DILATION THEORY AND CONTINUOUS TENSOR PRODUCT SYSTEMS OF HILBERT MODULES

MICHAEL SKEIDE
Lehrstuhl für Wahrscheinlichkeitstheorie und Statistik,
Brandenburgische Technische Universität Cottbus,
Postfach 10 13 44, D–03013 Cottbus, Germany, E-mail:
`skeide@math.tu-cottbus.de`, *Homepage:*
`http://www.math.tu-cottbus.de/INSTITUT/lswas/_skeide.html`

The investigation of products systems of Hilbert modules as introduced by Bhat and Skeide[6] has now reached a state where it seems appropriate to give a summary of what we know about the structure. After showing how product systems appear naturally in the theory of dilations of CP-semigroups, it is one of the goals of these notes to give a list of solved and open problems.

In contrast with Arveson[1], who starts his theory of product systems of Hilbert spaces (Arveson systems, for short) with a concise definition of measurability conditions (which are equivalent to similar continuity conditions), the theory of product systems of Hilbert modules (in the sense of Definition 3.1 below) developed so far works without such conditions. While the algebraic constructions which work in that framework behave nicely with respect to topological completions or closures at a fixed "time", we could show continuity results for time evolutions only in special cases. It is the second goal of these notes to launch a definition of *continuous* tensor product system (Definition 7.1)[(0.a)] and to show that this definition, although sufficiently general to contain all reasonable cases, does not have the described defect. In the case of type I and type II systems we found already a way to formulate continuity conditions in a less intrinsic way. Theorem 7.5 shows that the new definition is compatible with these special cases. We are, finally, able to define what we understand by a (continuous) *type III* product system, thus completing the classification scheme from Bhat and Skeide[6], Barreto, Bhat, Liebscher and Skeide[3] and Skeide[19].

1 Introduction

By *dilation* many authors understand slightly different things. The *denominator* common to all the different definitions may be described by the com-

(0.a)This definition was obtained in joint work with B.V.R. Bhat and V. Liebscher within a PPP-project supported by DAAD and DST.

mutative Diagram (1) below. Here $\mathcal{B}$ is a unital C^*–algebra with a unital *completely positive* (CP-) semigroup $T = \bigl(T_t\bigr)_{t\in\mathbb{R}_+}$ and $\mathcal{A}$ is another unital C^*–algebra with a semigroup $\vartheta = \bigl(\vartheta_t\bigr)_{t\in\mathbb{R}_+}$ of unital endomorphisms, i.e. an E_0–semigroup. The two are linked together by an *embedding* (i.e. an injective homomorphism) $\mathfrak{i}\colon \mathcal{B} \to \mathcal{A}$ of $\mathcal{B}$ into $\mathcal{A}$ and an *expectation* $\mathfrak{p}\colon \mathcal{A} \to \mathcal{B}$ back to $\mathcal{B}$ in such a way that $\varphi = \mathfrak{i} \circ \mathfrak{p}$ is a conditional expectation onto the range $\mathfrak{i}(\mathcal{B})$ of $\mathfrak{i}$, i.e. $\mathfrak{p}(\mathfrak{i}(b)a\mathfrak{i}(b')) = b\mathfrak{p}(a)b'$ for all $b, b' \in \mathcal{B}; a \in \mathcal{A}$.[1.a]

$$
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\;T_t\;} & \mathcal{B} \\
{\scriptstyle \mathfrak{i}}\big\downarrow & & \big\uparrow{\scriptstyle \mathfrak{p}} \\
\mathcal{A} & \xrightarrow[\;\vartheta_t\;]{} & \mathcal{A}
\end{array}
\tag{1}
$$

The idea of dilation is to understand the dynamics T of $\mathcal{B}$ as projection from the dynamics ϑ of $\mathcal{A}$. In statistical physics the algebras $\mathcal{B}$ and $\mathcal{A}$ may be considered as algebras of quantum mechanical observables so that $\mathcal{B}$ models the description of a small system embedded into a big one modelled by $\mathcal{A}$. In *the* classical example $\mathcal{B}$ is the algebra of random variables describing a brownian particle moving on a liquid in thermal equilibrium and $\mathcal{A}$ is the algebra of random variables describing both the molecules of the liquid and the particle. In both cases we say that the irreversible dynamics of the small system described by completely positive mappings is dilated to a more reversible one on the big system described by unital endomorphisms.[1.b]

Already in Bhat and Skeide[6] we showed how to construct from a CP-semigroup T on $\mathcal{B}$, i.e. the upper half of Diagram (1), a product system of Hilbert $\mathcal{B}$–$\mathcal{B}$–modules[1.c] and, in the unital case, how to complete the diagram to a dilation. More precisely, we constructed a dilation on a Hilbert module, i.e. in our case $\mathcal{A} = \mathcal{B}^a(E)$ is the C^*–algebra of all adjointable operators on a Hilbert $\mathcal{B}$–module E and E contains a unit vector ξ (i.e. $\langle \xi, \xi \rangle = \mathbf{1}$) such that

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[1.a] Of course, it is possible to identify $\mathcal{B}$ via $\mathfrak{i}$ as a subalgebra of $\mathcal{A}$ and to consider just the conditional expectation φ. We prefer, however, to keep the freedom to choose different embeddings $\mathfrak{i}$.

[1.b] The *most reversible* version, in fact, required by many authors, is a dilation to an automorphism semigroup (in fact, an automorphism group). However, in most cases there is a natural subalgebra $\mathcal{A}_+$ of $\mathcal{A}$, the algebra of *future observables*, which is left invariant by the automorphism semigroup. So the restriction to $\mathcal{A}_+$ gives rise to a proper E_0–semigroup. In any case, in order to speak of reversibility, the endomorphisms should be injective, but we do not need this additional requirement and, thus, omit it.

[1.c] The construction of product systems from CP-semigroups, historically the first, is a special case of the construction starting from so-called CPD-semigroups, which we mention only briefly in Footnote (4.d).

$\mathfrak{p}(a) = \langle \xi, a\xi \rangle$. The situation is illustrated in the following diagram.

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\quad T_t \quad} & \mathcal{B} \\ {\scriptstyle \mathfrak{i}}\big\downarrow & & \big\uparrow{\scriptstyle \mathfrak{p}=\langle \xi, \bullet \xi \rangle} \\ \mathcal{B}^a(E) & \xrightarrow[\quad \vartheta_t \quad]{} & \mathcal{B}^a(E) \end{array} \tag{2}$$

A dilation on a Hilbert module is a quadruple $(E, \vartheta, \mathfrak{i}, \xi)$ such that Diagram (2) commutes for all t. Actually, the dilation constructed in Reference[6] is a weak dilation, i.e. the embedding has the special form $\mathfrak{i} = j_0$ with $j_0(b) := \xi b \xi^*$ where ξ^* is the operator $x \mapsto \langle \xi, x \rangle$.[(1.d)] A weak dilation on a Hilbert module is a triple (E, ϑ, ξ) such that Diagram (3) commutes for all t.[(1.e)]

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\quad T_t \quad} & \mathcal{B} \\ {\scriptstyle j_0=\xi \bullet \xi^*}\big\downarrow & & \big\uparrow{\scriptstyle \mathfrak{p}=\langle \xi, \bullet \xi \rangle} \\ \mathcal{B}^a(E) & \xrightarrow[\quad \vartheta_t \quad]{} & \mathcal{B}^a(E) \end{array} \tag{3}$$

In Skeide[16] we showed (generalizing Bhat's[4] approach to tensor product systems of Hilbert spaces in the sense of Arveson[1]) how to construct a tensor product system of Hilbert $\mathcal{B}$–$\mathcal{B}$–modules from the triple (E, ϑ, ξ), i.e. the lower half of Diagram (1), at least, when the endomorphisms ϑ_t are *strict* (a condition replacing the normality assumption in the case of Hilbert spaces). It turns out that those Hilbert modules which have a unit vector form an important subclass of the class of all Hilbert modules.

It is the goal of these notes to describe the mentioned constructions of product systems in more detail. Where possible we explain the major ideas or even provide short proofs. For didactic reasons we reverse the historic order and start (after repeating in Section 2 some preliminaries about Hilbert

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(1.d) In this case, the family $j = (j_t)_{t \in \mathbb{R}_+}$ with $j_t = \vartheta_t \circ j_0$ defines a weak Markov flow for T in the sense of Bhat and Parthasarathy[5], i.e. $j_t(\mathbf{1}) j_{s+t}(b) j_t(\mathbf{1}) = j_t \circ T_s(b)$.

