E is a von Neumann $\mathcal{B}(G)$–$\mathcal{B}(G)$–module, then $H = G \bar{\otimes} \mathfrak{H}$ and $E = \mathcal{B}^a(G, G \bar{\otimes} \mathfrak{H}) = \mathcal{B}(G) \bar{\otimes}^s \mathfrak{H}$ where $\mathfrak{H}$ is a Hilbert space, Arveson's Hilbert space of intertwiners of the left and right multiplication (Example 3.3.4). In other words, $\mathfrak{H} = C_{\mathcal{B}(G)}(E)$, where generally $C_{\mathcal{B}}(E) = \{x \in E \colon bx = xb \ (b \in \mathcal{B})\}$ is the *$\mathcal{B}$–center* of a $\mathcal{B}$–$\mathcal{B}$–module. See Example 4.2.13 for the crucial interpretation of the tensor product of von Neumann modules in terms of mappings and the relation the composition of the centers under tensor product. This leads also to the crucial equivalence of Fock modules over von Neumann $\mathcal{B}(G)$–$\mathcal{B}(G)$–modules and Fock spaces tensorized with the initial space G (Examples 4.1.15 and 6.1.6).

Von Neumann modules are self-dual (Theorem 3.2.11). Consequently, each bounded right linear mapping on (or between) von Neumann modules is adjointable (Corollary 1.4.8) and von Neumann modules are *complementary* (i.e. for any von Neumann submodule F of a pre-Hilbert module E there exists a projection $p \in \mathcal{B}^a(E)$ onto F) (Proposition 1.5.9).

Let $T \colon \mathcal{A} \to \mathcal{B}$ be a (bounded) completely positive mapping between unital pre–C^*–algebras, and denote by (E, ξ) as the *GNS-construction* for T (Definition 4.1.7). If T is a normal mapping between von Neumann algebras, then $\overline{E}^s$ is a von Neumann $\mathcal{A}$–$\mathcal{B}$–module.

E.2 Matrices of Hilbert modules

Matrices with entries in a Hilbert module are probably the most crucial tool in Chapter 13. We developed these matrices in the form of examples and used them to show basic properties of tensor products. Here we proceed conversely and derive the properties of matrices by considering the tensor products as well-known. Recall that the C^*–algebras M_n are nuclear and that, therefore, the norms on the appearing exterior tensor products are unique.

For some Hilbert spaces G, H the space $\mathcal{B}(G, H)$ is a von Neumann $\mathcal{B}(H)$–$\mathcal{B}(G)$–module with inner product $\langle L, M\rangle = L^*M$ and the obvious module operations. In particular, the $n \times m$–matrices $M_{nm} = \mathcal{B}(\mathbb{C}^m, \mathbb{C}^n)$ are von Neumann M_n–M_m–modules. One easily checks that $M_{n\ell} \odot M_{\ell m} = M_{nm}$ where $X \odot Y = XY$ gives the canonical identification.

Let E be a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module. By $M_{nm}(E) = E \otimes M_{nm}$ we denote the spaces of $n \times m$–matrices with entries in a pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module. By construction $M_{nm}(E)$ is a pre-Hilbert $M_n(\mathcal{A})$–$M_m(\mathcal{B})$–module. It is a Hilbert and a von Neumann module, respectively, if and only if E is.

$M_{nm}(E)$ consists of matrices $X = \big(x_{ki}\big)$ whose inner product is

$$\langle X, Y\rangle_{ij} = \sum_{k=1}^{n} \langle x_{ki}, y_{kj}\rangle.$$

An element of $M_m(\mathcal{B})$ acts from the right on the right index and an element of $M_n(\mathcal{A})$ acts from the left on the left index of X in the usual way. Considering E as pre-Hilbert $\mathcal{B}^a(E)$–$\mathcal{B}$–module and making use of matrix units for $M_n(\mathcal{B}^a(E))$, one easily shows that $\mathcal{B}^a(M_{nm}(E)) = M_{nm}(\mathcal{B}^a(E))$. We have $M_{n\ell}(E) \odot M_{\ell m}(F) = M_{nm}(E \odot F)$ where $\big(X \odot Y\big)_{i,j} = \sum_k x_{ik} \odot y_{kj}$ gives the canonical identification. In particular, for square matrices we find $M_n(E) \odot M_n(F) = M_n(E \odot F)$.

Conversely, let E_{nm} be a pre-Hilbert $M_n(\mathcal{A})$–$M_m(\mathcal{B})$–module. Suppose for the time being that $\mathcal{A}$ and $\mathcal{B}$ are unital, and define Q_i as the matrix in $M_n(\mathcal{A})$ with $\mathbf{1}$ in the i–th place in the diagonal and $P_i \in M_m(\mathcal{B})$ defined analogously. Then all submodules $Q_iE_{nm}P_j$ are isomorphic to the same pre-Hilbert $\mathcal{A}$–$\mathcal{B}$–module E and $E_{nm} = M_{nm}(E)$. (Each of these *entries* $Q_iE_{nm}P_j$ takes its $\mathcal{A}$–$\mathcal{B}$–module structure by embedding $\mathcal{A}$ and $\mathcal{B}$ into that unique place in the diagonal of $M_n(\mathcal{A})$ and $M_m(\mathcal{B})$, respectively, where it acts non-trivially. The isomorphisms between the entries follows with the help of matrix units in M_n, M_m which are easily shown to restrict to isomorphisms between the entries.) The same shows to remains true, when $\mathcal{A}$ and $\mathcal{B}$ are not necessarily unital by appropriate use of approximate units.

Special forms are $E^n = M_{n1}(E)$ and $E_n = M_{1n}(E)$. Both consist of elements $X = (x_1, \ldots, x_n)$ $(x_i \in E)$. However, the former is an $M_n(\mathcal{A})$–$\mathcal{B}$–module with inner product $\langle X, Y\rangle = \sum\limits_i \langle x_i, y_i\rangle$ and $\mathcal{B}^a(E^n) = M_n(\mathcal{B}^a(E))$ (it is just the n–fold direct sum over E), whereas, the latter is an $\mathcal{A}$–$M_n(\mathcal{B})$–module with inner product $\langle X, Y\rangle_{i,j} = \langle x_i, y_i\rangle$ and $\mathcal{B}^a(E_n) = \mathcal{B}^a(E)$. Observe that $E_n \odot F^n = E \odot F$, whereas, $E^n \odot F_m = M_{nm}(E \odot F)$.

Let us set $X = \bigl(\delta_{ij} x_i\bigr) \in M_n(E)$ for some $x_i \in E$ $(i = 1, \ldots, n)$, and Y correspondingly. Then the mapping $T\colon M_n(\mathcal{A}) \to M_n(\mathcal{B})$, defined by setting $T(A) = \langle X, AY\rangle$ acts matrix-element-wise on A, i.e.

$$\bigl(T(A)\bigr)_{ij} = \langle x_i, a_{ij} y_j\rangle.$$

In particular, if $Y = X$, then T is completely positive. $T(A)$ may be considered as the *Schur product* of the matrix T of mappings $\mathcal{A} \to \mathcal{B}$ and the matrix A of elements in $\mathcal{A}$.

If S is another mapping coming in a similar manner from diagonal matrices X', Y' with entries in a pre-Hilbert $\mathcal{B}$–$\mathcal{C}$–module F, then we find as in Example 4.2.8 that the *Schur composition* of $S \circ T$ of the mappings T and S (i.e. the pointwise composition) is given by

$$S \circ T(A) \;=\; \langle X \odot X', AY \odot Y'\rangle.$$

This observation is crucial for the analysis of CPD-semigroups in Chapter 5.

Bibliography

[AB98] L. Accardi and M. Bozejko. Interacting Fock spaces and gaußanization of probability measures. *Infinite Dimensional Analysis, Quantum Probability & Related Topics*, 1:663–670, 1998.

[AB00] L. Accardi and A. Boukas. The square of white noise as a quantum semimartingale. Preprint, Rome, 2000.

[Acc74] L. Accardi. Noncommutative Markov chains. In *International School of Mathematical Physics*, pages 268–295, 1974.

[Acc75] L. Accardi. The noncommutative Markovian property. *Funct. Anal. Appl.*, 9:1–7, 1975.

[Acc78] L. Accardi. On the quantum Feynman-Kac formula. *Rendiconti Sem. Mat. e Fis. di Milano*, 48:135–179, 1978.