(1.e) Apparently, our set-up where $\mathcal{A} = \mathcal{B}^a(E)$ seems to be a restriction of the more general Diagram (1). However, in References[16,18] we point out that our notion of dilation is sufficiently wide to contain all explicit and most known abstract examples of dilations in the sense that $\mathcal{A}$ is contained in some $\mathcal{B}^a(E)$ to which the E_0–semigroup ϑ extends. (For the case of an automorphism white noise the statement follows from Footnote (2.b).) Moreover, all of these known dilations become weak dilations when $\mathfrak{i}$ (usually unital) is replaced by j_0 (usually non-unital). For the time being we content ourselves with the knowledge that considering dilations and weak dilations on Hilbert modules is a fairly general frame and the study of these by means of their product systems (for instance, via classification) showed already up to give many new insights.

modules) in Section 3 with the construction of a product system from a strict E_0–semigroup ϑ on $\mathcal{B}^a(E)$. This allows us to motivate related notions like *units* and *central* units for product systems directly from dilation theory. This way to construct product systems is also more directly related to the way how Arveson discovered product systems of Hilbert spaces.

Central units are tightly connected with *white noises* where we call the triple (E, ϑ, ξ) a white noise, if ϑ leaves invariant $\mathfrak{p} = \langle \xi, \bullet \xi \rangle$, i.e. if $\mathfrak{p} \circ \vartheta_t = \mathfrak{p}$ for all t. In other words, (E, ϑ, ξ) is a weak dilation of the trivial semigroup.[(1.f)] We will classify product systems admitting central unit as *spatial* product systems. Spatiality of product systems of von Neumann modules which have units is equivalent to the results by Christensen and Evans[7] on the form of the generator of a normal uniformly continuous CP-semigroup on a von Neumann algebra and, therefore, a deep problem.

In Section 4 we define units and central units. Then we set up our classification scheme, which is, like that for Arveson systems, based on units. We repeat simple Examples from Reference[3] which show that the refinement of Arveson's classification scheme (in that there are two types of units) and also the distinction into norm and strong topology are really necessary. The particular importance of spatiality we point out in Section 5. We show that a generator of a uniformly continuous CP-semigroup is a Christensen-Evans generator, if the associated GNS-system (see Footnote (4.d)) is spatial (or can be embedded into a spatial one). This allows to identify in the following Section 6 spatial type I systems as time ordered Fock modules. It also shows us that spatial product systems have an *index* generalizing that of Arveson systems.

In the final Section 7 we solve the outstanding problem to define *continuous* product systems. We show that our definition extends our preliminary definition for type I and type II systems based on the extistence of a *continuous* unit. The definition is motivated from properties of product systems constructed from *strictly continuous* E_0–semigroups and, of course, also such product systems fulfill our definition. Now we, finally, have a chance to solve also the reverse problem, namely, to construct an E_0–semigroup from a continuous product system (known for non-type III) in full generality.

[(1.f)] We know that it is in some sense provocant to call this a white noise as 'white noise' is something which has to do with 'independence' and, in fact, our white noises come shipped with subalgebras $\mathcal{A}_{[s,t]} \subset \mathcal{B}^a(E)$ which are *monotone independent* (see References[14,15]) with *amalgamation over* $\mathcal{B}$ in the conditional expectation $\mathfrak{p}$ (to be published elsewhere). However, we do not intend to speak about 'independence', and if we subtract 'independence' from existing definitions of 'white noise' (over $\mathcal{B}$) like, for instance, that in Kümmerer[10], then our definition is, more or less, what remains.

2 Preliminaries on Hilbert modules and conventions

We repeat the basic definitions and constructions for Hilbert modules. For a detailed introduction to Hilbert modules (adapted to our needs) we refer to Skeide[18], for a quick reference to Bhat and Skeide[6]. The book of Lance[11] provides a general introduction to Hilbert modules. Throughout these notes $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ denote unital C^*–algebras.

2.1 A pre-Hilbert $\mathcal{B}$–module is a right $\mathcal{B}$–module E with a sesquilinear inner product $\langle \bullet, \bullet \rangle : E \times E \to \mathcal{B}$ which is positive ($\langle x, x \rangle \geq 0$), right linear ($\langle x, yb \rangle = \langle x, y \rangle b$) and definite ($\langle x, x \rangle = 0 \Rightarrow x = 0$). If the inner product fails to be definite, then the Cauchy-Schwarz inequality

$$\langle x, y \rangle \langle y, x \rangle \ \leq \ \| \langle y, y \rangle \| \, \langle x, x \rangle \tag{4}$$

tells us that we may divide out the submodule $\mathcal{N}_E = \{x : \langle x, x \rangle = 0\}$ of length-zero elements and obtain a pre-Hilbert module. It tells us also that $\|x\| = \sqrt{\|\langle x, x \rangle\|}$ defines a (semi-)norm. A Hilbert $\mathcal{B}$–module is a pre-Hilbert $\mathcal{B}$–module which is complete in this norm. Every pre-Hilbert $\mathcal{B}$–module E may be completed and we denote the completion (as with all other normed spaces) by $\overline{E}$. The isomorphisms among (pre-)Hilbert $\mathcal{B}$–modules are the unitary (i.e. surjective inner product preserving) mappings.[(2.a)]

A Hilbert $\mathcal{A}$–$\mathcal{B}$–module (or just two-sided Hilbert module) is a Hilbert $\mathcal{B}$–module with a non-degenerate ($*$–)representation of $\mathcal{A}$ by elements in the (C^*–algebra) $\mathcal{B}^a(E)$ of adjointable (and, therefore, bounded and right linear) mappings on E. By $\mathcal{B}^{a,bil}(E)$ we denote the subspace of bilinear or two-sided mappings. In particular, an isomorphism of two-sided Hilbert modules is a two-sided unitary.

2.2 The, by far, most important way how Hilbert modules, in particular, two-sided Hilbert modules appear in dilation theory is the GNS-construction for a completely positive mapping $T : \mathcal{A} \to \mathcal{B}$. The GNS-module of T is that Hilbert $\mathcal{A}$–$\mathcal{B}$–module E generated by a vector ξ (i.e. $E = \overline{\operatorname{span}} \mathcal{A} \xi \mathcal{B}$) and with inner product $\langle a\xi b, a'\xi b' \rangle = b^* T(a^* a') b'$. This module is determined uniquely by the requirement $\langle \xi, a\xi \rangle = T(a)$.[(2.b)]

[(2.a)] Observe that a unitary u has an adjoint, namely, $u^* = u^{-1}$. Therefore, it is right linear and (by isometry) bounded. Adjointable mappings which have a Hilbert module as domain or as range are bounded by the closed graph theorem. For pre-Hilbert modules this need not be so.

[(2.b)] If $\mathfrak{i}$ is an embedding and $\mathfrak{p}$ an expectation as required in a dilation, then we have even $E = \overline{\mathcal{A}\xi}$ and $\mathfrak{i}(b)\xi = \xi b$. (This situation is most similar to usual GNS-construction for a

2.3 The tensor product $E \odot F$ of a Hilbert $\mathcal{A}$–$\mathcal{B}$–module E and a Hilbert $\mathcal{B}$–$\mathcal{C}$–module F is the Hilbert $\mathcal{A}$–$\mathcal{C}$–module which is the closed linear span of elementary tensors $x \odot y$ whose inner product is defined by $\langle x \odot y, x' \odot y' \rangle = \big\langle y, \langle x, x' \rangle y' \big\rangle$. (By '$\otimes$' we always denote the tensor product of vector spaces, usually, completed in some natural norm.)

If T, S are completely positive mappings $\mathcal{A} \xrightarrow{T} \mathcal{B} \xrightarrow{S} \mathcal{C}$ and if (E, ξ) and (F, ζ) denote their GNS-constructions, then $S \circ T(a) = \langle \xi \odot \zeta, a\xi \odot \zeta \rangle$. In other words, the submodule $\overline{\mathsf{span}}\mathcal{A}\xi \odot \zeta\mathcal{C}$ of $E \odot F$ with cyclic vector $\xi \odot \zeta$ is the GNS-module of $S \circ T$.

2.4 If $\mathcal{B}$ is a von Neumann algebra acting (non-degenerately) on a Hilbert space G, then G is a Hilbert $\mathcal{B}$–$\mathbb{C}$–module. For some Hilbert $\mathcal{B}$–module E we construct the Hilbert space $H = E \odot G$. For every $x \in E$ we define the mapping $L_x \colon g \mapsto x \odot g$ in $\mathcal{B}(G, H)$ whose adjoint is determined by $L_x^* \colon y \odot g \mapsto \langle x, y \rangle g$. Moreover, $L_x^* L_y = \langle x, y \rangle$ so that we may identify E (isometrically) as a subset of $\mathcal{B}(G, H)$. We call the mapping $\eta \colon x \mapsto L_x$ the Stinespring representation of E. Following Skeide[17], we say E is a von Neumann $\mathcal{B}$–module, if it is strongly closed in $\mathcal{B}(G, H)$.

In contrast with general Hilbert modules, von Neumann modules are always self-dual, i.e. for every bounded right linear mapping $\Phi \colon E \to \mathcal{B}$ there exists a (unique) element $x \in E$ generating Φ as $\Phi(y) = x^* y := \langle x, y \rangle$. Like for Hilbert spaces one shows that bounded right linear operators on (or between) von Neumann modules are adjointable. Self-duality also guarantees that for any strongly closed submodule $F \subset E$ there is a (unique) projection $p \in \mathcal{B}^a(E)$ onto F. Also this is a fact that need not be true for general Hilbert modules.

If E is a Hilbert $\mathcal{A}$–$\mathcal{B}$–module, then $\rho(a)(x \odot g) = (ax) \odot g$ defines a representation $\rho \colon \mathcal{A} \to \mathcal{B}(H)$ which we call the Stinespring representation of $\mathcal{A}$ associated with E. (In particular, if $\mathcal{A} = \mathcal{B}^a(E)$, then ρ identifies $\mathcal{B}^a(E)$ as a subalgebra of $\mathcal{B}(H)$, even a von Neuman algebra on H, if E is a von Neumann module.) We say E is a von Neumann $\mathcal{A}$–$\mathcal{B}$–module (or a two-sided von Neumann module), if it is a von Neumann $\mathcal{B}$–module and if the

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state.) What happens, if there is an automorphism α of $\mathcal{A}$ leaving $\mathfrak{p}$ invariant, i.e. $\mathfrak{p} \circ \alpha = \mathfrak{p}$? Then two short computations show that the mapping $u \colon a\xi \to \alpha(a)\xi$ defines a unitary on E such that $\alpha(a)x = uau^*x$ for all $x \in E$. If the GNS-representation is faithful so that we may identify $\mathcal{A}$ as a subset of $\mathcal{B}^a(E)$, then $\alpha(a) = uau^*$ and the automorphism α extends to a unitarily implemented automorphism of $\mathcal{B}^a(E)$. If the GNS-representation is not faithful, then the computations show that α repects the kernel of the GNS-representation so that we may divide out this kernel. Therefore, as soon as we are concerned with a white noise of automorphisms, we may divide out the kernel of the expectation $\mathfrak{p}$ and pass to a (unitarily implemented) white noise on a Hilbert module.

Stinespring representation of $\mathcal{A}$ is normal.

The strong closure $E \bar{\odot}^s F$ of the tensor product of two-sided von Neumann modules is again a two-sided von Neumann module.

If E is the GNS-module of a normal completely positive mapping T between von Neumann algebras $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{B}(G)$, then the strong closure $\overline{E}^s \subset \mathcal{B}(G, H)$ of E is a von Neumann $\mathcal{A}$–$\mathcal{B}$–module. Moreover, $\rho: \mathcal{A} \rightarrow \mathcal{B}(H)$ is, indeed, the original Stinespring representation of $\mathcal{A}$ and the mapping L_ξ for the cyclic vector $\xi \in E$ fulfills $T(a) = L_\xi^* \rho(a) L_\xi$.[2.c]

2.5 By far, the most concrete results in dilation theory are obtained for the von Neumann algebra $\mathcal{B} = \mathcal{B}(G)$ and the dilations act on the algebra $\mathcal{B}(H)$ where the Hilbert space H usually has the form $H = G \otimes \mathfrak{H}$ for some other Hilbert space $\mathfrak{H}$. Why is this so? The answer lies in the simple structure of von Neumann $\mathcal{B}(G)$–modules and, in particular, of von Neumann $\mathcal{B}(G)$–$\mathcal{B}(G)$–modules. Since $\mathcal{B}(G)$ contains the finite-rank operators, the von Neumann $\mathcal{B}(G)$–module $E \subset \mathcal{B}(G, H)$ contains the finite-rank operators of $\mathcal{B}(G, H)$ (or, to be more precise, at least those to elements in the total subset EG of H) and, because E is strongly closed, we find $E = \mathcal{B}(G, H)$. One easily checks that also $\mathcal{B}^a(E) = \mathcal{B}(H)$. Therefore, dilations of CP-semigroups on a von Neumann $\mathcal{B}(G)$–module act on $\mathcal{B}(H)$.

Moreover, if E is a two-sided von Neumann $\mathcal{B}(G)$–module, then the representation ρ of $\mathcal{B}(G)$ on H is normal (and non-degenerate). Therefore, $H = G \otimes \mathfrak{H}$ for a suitable Hilbert space $\mathfrak{H}$ and $\rho = \mathsf{id}_G \otimes \mathbf{1}$ in that identification. This explains clearly why problems which, for general C^*–algebras, require Hilbert module methods can be solved on $G \otimes \mathfrak{H}$ in the case $\mathcal{B}(G)$.

Can we specify $\mathfrak{H}$ further? The answer is simple. The elements h of $\mathfrak{H}$ are in one-to-one correspondence with those mappings $\mathsf{id}_G \otimes h \in \mathcal{B}(G, G \otimes \mathfrak{H})$ which *commute* with all elements in $\mathcal{B}(G)$.[2.d]

2.6 The strict topology of $\mathcal{B}^a(E)$ arises by the observation due to Kasparov[9] that $\mathcal{B}^a(E)$ is the multiplier algebra of the C^*–subalgebra of compact operators

[2.c] Notice that the Stinespring representations of two completely positive mappings T, S do not help us in recovering the Stinespring representation of $S \circ T$. On the contrary, the GNS-modules, being functors which send representations of the algebra to the right to representations of the algebra to the left, compose under tensor product to the functor for the composed mapping $S \circ T$; cf. Section 2.3.

[2.d] Defining the center of a two-sided $\mathcal{B}$–module E as $C_\mathcal{B}(E) = \{x \in E: bx = xb \ (b \in \mathcal{B})\}$, we notice that for central elements x_1, x_2 the inner product takes values in the center of $\mathcal{B}$. If $\mathcal{B} = \mathcal{B}(G)$, then the center of $\mathcal{B}$ is trivial so that the inner product of central elements in E is a scalar multiple of $\mathbf{1}$. Identifying this scalar with the inner product of the corresponding (unique) elements of $f_1, f_2 \in \mathfrak{H}$ such that $x_i = \mathsf{id}_G \otimes f_i$ gives us the isomorphism.