[AFL82] L. Accardi, A. Frigerio, and J.T. Lewis. Quantum stochastic processes. *Publ. RIMS Kyoto Univ.*, 18:97–133, 1982.

[AFQ92] L. Accardi, F. Fagnola, and J. Quaegebeur. A representation-free quantum stochastic calculus. *J. Funct. Anal.*, 104:149–197, 1992.

[AFS00] L. Accardi, U. Franz, and M. Skeide. Squared white noise and other non-gaußian noises as Lévy processes on real Lie algebras. Preprint, 2000.

[AH84] D. Applebaum and R.L. Hudson. Fermion Ito's formula and stochastic evolutions. *Commun. Math. Phys.*, 96:473–496, 1984.

[AHK00] L. Accardi, T. Hida, and H.-H. Kuo. The Ito table of the square of white noise. Preprint, Rome, 2000. To appear in Infinite Dimensional Analysis, Quantum Probability & Related Topics.

[AK99] L. Accardi and S. Kozyrev. On the structure of Markov flows. Preprint, Rome, 1999.

[AK00] L. Accardi and S. Kozyrev. Quantum interacting particle systems. Preprint, Rome, 2000. To appear in the lecture notes of the Volterra International School of the same title, held in Levico.

[AL93] L. Accardi and Y.G. Lu. Wiener noise versus Wigner noise in quantum electrodynamics. In L. Accardi, editor, *Quantum Probability & Related Topics VIII*, pages 1–18. World Scientific, 1993.

[AL96] L. Accardi and Y.G. Lu. The Wigner semi-circle law in quantum electro dynamics. *Commun. Math. Phys.*, 180:605–632, 1996.

[ALV97] L. Accardi, Y.G. Lu, and I.V. Volovich. *Interacting Fock spaces and Hilbert module extensions of the Heisenberg commutation relations.* IIAS Publications, Kyoto, 1997.

[ALV99] L. Accardi, Y.G. Lu, and I.V. Volovich. White noise approach to classical and quantum stochastic calculi. Preprint, Rome, 1999. To appear in the lecture notes of the Volterra International School of the same title, held in Trento.

[ALV01] L. Accardi, Y.G. Lu, and I.V. Volovich. *Quantum theory and its stochastic limit.* Springer, 2001. To appear.

[AP96] S. Attal and K.R. Parthasarathy. Strong Markov processes and the Dirichlet problem on C^*–algebras. Prepublications de l'Institut Fourier, Grenoble 357, 1996.

[Arv69] W. Arveson. Subalgebras of C^*–algebras. *Acta Math.*, 123:141–224, 1969.

[Arv89a] W. Arveson. *Continuous analogues of Fock space.* Number 409 in Memoires of the American Mathematical Society. American Mathematical Society, 1989.

[Arv89b] W. Arveson. Continuous analogues of Fock space III: Singular states. *J. Operator Theory*, 22:165–205, 1989.

[Arv90a] W. Arveson. Continuous analogues of Fock space II: The spectral C^*–algebra. *J. Funct. Anal.*, 90:138–205, 1990.

[Arv90b] W. Arveson. Continuous analogues of Fock space IV: essential states. *Acta Math.*, 164:265–300, 1990.

[Arv96] W. Arveson. Dynamical invariants for noncommutative flows. In S. Doplicher, R. Longo, J.E. Roberts, and L. Zsido, editors, *Operator algebras and quantum field theory*, pages 476–514. International Press, 1996.

[AS98] L. Accardi and M. Skeide. Interacting Fock space versus full Fock module. Preprint, Rome, 1998. Revised 2000.

[AS00a] L. Accardi and M. Skeide. Hilbert module realization of the square of white noise and the finite difference algebra. *Math. Notes*, 86:803–818, 2000.

[AS00b] L. Accardi and M. Skeide. On the relation of the square of white noise and the finite difference algebra. *Infinite Dimensional Analysis, Quantum Probability & Related Topics*, 3:185–189, 2000.

[BBLS00] S.D. Barreto, B.V.R. Bhat, V. Liebscher, and M. Skeide. Type I product systems of Hilbert modules. Preprint, Cottbus, 2000.