$\mathcal{K}(E)$ which is generated by the rank-one operators $xy^*: z \mapsto x\langle y, z\rangle$. In other words, $\mathcal{B}^a(E)$ is the completion of $\mathcal{K}(E)$ in the topology generated by the two families $a \mapsto \|ak\|$ and $a \mapsto \|ka\|$ $(k \in \mathcal{K}(E))$ of seminorms. Here we follow Lance' convention[11] and by the strict topology we mean always the restriction to bounded subsets of $\mathcal{B}^a(E)$. One can show that on the ball the strict topology coincides with the $*$–strong topology. In the case of Hilbert spaces the strict topology is the $*$–σ–strong topology. It is well-known that normal representations of $\mathcal{B}(H)$ are also $*$–σ–strong, so for Hilbert modules the strict topology on the ball is, indeed, an appropriate substitute of the normal topology.[(2.e)]

We used already the well-known fact that normal (non-degenerate) repesentations of $\mathcal{B}(G)$ on H decompose H as $G \otimes \mathfrak{H}$. Can we do the same for Hilbert modules? The answer is yes, if the (non-degenerate) representation $\vartheta: \mathcal{B}^a(E) \to \mathcal{B}^a(F)$ is strict (cf. also Footnote (2.e)), and if E contains a unit vector ξ.[(2.f)] Here we restrict to the case $F = E$, i.e. ϑ is a strict unital endomorphism of $\mathcal{B}^a(E)$.[(2.g)]

So let E be a unital Hilbert $\mathcal{B}$–module and ϑ a strict unital endomorphism of $\mathcal{B}^a(E)$. Put $E_\xi = \vartheta(\xi\xi^*)E$ and define a (unital) left multiplication on this Hilbert submodule of E by $bx = \vartheta(\xi b \xi^*)x$. One easily shows that the mapping

$$u: x \odot y \longmapsto \vartheta(x\xi^*)y$$

defines an isometry $E \odot E_\xi \to E$. Using an approximate unit for $\mathcal{F}(E)$ and strictness of ϑ one shows that u is surjective, hence, unitary.[(2.h)] In the identification $E = E \odot E_\xi$ we find that $\vartheta(a) \in \mathcal{B}^a(E)$ acts as $\vartheta(a) = a \odot \mathrm{id}_{E_\xi}$.[(2.i)]

[(2.e)] In the sequel, what we need from the strict topology is the fact that a (bounded) approximate unit for the compact operators, or even the finite-rank operators $\mathcal{F}(E) := \mathrm{span} EE^*$, converges to id_E. Any other topology which also has this property will also serve this purpose and may replace the strict topology.

[(2.f)] This result and others which are valid only under the assumption of a unit vector make this property so important that we give a name to it, and call E a unital Hilbert module.

[(2.g)] We discuss in Skeide[16] that the result is valid for arbitrary Hilbert modules F, even over a different C^*–algebra $\mathcal{C}$.

[(2.h)] In both computations it is an important step to insert a one in the rank-one operator $xy^* = x\mathbf{1}y^* = x\xi^*\xi y^*$ and to use then the $*$–homomorphism property of ϑ, when applied to that product of operators $x\xi^*$ and ξy^*.

[(2.i)] One can show that up to two-sided isomorphism E_ξ does not depend on the choice of ξ. If ϑ, ϑ' are two strict unital endomorphisms of $\mathcal{B}^a(E)$ than E_ξ and E'_ξ are isomorphic, if and only if ϑ and ϑ' are conjugate, i.e. if $\vartheta' = u\vartheta u^*$ for some unitary $u \in \mathcal{B}^a(E)$.

3 From E_0–semigroups to product systems of Hilbert modules

Let (E, ϑ, ξ) be a unital Hilbert $\mathcal{B}$–module $E \ni \xi$ and and let ϑ be a strict E_0–semigroup on $\mathcal{B}^a(E)$. A simultaneous application of our representation theory in Section 2.6 to all ϑ_t provides us with a family $E^\odot = \big(E_t\big)_{t\in\mathbb{R}_+}$ of Hilbert $\mathcal{B}$–$\mathcal{B}$–modules $E_t = p_t E$ ($p_t := \vartheta_t(\xi\xi^*)$, left multiplications $bx_t = \vartheta_t(\xi b \xi^*)x_t$) and unitaries $u_t\colon E \odot E_t$ defined by $u_t(x \odot y_t) = \vartheta_t(x\xi^*)y_t$. Observe that $E_0 \cong \mathcal{B}$ as two-sided module via $\xi \mapsto \mathbf{1}$.

We define the restrictions $u_{st} = u_t \restriction (E_s \odot E_t)$. These map into E_{s+t}, because $p_{s+t}\vartheta_t(x_s\xi^*)y_t = \vartheta_t(p_s x_s \xi^*)y_t = \vartheta_t(x_s\xi^*)y_t$. On the other hand, they are onto E_{s+t}. (Write $z \in E_{s+t} \subset E = u_t(E \odot E_t)$ as $z = \sum\limits_i u_t(x^i \odot y_t^i)$. Since $p_{s+t}z = z$ we have

$$z \;=\; p_{s+t}\sum_i u_t(x^i \odot y_t^i) \;=\; \sum_i \vartheta_t(p_s x^i \xi^*)y_t^i \;=\; u_{st}\Big(\sum_i p_s x^i \odot y_t^i\Big)$$

$\in u_{st}(E_s \odot E_t)$.) Therefore, $u_{st}\colon E_s \odot E_t \to E_{s+t}$ are unitaries. Moreover, from $u_{st}(bx_s \odot y_t) = \vartheta_t(\vartheta_s(\xi b \xi^*)x_s\xi^*)y_t = bu_{st}(x_s \odot y_t)$ we see that the u_{st} are two-sided. Finally, we easilly verify the associativity conditions

$$u_{s+t}(\mathsf{id}_E \odot u_{st}) = u_t(u_s \odot \mathsf{id}_{E_t}) \tag{5}$$

$$u_{r(s+t)}(\mathsf{id}_E \odot u_{st}) = u_{(r+s)t}(u_{rs} \odot \mathsf{id}_{E_t}). \tag{6}$$

We observe that $u_{0t}\colon b \odot x_t \mapsto bx_t$ and $u_{t0}\colon x_t \odot b \mapsto x_t b$ give us back the canonical indentifications $\mathcal{B} \odot E_t = E_t = E_t \odot \mathcal{B}$.

Collecting the majority of these results we see that $E^\odot$ is a product system in the sense of the following definition from Bhat and Skeide[6].

3.1 Definition. A product system of Hilbert modules is a family $E^\odot = \big(E_t\big)_{t\in\mathbb{R}_+}$ of Hilbert $\mathcal{B}$–$\mathcal{B}$–modules E_t ($E_0 = \mathcal{B}$) with a family u_{st} of unitaries in $\mathcal{B}^{a,bil}(E_s \odot E_t, E_{s+t})$ fulfilling the associativity condition (6) (u_{0t}, u_{t0} being the canonical identifications).[(3.a)]

[(3.a)] This is the definition of product systems of Hilbert modules. It has an obvious version for von Neumann modules, where all appearing (operator) spaces should be strongly closed. However, we do not intend to go too much into the technicalities of von Neumann modules. It was our intention to give a precise definition in Section 2.4, because some of our classification results hold only for von Neumann modules, and because of the importance of the case $\mathcal{B} = \mathcal{B}(G)$ in Section 2.5 when we want to compare with existing results. Let us mention, however, that, starting from an E_0–semigroup ϑ on $\mathcal{B}^a(E)$ where E is some unital von Neumann module, we may construct a product system of von Neumann modules as before, provided that all ϑ_t fulfill the weaker condition to be normal mappings.

Once the mappings u_{st} (and in the case of the preceding product system coming from an E_0–semigroup the mappings u_t) are fixed, we use the identifications

$$E_s \odot E_t \;=\; E_{s+t} \;\; (\text{and } E \odot E_t \;=\; E). \tag{7}$$

(Obviously, if $E^{\odot}$ is the product system constructed from an E_0–semigroup ϑ, then we recover ϑ in this identification as $\vartheta_t(a) = a \odot \mathsf{id}_{E_t}$.)

Several natural questions arise.

3.2 Question. What is the connection with Arveson's product systems of Hilbert spaces (Arveson systems for short) which start from normal E_0–semigroups on $\mathcal{B}(G)$? Our construction (including the representation theory for $\mathcal{B}^a(E)$) is a direct generalization from Bhat's[4] approach to Arveson systems.$^{(3.b)}$ Arveson requires additional measurability conditions on a product system which are fulfilled, if we start with an E_0–semigroup which is continuous (pointwise on $\mathcal{B}(G)$) in the strong operator topology. Additionally, the Hilbert spaces of an Arveson system are all isomorphic (infinite-dimensional separable). We will see in Example 4.7 that we cannot hold this condition. In Section 7 we propose a suitable definition of continuous tensor product systems.

3.3 Question. Do we obtain all product systems by the preceding construction?$^{(3.c)}$ This question is closely related to the correct notion of measurability. It can be answered in the affirmative sense for Arveson systems; see Reference[2]. Certainly our answer to the measurability problem should be judged by checking whether it allows to preserve Arveson's result that all Arveson systems arise from E_0–semigroups on $\mathcal{B}(G)$ also in the case of product systems of Hilbert modules. We are not yet able to solve that problem, however, we show at least that our definition of continuous product system is not too restrictive.

3.4 Question. How can we classify product systems? Like for Arveson systems our classification scheme is based on how many units we have; see Section

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$^{(3.b)}$Arveson's[1] approach is based on Footnote (2.d) and relies on the simple structure of $\mathcal{B}(G)$–modules. It does not generalize to the module case. When we construct as in Example 4.5 the product system $E^{\odot}$ of two-sided von Neumann $\mathcal{B}(G)$–modules for an E_0–semigroup on $\mathcal{B}(G)$, then the Hilbert spaces $\mathfrak{H}_t$ such that $E_t = \mathcal{B}(G, G \otimes \mathfrak{H}_t)$ form the corresponding Arveson system.

$^{(3.c)}$The second construction of product systems starting from CP- or CPD-semigroups (cf. Footnote (4.d)) is even less exhaustive as it leads necessarily to *type I systems* (cf. Footnote (4.f)).

4. However, it turns out that we have to distinguish between general units and central units.$^{(3.d)}$ This leads to a refined classification as compared with that of Arveson systems. However, we are able to present simple examples (even of type III systems — a difficult issue for Arveson systems) which show that our refinement is necessary. Of course, our classification is made in such a way that most results by Arveson show to remain true also for product systems of Hilbert modules. In the first place, we are able to preserve the distinguished role played by the symmetric Fock space which becomes now the time ordered Fock module.

4 Units in products systems

Arveson systems, so far, are classified by their *units* (families of vectors in the members of the Arveson system which factorize into elementary tensors in a stationary way). The basic example of an Arveson system is the family $\Gamma^{\otimes}(K) = \big(\Gamma_t(K)\big)_{t\in\mathbb{R}_+}$ (K some Hilbert space) of boson Fock spaces $\Gamma_t(K) = \Gamma(L^2([0,t],K))$ which factorize as

$$\Gamma_s(K)\otimes\Gamma_t(K) \;\cong\; \Gamma(L^2([t,t+s],K))\otimes\Gamma_t(K) \;\cong\; \Gamma_{s+t}(K).$$

The units have the form $u_t = e^{ct}\psi(\mathbb{I}_{[0,t]}f)$ ($c\in\mathbb{C}, f\in K$) where $\psi(x)$ denotes the exponential vector to $x\in L^2([0,t],K)$.

$$(e^{cs}\psi(\mathbb{I}_{[0,s]}f))\otimes(e^{ct}\psi(\mathbb{I}_{[0,t]}f)) \;=\; e^{c(s+t)}\psi(\mathbb{I}_{[0,s+t]}f)$$

The Arveson system $\Gamma^{\otimes}(K)$ is generated by its units (there is no proper subsystem containing all the units). Such Arveson systems are said to be type I and Arveson showed that all type I Arveson systems have the form $\Gamma^{\otimes}(K)$ for a (unique up to isomorphism, i.e. up to dimension) Hilbert space K. The dimension of K, called index, is a complete isomorphism invariant of type I Arveson systems.

An Arveson system is type II, if it has a unit, but is not type I. It contains a unique maximal type I subsystem and the index of a type II system is that of its maximal type I subsystem.$^{(4.a)}$ Recent work of Tsirelson[20] and its systematic extension by Liebscher[12] show that there is an abundance of (mutually non-isomorphic) type II systems having the same index. So the index is certainly not a complete isomorphism invariant for type II systems.

$^{(3.d)}$Central units for a product system of von Neumann $\mathcal{B}(G)$–modules correspond precisely to units for the central Arveson subsystem as discussed in Footnote (3.b).

$^{(4.a)}$Indices behave additive under tensor product of Arveson systems, thus, justifying the name ‘index’.

Finally, an Arveson systems is type III, if it has no units. Existence of type III Arveson systems is known since ever, but also here only recently Tsirelson[21] has constructed an explicit example.

Unlike, for Arveson systems, where the notion of unit is put into evidence by the importance of the results derived from it, for product systems of Hilbert modules we have a possibility to motivate this notion. Let us recall that we are particularly interested in the case when the (strict) E_0–semigroup ϑ is a (weak) dilation of some (unital) CP-semigroup, or even a white noise. We repeat a result from Skeide[16].

4.1 Proposition. *For the triple (E, ϑ, ξ) the following conditions are equivalent.*

1. *The family $p_t = \vartheta_t(\xi\xi^*)$ of projections is increasing, i.e. $p_t \geq p_0$ for all $t \in \mathbb{T}$.*
2. *The mappings $T_t(b) = \langle \xi, \vartheta_t(\xi b \xi^*)\xi \rangle$ define a unital CP-semigroup T, i.e. (E, ϑ, ξ) is a weak dilation.*
3. *$T_t(\mathbf{1}) = \mathbf{1}$ for all $t \in \mathbb{T}$.*

Under any of these conditions the elements $\xi_t = \xi \in E_t \subset E$ fulfill

$$\xi_s \odot \xi_t \;=\; \xi_{s+t}, \tag{8}$$

$\xi_0 = \mathbf{1}$ and $T_t(b) = \langle \xi_t, b\xi_t \rangle$. Moreover, T is the trivial semigroup, i.e. (E, ϑ, ξ) is a (weak) white noise, if and only if all ξ_t commute with all $b \in \mathcal{B}$.

This encourages the following Definition from Bhat and Skeide[6].

4.2 Definition. A unit for a product system $E^\odot$ is a family $\xi^\odot = \big(\xi_t\big)_{t\in\mathbb{R}_+}$ of vectors $\xi_t \in E_t$ fulfilling (8) and $\xi_0 = \mathbf{1}$. The unit $\xi^\odot$ is unital, if it consists of unit vectors. It is central, if all ξ_t commute with all $b \in \mathcal{B}$. By $\mathcal{U}(E^\odot)$ we denote the set of all units

So far, this is a purely algebraic definition. Even the case that a unit is 0 except at $t = 0$ is allowed. Units in an Arveson system must satisfy certain measurability conditions. These conditions imply, in particular, that for any two (non-zero) units $u^\otimes, u'^\otimes$ the mapping $t \mapsto \langle u_t, u'_t \rangle$, which obviously is a semigroup in $\mathbb{C}$, is measurable, hence, continuous.[(4.b)]

[(4.b)] The *covariance function* defined on the pairs of all measurable Arveson units as the derivative of $\langle u_t, u'_t \rangle$ at $t = 0$ is a *conditionally positive definite kernel.* From here it is quite easy to show that an Arveson system generated by its units consists of symmetric Fock spaces.

In our frame it turns out that we obtain the most satsifactory results, if we base our classification on *continuous* sets of units.$^{(4.c)}$

Like for Arveson units it turns out that matrix elements of units have a semigroup property. However, as we learned already in 2.3, instead of looking just at matrix elements $\langle \xi_t, \xi'_t \rangle$ we have to switch our interest to (bounded) mappings $b \mapsto \langle \xi_t, b\xi'_t \rangle$ which, clearly, form a semigroup on $\mathcal{B}$. The collection of all these semigroups fulfills a positivity condition, namely, it is a *completely positive definite* (CPD) *kernel* in the sense of References[3,18].$^{(4.d)}$

4.3 Definition. By $\mathfrak{U} = \big(\mathfrak{U}_t\big)_{t\in\mathbb{R}_+}$, where the completely positive definite kernel $\mathfrak{U}_t \colon \mathcal{U}(E^\odot) \times \mathcal{U}(E^\odot) \to \mathcal{B}(\mathcal{B})$ is defined by

$$\mathfrak{U}_t^{\xi,\xi'}(b) \;=\; \langle \xi_t, b\xi'_t \rangle,$$

we denote the CPD-semigroup associated with $E^\odot$.

A set $S \subset \mathcal{U}(E^\odot)$ of units is continuous, if the CPD-semigroup $\mathfrak{U} \upharpoonright S$ is uniformly continuous, i.e. if the semigroup $\mathfrak{U}^{\xi,\xi'}$ is uniformly continuous for all $\xi^\odot, \xi'^\odot \in S$. In particular, a single unit $\xi^\odot$ is continuous, if the set $\{\xi^\odot\}$ is.$^{(4.e)}$

Now we are ready to set up our classification scheme.