[Bel85] V.P. Belavkin. Reconstruction theorem for a quantum stochastic process. *Theoret. and Math. Phys.*, 62:275–289, 1985.

[Bel92] V.P. Belavkin. Chaotic states and stochastic integration in quantum systems. *Russian Math. Surveys*, 47:53–116, 1992.

[BGS99] A. Ben Ghorbal and M. Schürmann. On the algebraic foundations of non-commutative probability theory. Preprint, Nancy, 1999.

[Bha93] B.V.R. Bhat. *Markov dilations of nonconservative quantum dynamical semigroups and a quantum boundary theory.* PhD thesis, New Delhi, 1993.

[Bha96] B.V.R. Bhat. An index theory for quantum dynamical semigroups. *Trans. Amer. Math. Soc.*, 348:561–583, 1996.

[Bha98] B.V.R. Bhat. Product systems of one-dimensional Evans-Hodson flows. In R.L. Hudson and J.M. Lindsay, editors, *Quantum Probability Communications X*, pages 187–194. World Scientific, 1998.

[Bha99] B.V.R. Bhat. Minimal dilations of quantum dynamical semigroups to semigroups of endomorphisms of C^*–algebras. *J. Ramanujan Math. Soc.*, 14:109–124, 1999.

[Bha01] B.V.R. Bhat. *Cocycles of CCR-flows.* Number 709 in Memoires of the American Mathematical Society. American Mathematical Society, 2001.

[Bla86] B. Blackadar. *K–Theory for operator algebras.* Springer, 1986.

[Ble97] D.P. Blecher. A new approach to Hilbert C^*–modules. *Math. Ann.*, 307:253–290, 1997.

[Bou88] A. Boukas. *Quantum stochastic analysis: a non-Brownian case.* PhD thesis, Southern Illinois University, 1988.

[Bou91a] A. Boukas. An example of quantum exponential process. *Mh. Math.*, 112:209–215, 1991.

[Bou91b] A. Boukas. Stochastic calculus on the finite-difference Fock space. In L. Accardi, editor, *Quantum Probability & Related Topics VI.* World Scientific, 1991.

[BP94] B.V.R. Bhat and K.R. Parthasarathy. Kolmogorov's existence theorem for Markov processes in C^*–algebras. *Proc. Indian Acad. Sci. (Math. Sci.)*, 104:253–262, 1994.

[BP95] B.V.R. Bhat and K.R. Parthasarathy. Markov dilations of nonconservative dynamical semigroups and a quantum boundary theory. *Ann. I.H.P. Prob. Stat.*, 31:601–651, 1995.

[BS00] B.V.R. Bhat and M. Skeide. Tensor product systems of Hilbert modules and dilations of completely positive semigroups. *Infinite Dimensional Analysis, Quantum Probability & Related Topics*, 3:519–575, 2000.

[BSW82] C. Barnett, R.F. Streater, and I.F. Wilde. The Ito-Clifford integral. *J. Funct. Anal.*, 48:172–212, 1982.

[CE79] E. Christensen and D.E. Evans. Cohomology of operator algebras and quantum dynamical semigroups. *J. London Math. Soc.*, 20:358–368, 1979.

[Cun77] J. Cuntz. Simple C^*–algebras generated by isometries. *Commun. Math. Phys.*, 57:173–185, 1977.

[Dav76] E.B. Davies. *Quantum theory of open systems.* Academic Press, 1976.

[Die85] J. Dieudonné. *Grundzüge der modernen Analysis 1.* Deutscher Verlag der Wissenschaften, 1985.

[DU77] J. Diestel and J.J. Uhl. *Vector measures.* American Mathematical Society, 1977.

[EL77] D.E. Evans and J.T. Lewis. *Dilations of irreversible evolutions in algebraic quantum theory.* Number 24 in Commun. Dublin Inst. Adv. Studies, Ser. A. Dublin Institute for Advanced Studies, 1977.

[Fag91] F. Fagnola. On quantum stochastic integration with respect to “free” noises. In L. Accardi, editor, *Quantum Probability & Related Topics VI*, pages 285–304. World Scientific, 1991.

[Fei87] P.J. Feinsilver. Discrete analogues of the Heisenberg-Weyl algebra. *Mh. Math.*, 104:89–108, 1987.