4.4 Definition. A product systems of Hilbert modules $E^\odot$ is type I, if there is a continuous set S of units which generates $E^\odot$ (i.e. $E^\odot$ is the smallest subsystem of $E^\odot$ containing all units of S). $E^\odot$ is type II, if it has a continuous unit, but is not type I. It is type III if it has no continuous unit.$^{(4.f)}$

$^{(4.c)}$We are speaking about norm continuity. It is an open problem to decide, whether this may be weakened to norm measurability. Also weaker topologies coming from weaker topologies on $\mathcal{B}$ are thinkable. However, Example 4.7, which has only strongly continuous units, tells us that we may not expect to derive similar results for weaker topologies.

$^{(4.d)}$We show also that every CPD-semigroup, continuous or not, arises in this way from matrix elements of units in a product system. The construction of that product system is very much like a GNS-construction and, therefore, we call it the GNS-system of the CPD-semigroup. The first construction of a product system from a CP-semigroup in Reference[6] appears as a special case of the GNS-construction for CPD-semigroups.

$^{(4.e)}$One may show that for checking continuity of S it is sufficient that E contains one continuous unit $\xi^\odot$ and that the matrix elements $\langle \xi_t, \xi'_t \rangle$ and $\langle \xi'_t, \xi'_t \rangle$ depend continuously (in $\mathcal{B}$) on t for all $\xi'^\odot \in S$; see References[3,18].

$^{(4.f)}$The GNS-system of a CPD-semigroup (cf. Footnote (4.d)) is, by definition, generated by its units. Therefore, if the CPD-semigroup is uniformly continuous, then the GNS-system is type I, automatically.

A product system is spatial, if it has a central unital unit $\omega^\odot$, and completely spatial, if it is also type I and the generating subset S can be chosen to contain $\omega^\odot$.$^{(4.g)}$

Clearly, a central unital unit is continuous ($\langle \omega_t, \bullet\omega_t \rangle = \mathrm{id}_{\mathcal{B}}$ is constant and, therefore, continuous). So, a spatial product system is clearly non-type III.$^{(4.h)}$ One main result of Reference[3] asserts that non-type III product systems of von Neumann modules are spatial, automatically. We can determine completely the form of completely spatial systems and, therefore, also of type I systems of von Neumann modules: They are all (systems of) time ordered Fock modules; see Section 6. In Example 4.6 we describe a type I product system without central unit. This shows us that non-type III product systems of Hilbert modules need not be spatial and that type I product systems need not be time ordered Fock modules. On the other hand, one of the main results of Reference[3] asserts that non-type III product systems of von Neumann modules are always spatial. Therefore, type I products systems of von Neumann modules are always time ordered Fock modules.

Example 4.7 shows us that it is easy to write down product systems even of von Neumann modules which have not a single continuous unit. Nevertheless, this product system is generated by a single strongly continuous unit. This shows that classifications based on units which are continuous in a weaker topology only may be quite different. As a typical feature we find that, in particular, commutative algebras (in contrast with the extremely non-commutative $\mathcal{B}(G)$) provide us with interesting counter examples.

4.5 Example[3]. Let $\mathcal{B}$ denote a unital C^*–algebra. Then $\mathcal{B}$ is itself a Hilbert $\mathcal{B}$–module (with inner product $\langle b, b' \rangle = b^*b'$) with unit vector $\mathbf{1}$ and $\mathcal{B}^a(\mathcal{B}) = \mathcal{B}$. Let ϑ be an E_0–semigroup on $\mathcal{B}$. Then the associated product system is $E_t = \mathcal{B}$ as Hilbert $\mathcal{B}$–module, but with left multiplication $b.x_t = \vartheta_t(b)x_t$. Clearly, $E^\odot$ has a unital unit $\xi^\odot$ with $\xi_t = \mathbf{1}$. This unit is continuous, if and only ϑ is uniformly continuous.$^{(4.i)}$

In particular, if $\vartheta_t(b) = u_t b u_t^*$ is a (semi-)group of inner automorphisms

$^{(4.g)}$We are speaking about Hilbert modules. The preceding definition has analogues for algebraic product systems of pre-Hilbert modules (with types denoted by $\underline{\mathrm{I}}$, and so on) and for product systems of von Neumann modules (with types denoted by I^s, and so on. The continuity required for the units is, however, the same in all cases.

$^{(4.h)}$By Reference[3] a continuous unit may be *normalized* to consist of unit vectors. Therefore, in the definition of spatial we may replace central unital unit by central continuous unit.

$^{(4.i)}$This shows clearly that for checking continuity of a unit it is not sufficient to look only at matrix elements $\langle \xi_t, \xi_t \rangle$; cf. Footnote (4.e).

(for some unitary group u in $\mathcal{B}$), then $u_t : \mathcal{B} \to E_t, x \mapsto u_t x$ establishes an isomorphism from the trivial product system $(\mathcal{B})_{t \in \mathbb{R}_+}$ to $E^\odot$. One can show that the product system $E^\odot$ is isomorphic to the trivial product system, if and only if the automorphisms ϑ_t are inner. Therefore, the fact that automorphism semigroups on $\mathcal{B}(G)$ have trivial Arveson systems is entirely due to the fact that $\mathcal{B}(G)$ admits only inner automorphisms. Product systems of non-inner automorphism semigroups have interesting product systems and should not be excluded.

4.6 Example[3]**.** Let $\mathcal{B} = \mathcal{K}(G) + \mathbb{C}\mathbf{1} \subset \mathcal{B}(G)$ be the unitization of the compact operators on some infinite-dimensional Hilbert space G. Let e^{ith} be a unitary group on G for some self-adjoint element $h \in \mathcal{B}(G)$. For the automorphism semigroup $\vartheta_t = e^{ith} \bullet e^{-ith}$ we construct the product systems $E^\odot$ as in Example 4.5.

Suppose $\omega_t \in E_t$ is a central element, i.e. $e^{ith} b e^{-ith} \omega_t = \omega_t b$ or $b e^{-ith} \omega_t = e^{-ith} \omega_t b$ for all $b \in \mathcal{B}$. In other words, $e^{-ith}\omega_t$ is in the center of $\mathcal{B}$ and, therefore, a scalar multiple $c_t\mathbf{1}$ of $\mathbf{1}$ so that $\omega_t = c_t e^{ith}$. If $c_t \neq 0$, then it is not difficult to see that the requirment $e^{ith} \in \mathcal{B}$ puts very severe restrictions on h. For a single time t we cannot exclude completely that $h \notin \mathcal{B}$. However, if ω_t is a whole family of central elements in E_t (for instance, a unit) with $c_t \neq 0$, then differentiating $\frac{\omega_t}{c_t}$ at $t = 0$ tells us that h must be in $\mathcal{B}$ (as norm limit of elements in $\mathcal{B}$). Consequently, if $h \notin \mathcal{B}$, then $E^\odot$ has no central continuous units, although it is generated by the single continuous unit $\xi_t = \mathbf{1}$. In accordance with our result that type I systems of von Neumann modules are spatial, the problem dissappears, if we pass to the strong closure $\mathcal{B}(G)$ of $\mathcal{B}$.

4.7 Example[3]**.** Let $\mathcal{B} = \mathcal{C}_0(\mathbb{R}) + \mathbb{C}\mathbf{1} \subset \mathcal{C}_b(\mathbb{R})$ the unitization of the continuous functions on $\mathbb{R}$ vanishing at infinity. On $\mathcal{B}$ we define the time shift automorphism (semi-)group $\mathfrak{s} = (\mathfrak{s}_t)_{t \in \mathbb{R}_+}$ by setting $\mathfrak{s}_t f(s) = f(s-t)$. Clearly, $\mathfrak{s}_t$ is not uniformly continuous.

We construct the product system $E^\odot$ as in Example 4.5. Suppose now that $\xi^\odot$ is a unit. Then $\langle \xi_t, f\xi_t \rangle = \langle \xi_t, \xi_t \rangle \mathfrak{s}_t f$. Suppose $\xi^\odot$ was continuous. Then $f - \mathfrak{s}_t f = (\langle \xi_0, f\xi_0 \rangle - \langle \xi_t, f\xi_t \rangle) + (\langle \xi_t, \xi_t \rangle - \langle \xi_0, \xi_0 \rangle)\mathfrak{s}_t f$ implies, a contradiciton, that $\mathfrak{s}_t$ is norm continuous. Therefore, $E^\odot$ does not have continuous units.