[FM92] F. Fagnola and M. Mancino. Free noise dilation of semigroups of countable state Markov processes. In L. Accardi, editor, *Quantum Probability & Related Topics VII*, pages 149–163. World Scientific, 1992.

[Fow95] N.J. Fowler. Free E_0–semigroups. *Can. J. Math.*, 47:744–785, 1995.

[Fra97] M. Frank. *Beiträge zur Theorie der Hilbert C^*–Moduln.* Shaker, 1997.

[Fra99] U. Franz. Classical Markov processes from quantum Lévy processes. *Infinite Dimensional Analysis, Quantum Probability & Related Topics*, 2:105–129, 1999.

[FS93] F. Fagnola and K.B. Sinha. Quantum flows with unbounded structure maps and finite degrees of freedom. *J. London Math. Soc.*, 48:537–551, 1993.

[GL99] S. Goldstein and J.M. Lindsay. Markov semigroups KMS-symmetric for a weight. *Math. Ann.*, 313:39–67, 1999.

[Gou96] J. Gough. On the emergence of a free noise limit from quantum field theory. Preprint, Rome, 1996.

[GS99] D. Goswami and K.B. Sinha. Hilbert modules and stochastic dilation of a quantum dynamical semigroup on a von Neumann algebra. *Commun. Math. Phys.*, 205:377–403, 1999.

[Gui72] A. Guichardet. *Symmetric Hilbert spaces and related topics.* Number 261 in Lect. Notes Math. Springer, 1972.

[Hel01] J. Hellmich. *Quantenstochastische Integration in Hilbertmoduln.* PhD thesis, Tübingen, 2001.

[HKK98] J. Hellmich, C. Köstler, and B. Kümmerer. Stationary quantum Markov processes as solutions of stochastic differential equations. In R. Alicki, M. Bozejko, and W.A. Majewski, editors, *Quantum probability*, volume 43 of *Banach Center Publications*, pages 217–229. Polish Academy of Sciences — Institute of Mathematics, 1998.

[HP84] R.L. Hudson and K.R. Parthasarathy. Quantum Ito's formula and stochastic evolutions. *Commun. Math. Phys.*, 93:301–323, 1984.

[Kas80] G.G. Kasparov. Hilbert C^*–modules, theorems of Stinespring & Voiculescu. *J. Operator Theory*, 4:133–150, 1980.

[Kös00] C. Köstler. *Quanten-Markoff-Prozesse und Quanten- Brownsche Bewegungen.* PhD thesis, Tübingen, 2000.

[KS92] B. Kümmerer and R. Speicher. Stochastic integration on the Cuntz algebra O_∞. *J. Funct. Anal.*, 103:372–408, 1992.

[Küm85] B. Kümmerer. Markov dilations on W^*–algebras. *J. Funct. Anal.*, 63:139–177, 1985.

[Lan95] E.C. Lance. *Hilbert C^*–modules.* Cambridge University Press, 1995.

[Lan99] N.P. Landsman. Compact quantum groupoids. Preprint, math-ph/9912006, 1999.

[Lie98] V. Liebscher. A generalization of the conservation integral. In R. Alicki, M. Bozejko, and W.A. Majewski, editors, *Quantum probability*, volume 43 of *Banach Center Publications*, pages 273–284. Polish Academy of Sciences — Institute of Mathematics, 1998.

[Lie00a] V. Liebscher. Random sets and invariants for (type II) continuous tensor product systems of Hilbert spaces. Preprint, in preparation, 2000.

[Lie00b] V. Liebscher. Two types of markovian cocycles on symmetric Fock spaces. Preprint, in preparation, 2000.

[Lin76] G. Lindblad. On the generators of quantum dynamical semigroups. *Commun. Math. Phys.*, 48:119–130, 1976.

[LR98] Y.G. Lu and S. Ruggeri. A new example of interacting free Fock space. Preprint, Rome, 1998.

[LS00a] V. Liebscher and M. Skeide. Markovian systems of transition expectations. Preprint, Cottbus, 2000.

[LS00b] V. Liebscher and M. Skeide. Units for the time ordered Fock module. Preprint, Rome, 2000. Submitted to Infinite Dimensional Analysis, Quantum Probability & Related Topics.

[Lu94] Y.G. Lu. A note on free stochastic calculus on Hilbert modules. Preprint, Rome, 1994.

[LW00] J.M. Lindsay and S.J. Wills. Existence, positivity and contractivity for quantum stochastic flows with infinite dimensional noise. *Probab. Th. Rel. Fields*, 116:505–543, 2000.

[Mey93] P.A. Meyer. *Quantum probability for probabilists.* Number 1538 in Lect. Notes Math. Springer, 1993.

[MP84] J.A. Mingo and W.J. Phillips. Equivariant triviality theorems for Hilbert C^*–modules. *Proc. Amer. Math. Soc.*, 91:225–230, 1984.

[MS90] A. Mohari and K.B. Sinha. Quantum stochastic flows with infinite degrees of freedom and countable state Markov processes. *Sankhya, Ser. A*, 52:43–57, 1990.

[Mur90] G.J. Murphy. *C^*–Algebras and operator theory.* Academic Press, 1990.

[Mur97] G.J. Murphy. Positive definite kernels and Hilbert C^*–modules. *Proc. Edinburgh Math. Soc.*, 40:367–374, 1997.

[Mur00] N. Muraki. Monotonic independence, monotonic central limit theorem and monotonic law of small numbers. Preprint, Takizawa, 2000. To appear in Infinite Dimensional Analysis, Quantum Probability & Related Topics.

[Par92] K.R. Parthasarathy. *An introduction to quantum stochastic calculus.* Birkhäuser, 1992.

[Pas73] W.L. Paschke. Inner product modules over B^*–algebras. *Trans. Amer. Math. Soc.*, 182:443–468, 1973.

[Pet90] D. Petz. *An invitation to the algebra of canonical commutation relations.* Leuven University Press, 1990.

[Pim97] M.V. Pimsner. A class of C^*–algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbf{Z}$. In D.V. Voiculescu, editor, *Free probability theory*, number 12 in Fields Institute Communications, pages 189–212, 1997.

[Pow87] R.T. Powers. A non-spatial continuous semigroup of $*$–endomorphisms of $\mathcal{B}(H)$. *Publ. RIMS Kyoto Univ.*, 23:1053–1069, 1987.

[PS72] K.R. Parthasarathy and K. Schmidt. *Positive definite kernels, continuous tensor products, and central limit theorems of probability theory.* Number 272 in Lect. Notes Math. Springer, 1972.

[PS91] K.R. Parthasarathy and K.B. Sinha. Unification of quantum noise processes in Fock spaces. In L. Accardi, editor, *Quantum Probability & Related Topics VI*, pages 371–384. World Scientific, 1991.

[PS98] K.R. Parthasarathy and V.S. Sunder. Exponentials of indicater functions are total in the boson Fock space $\Gamma(L^2[0,1])$. In R.L. Hudson and J.M. Lindsay, editors, *Quantum Probability Communications X*, pages 281–284. World Scientific, 1998.

[Rie74] M.A. Rieffel. Induced representations of C^*-algebras. *Adv. Math.*, 13:176–257, 1974.

[RSN82] F. Riesz and B. Sz.-Nagy. *Vorlesungen über Funktionalanalysis.* Deutsch, 1982.

[Sak71] S. Sakai. *C^*–Algebras and W^*–algebras.* Springer, 1971.

[Sau86] J.-L. Sauvageot. Markov quantum semigroups admit covariant Markov C^*–dilations. *Commun. Math. Phys.*, 106:91–103, 1986.

[Sch93] M. Schürmann. *White noise on bialgebras.* Number 1544 in Lect. Notes Math. Springer, 1993.

[Sch95] M. Schürmann. Non-commutative probability on algebraic structures. In H. Heyer, editor, *Probability measures on groups and related structures XI*, pages 332–356. World Sci. Publishing, 1995.

[Sch96] J. Schweitzer. *Interplay between noncommutative topology and operators on C^*–algebras.* PhD thesis, Tübingen, 1996.

[Shl96] D. Shlyakhtenko. Random gaußian band matrices and freeness with amalgamation. *International Mathematics Research Notices*, 20:1013–1025, 1996.

[Ske96] M. Skeide. A note on Bose $\mathfrak{Z}$–independent random variables fulfilling q–commutation relations, 1996. Submitted to Quantum Probability Communications XII.