This example can be extended in two directions. Firstly, we may restrict to $\mathcal{C}_0(\mathbb{R}_-) + \mathbb{C}\mathbf{1}$ so that $I\!I_{\mathbb{R}_-} \mathfrak{s}_t$ defines a proper E_0–semigroup. Secondly, we may pass to the strong closures $L^\infty(\mathbb{R})$ and $L^\infty(\mathbb{R}_-)$, respectively, providing us with analogue examples of product systems of von Neumann modules.

The structure of type I systems is remarkably invariant under the choice of the generating continuous subset S of units. In type II systems (or spatial systems) we have to fix a continuous (central) *reference* unit and in how far there are other units extending the reference unit to a continuous set of units may depend on the choice of the reference unit. The question in how far the continuity (or measurability) structure on the product system depends on the choice of the unit is an open problem even for Arveson systems. Therefore, if we speak about type II product systems, we often include the reference unit into the definition a speak of pairs $(E^\odot, \xi^\odot)$. We discuss in Section 7 the relation to a definition of continuous tensor product system.

5 The CPD-semigroup of spatial product systems

We mentioned that the crucial point in our classification of type I systems is to establish spatiality (where possible, of course). Let us see why this is so important. So let $\omega^\odot$ be a central unit for a product system $E^\odot$ and let $\xi^\odot$ be any other unit. Then

$$\mathfrak{U}_t^{\xi,\omega}(b) \;=\; \langle \xi_t, b\omega_t \rangle \;=\; \langle \xi_t, \omega_t \rangle b \;=\; \mathfrak{U}_t^{\xi,\omega}(\mathbf{1})b \tag{9}$$

and

$$\mathfrak{U}_{s+t}^{\xi,\omega}(\mathbf{1}) \;=\; \mathfrak{U}_t^{\xi,\omega}(\mathfrak{U}_s^{\xi,\omega}(\mathbf{1})) \;=\; \mathfrak{U}_t^{\xi,\omega}(\mathbf{1})\mathfrak{U}_s^{\xi,\omega}(\mathbf{1}).$$

In other words, $\mathfrak{U}^{\xi,\omega}(\mathbf{1})$ is a semigroup *in* $\mathcal{B}$ and determines $\mathfrak{U}^{\xi,\omega}$ by (9). In particular, $\mathfrak{U}^{\omega,\omega}(\mathbf{1})$ is a semigroup in $C_\mathcal{B}(\mathcal{B})$. If $\omega^\odot$ is continuous, then all $\mathfrak{U}_t^{\omega,\omega}(\mathbf{1})$ are invertible. Henceforth, we may assume without loss of generality that $\omega^\odot$ is unital, i.e. $\mathfrak{U}^{\omega,\omega} = \mathsf{id}$ is the trivial semigroup.

5.1 Lemma. *Let $\omega^\odot$ be a central unital unit and let $\xi^\odot$ be another unit for a product system $E^\odot$ such that the CPD-semigroup $\mathfrak{U} \restriction \{\omega^\odot, \xi^\odot\}$ is uniformly continuous. Let β denote the generator of the semigroup $\mathfrak{U}^{\omega,\xi}(\mathbf{1})$ in $\mathcal{B}$, i.e. $\mathfrak{U}_t^{\omega,\xi}(\mathbf{1}) = e^{t\beta}$, and let $\mathcal{L}^\xi$ denote the generator of the CP-semigroup $\mathfrak{U}^{\xi,\xi}$ on $\mathcal{B}$. Then the mapping*

$$b \;\longmapsto\; \mathcal{L}^\xi(b) - b\beta - \beta^* b \tag{10}$$

is completely positive, i.e. $\mathcal{L}^\xi$ is a CE-generator.[(5.a)]

[(5.a)] Christensen and Evans[7] established that every generator of a unifomly continuous normal CP-semigroup on a von Neumann algebra decomposes into a completely positive part and a part $b \mapsto b\beta + \beta^* b$.

PROOF. Since $\mathfrak{U}$ is a CPD-semigroup, the semigroup $\mathfrak{U}^{(2)} = \big(\mathfrak{U}_t^{(2)}\big)_{t\in\mathbb{R}_+}$ on $M_2(\mathcal{B})$ with $\mathfrak{U}_t^{(2)} = \left(\begin{smallmatrix}\mathfrak{U}_t^{\omega,\omega} & \mathfrak{U}_t^{\omega,\xi}\\ \mathfrak{U}_t^{\xi,\omega} & \mathfrak{U}_t^{\xi,\xi}\end{smallmatrix}\right)$ is competely positive. Its generator is

$$\mathfrak{L}^{(2)}\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix} \;=\; \frac{d}{dt}\Big|_{t=0}\begin{pmatrix}\mathfrak{U}_t^{\omega,\omega}(b_{11}) & \mathfrak{U}_t^{\omega,\xi}(b_{12})\\ \mathfrak{U}_t^{\xi,\omega}(b_{21}) & \mathfrak{U}_t^{\xi,\xi}(b_{22})\end{pmatrix} \;=\; \begin{pmatrix}0 & b_{12}\beta\\ \beta^*b_{21} & \mathcal{L}^{\xi}(b_{22})\end{pmatrix}.$$

As generator of a CP-semigroup $\mathfrak{L}^{(2)}$ is conditionally completely positive. Let $A_i = \left(\begin{smallmatrix}0 & 0\\ a_i & a_i\end{smallmatrix}\right)$ and $B_i = \left(\begin{smallmatrix}0 & -b_i\\ 0 & b_i\end{smallmatrix}\right)$. Then $A_iB_i = 0$, i.e. $\sum\limits_i A_iB_i = 0$, so that

$$0 \;\le\; \sum_{i,j} B_i^*\mathfrak{L}^{(2)}(A_i^*A_j)B_j \;=\; \sum_{i,j} B_i^*\begin{pmatrix}0 & a_i^*a_j\beta\\ \beta^*a_i^*a_j & \mathcal{L}^{\xi}(a_i^*a_j)\end{pmatrix}B_j$$

$$=\; \sum_{i,j}\begin{pmatrix}0 & 0\\ 0 & b_i^*(\mathcal{L}^{\xi}(a_i^*a_j) - a_i^*a_j\beta - \beta^*a_i^*a_j)b_j\end{pmatrix}.$$

This means that (10) is completely positive. ■

It is not difficult to prove the multi index version for CPD-semigroups; see References[3,18].

5.2 Theorem. *Let $E^{\odot}$ be a product system with a subset $S \subset \mathcal{U}(E^{\odot})$ of units and a central (unital) unit $\omega^{\odot}$ such that $\mathfrak{U} \restriction S \cup \{\omega^{\odot}\}$ is a uniformly continuous CPD-semigroup and denote by $\mathfrak{L} = \frac{d}{dt}\big|_{t=0}\mathfrak{U}_t \restriction S$ the generator of $\mathfrak{U} \restriction S$. Then there exists a mapping $S \to \mathcal{B}, \xi^{\odot} \mapsto \beta_\xi$ such that the kernel*

$$\mathfrak{L}_0^{\xi,\xi'}(b) \;=\; \mathfrak{L}^{\xi,\xi'}(b) - b\beta_{\xi'} - \beta_\xi^*b$$

is completely positive definite. In other words, doing the Kolmogorov decomposition for $\mathfrak{L}_0$ we find a Hilbert $\mathcal{B}$–$\mathcal{B}$–module F and a mapping $S \to F, \xi^{\odot} \mapsto \zeta_\xi$ such that

$$\mathfrak{L}^{\xi,\xi'}(b) \;=\; \langle\zeta_\xi, b\zeta_{\xi'}\rangle + b\beta_{\xi'} + \beta_\xi^*b. \tag{11}$$

We say $\mathfrak{L}$ has CE-form.

6 The time ordered Fock module and its CPD-semigroup

Let F be a Hilbert $\mathcal{B}$–$\mathcal{B}$–module F. Then $L^2(\mathbb{R}_+, F)$ is defined as norm completion of the space of F–valued step functions with inner product $\langle x, y\rangle =$

$\int \langle x(t), y(t)\rangle \, dt$. Higher tensor powers fulfill $L^2(\mathbb{R}_+, F)^{\odot n} = L^2(\mathbb{R}_+^n, F^{\odot n})$. The time ordered Fock module is defined as

$$\mathrm{I\Gamma}(F) = \bigoplus_{n=0}^{\infty} \Delta_n L^2(\mathbb{R}_+, F)^{\odot n}$$

where Δ_n is the indicator function of the set $\{t_n > \ldots > t_1 > 0\} \subset \mathbb{R}_+^n$ which acts as projection in the obvious way.