[Ske98a] M. Skeide. Hilbert modules in quantum electro dynamics and quantum probability. *Commun. Math. Phys.*, 192:569–604, 1998.

[Ske98b] M. Skeide. Quantum stochastic calculus on full Fock space. In R. Alicki, M. Bozejko, and W.A. Majewski, editors, *Quantum probability*, volume 43 of *Banach Center Publications*, pages 369–379. Polish Academy of Sciences — Institute of Mathematics, 1998.

[Ske99a] M. Skeide. A central limit theorem for Bose $\mathfrak{Z}$–independent quantum random variables. *Infinite Dimensional Analysis, Quantum Probability & Related Topics*, 2:289–299, 1999.

[Ske99b] M. Skeide. Infinitesimal generators on $SU_q(2)$ in the classical and anti-classical limit. *Information Dynamics & Open Systems*, 6:375–414, 1999.

[Ske99c] M. Skeide. Tensor product systems of CP-semigroups on $\mathbb{C}^2$. Preprint, Rome, 1999. To appear in Journal of Mathematical Sciences.

[Ske00a] M. Skeide. Dilations, product systems and weak dilations. Preprint, Cottbus, 2000. Submitted to Math. Notes.

[Ske00b] M. Skeide. Generalized matrix C^*–algebras and representations of Hilbert modules. *Mathematical Proceedings of the Royal Irish Academy*, 100A:11–38, 2000.

[Ske00c] M. Skeide. Indicator functions of intervals are totalizing in the symmetric Fock space $\Gamma(L^2(\mathbb{R}_+))$. In L. Accardi, H.-H. Kuo, N. Obata, K. Saito, Si Si, and L. Streit, editors, *Trends in contemporary infinite dimensional analysis and quantum probability*, volume 3 of *Natural and Mathematical Sciences Series*. Istituto Italiano di Cultura (ISEAS), Kyoto, 2000. Volume in honour of Takeyuki Hida.

[Ske00d] M. Skeide. Quantum stochastic calculus on full Fock modules. *J. Funct. Anal.*, 173:401–452, 2000.

[Sni00] P. Sniady. Quadratic bosonic and free white noise. *Commun. Math. Phys.*, 211:615–628, 2000.

[Spe91] R. Speicher. Stochastic integration on the full Fock space with the help of a kernel calculus. *Publ. RIMS Kyoto Univ.*, 27:149–184, 1991.

[Spe97] R. Speicher. On universal products. *Fields Institute Communications*, 12:257–266, 1997.

[Spe98] R. Speicher. *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory.* Number 627 in Memoires of the American Mathematical Society. American Mathematical Society, 1998.

[Sto72] E. Stormer. On projection maps of von Neumann algebras. *Math. Scand.*, 30:46–50, 1972.

[Tak79] M. Takesaki. *Theory of operator algebras I.* Springer, 1979.

[Tsi00a] B. Tsirelson. From random sets to continuous tensor products: answers to three questions of W. Arveson. Preprint, ArXiv:math.FA/0001070, 2000.

[Tsi00b] B. Tsirelson. From slightly coloured noises to unitless product systems. Preprint, ArXiv:math.FA/0006165, 2000.

[Voi87] D. Voiculescu. Dual algebraic structures on operator algebras related to free products. *J. Operator Theory*, 17:85–98, 1987.

[Voi95] D. Voiculescu. Operations on certain non-commutative operator-valued random variables. *Astérisque*, 232:243–275, 1995.

[Wal73] W. von Waldenfels. An approach to the theory of pressure broadening of spectral lines. In M. Behara, K. Krickeberg, and J. Wolfowitz, editors, *Probability and Information Theory II*, number 296 in Lect. Notes Math. Springer, 1973.

[WO93] N.E. Wegge-Olsen. *K–Theory and C^*–algebras.* Oxford University Press, 1993.

[Yos80] K. Yosida. *Functional analysis.* Springer, 1980.

[Zac96] J. Zacharias. *Continuous tensor products and Arveson's spectral algebras.* PhD thesis, Heidelberg, 1996.

[Zac00] J. Zacharias. Pure infiniteness of spectral algebras. Preprint, Nottingham, 2000.

Index