By Reference[6] the family $\mathrm{I\Gamma}^{\odot}(F) = \big(\mathrm{I\Gamma}_t(F)\big)_{t\in\mathbb{R}_+}$ of restrictions $\mathrm{I\Gamma}_t(F)$ of $\mathrm{I\Gamma}(F)$ to the interval $[0, t]$ forms a product system via the identification

$$[X_s \odot Y_t] = [\mathfrak{s}_t X_s] \odot [Y_t]$$

where $[X]$ means the function obtained by pointwise "evaluation" of the element $X \in \mathrm{I\Gamma}(F)$. Liebscher and Skeide[13] show that the set of continuous units (with the vacuum unit $\omega^{\odot} = (\omega_t)$, $\omega_t = \mathbf{1} \in L^2(\mathbb{R}_+, F)^{\odot 0} = \mathcal{B}$ as reference unit) consists of the units $\xi^{\odot}(\beta, \zeta) = \big(\xi_t\big)_{t\in\mathbb{R}_+}$ with $\xi_t = \bigoplus_{n=0}^{\infty} \xi_t^n$ defined by $\xi_t^0 = e^{t\beta}$ and

$$\xi_t^n(t_n, \ldots, t_1) = \xi_{t-t_n}^0 \zeta \odot \ldots \odot \xi_{t_2-t_1}^0 \zeta \xi_{t_1}$$

where the parametrization by pairs $(\beta, \zeta) \in \mathcal{B}\times F$ is one-to-one. The generator of the associated CPD-semigroup is

$$\mathfrak{L}^{(\beta,\zeta),(\beta',\zeta')}(b) = \langle \zeta, b\zeta'\rangle + b\beta' + \beta^* b.$$

In other words, it is a CE-generator.

6.1 Corollary. *Let $E^{\odot}$ be a completely spatial product system with a generating set $S \subset \mathfrak{U}(E^{\odot})$ of continuous units. By Theorem 5.2 the CPD-semigroup $\mathfrak{U} \upharpoonright S$ has the CE-generator (11) so that the the mapping*

$$\xi^{\odot} \longmapsto \xi^{\odot}(\beta_{\xi}, \zeta_{\xi})$$

determines an isometric embedding $E^{\odot} \to \mathrm{I\Gamma}^{\odot}(F)$.

A result from Skeide[19] asserts that F can be chosen minimal and that in this case the embedding is onto, i.e. an isomorphism. In other words, completely spatial product systems are time ordered Fock modules. The module F is a complete isomorphism invariant.

A further result from Reference[19] asserts that every spatial product system contains a unique maximal completely spatial subsystem.

6.2 Definition. The index of a spatial product system is the module F such that the maximal completely spatial subsystem is isomorphic to $\mathrm{I\Gamma}^{\odot}(F)$.

Reference[19] provides also a *product* of product systems under which the index behaves *additive* (direct sum), thus, justifying the name 'index'. In the case of Arveson systems this product gives us back the tensor product, if at least one factor is type I. In general, it gives only a subsystem of the tensor product.

We mentioned already the result from Reference[3] which states that non-type III systems of von Neumann modules are spatial. Thus, type I systems of von Neumann modules are completely spatial and, therefore, isomorphic to (strong closures of) time ordered Fock modules. It is not possible to explain the proof of this result in a few words. We mention that it requires a complete understanding of the endomorphisms of a time ordered product system. The result shows then to be equivalent to the result by Christensen and Evans[7].

7 Continuous product systems

So far, we know completely the structure of completely spatial systems (i.e. also of type I systems of von Neumann modules) and we have simple examples of other types which show that the refinement of our classification scheme, as compared with that of Arveson, is necessary. Applying a technique from Liebscher[12], which associates with each Arveson system one-to-one a type II Arveson system, we should be able to produce also lots of type II systems of Hilbert modules.

A result missing so far, is that any Arveson system comes from an E_0–semigroup. σ–Weakly continuous normal E_0–semigroups on $\mathcal{B}(G)$ for an infinite-dimensional separable Hilbert space G are classified one-to-one up to cocycle conjugacy by product systems. We see that this result depends on both sides on technical conditions: On the semigroup side σ–weak continuity and on the product system side measurability conditions.

In this section we start from strictly continuous E_0–semigroups and simply look which restrictions this implies for the structure of the associated product system. If we want to give a definition of *continuous* product system in such a way that every such product system comes from a strictly continuous strict E_0–semigroup on $\mathcal{B}^a(E)$ for some unital Hilbert $\mathcal{B}$–module E, then we are certainly not comitting an error requiring these conditions. Looking, in how far the conditions may be weakened (for instance, measurability instead of continuity) is a different problem. We will also not show that our conditions will be sufficient to assure that every such product system comes from an E_0–semigroup[(7.a)]. These questions we leave for future work.

[(7.a)] Although we, certainly, believe that this should be the case.

So let ϑ be a strict and strictly continuous E_0–semigroup on a Hilbert $\mathcal{B}$–module E with unit vector ξ and associated product system $E^\odot$. A first observation is that every member E_t of the product system is contained as a right submodule in E. Arveson requires that all members of an Arveson system (for $t > 0$) are isomorphic (infinite-dimensional and separable). This is true also for the members of a time ordered system which are isomorphic (for $t > 0$) as Hilbert bimodules. However, Example 4.7 shows that it need not be so. (All members of this product system are isomorphic as right modules but not two different of them are isomorphic as bimodules.) Consequently, we will require that all members of a product system may be embedded as right Hilbert modules into a fixed one.

Let us fix $x \in E$ and consider the family $\bigl(x_t\bigr)_{t\in\mathbb{R}_+}$ with $x_t = p_t x$. Since ϑ is strictly continuous, the function $t \mapsto x_t \in E$ is norm continuous. Moreover, $E^\odot$ is generated by such *sections*. (Each $y_t \in E_t$ can be written as $p_t x$ for a suitable $x \in E$.) Furthermore, if $\bigl(x_t\bigr)_{t\in\mathbb{R}_+}, \bigl(y_t\bigr)_{t\in\mathbb{R}_+}$ $(x_t, y_t \in E_t)$ are continuous families in E, then the function $(s,t) \mapsto x_s \odot y_t$ is also continuous. (We have

$$\begin{aligned} x_{s+\delta} \odot y_{t+\varepsilon} - x_s \odot y_t \;&=\; \vartheta_{t+\varepsilon}(x_{s+\delta}\xi^*)y_{t+\varepsilon} - \vartheta_t(x_s\xi^*)y_t \\ &=\; \vartheta_{t+\varepsilon}(x_{s+\delta}\xi^*)(y_{t+\varepsilon} - y_t) \\ &\quad+\; \vartheta_{t+\varepsilon}((x_{s+\delta} - x_s)\xi^*)y_t \\ &\quad+\; (\vartheta_{t+\varepsilon}(x_s\xi^*) - \vartheta_t(x_s\xi^*))y_t \end{aligned}$$

which is small, if (δ, ε) is small in $\mathbb{R}^2$.)

We obtain the following definition by passing from the concrete identification $E_t \subset E$ to a more arbitrary one $i_t \colon E_t \to E$, and expressing the preceding properties in terms of i_t.

7.1 Definition. Let $E^\odot$ be a product system of Hilbert $\mathcal{B}$–$\mathcal{B}$–modules with a family $i = \bigl(i_t\bigr)_{t\in\mathbb{R}_+}$ of isometric embeddings $i_t \colon E_t \to E$ into a unital Hilbert $\mathcal{B}$–module. Denote by

$$CS_i(E^\odot) \;=\; \Bigl\{x = \bigl(x_t\bigr)_{t\in\mathbb{R}_+} : x_t \in E_t, t \mapsto i_t x_t \text{ is continuous}\Bigr\}$$

the set of continuous sections of $E^\odot$ (with respect to i). We say $E^\odot$ is continuous (with respect to i), if the following conditions are satisfied.

1. For every $y_t \in E_t$ we can find a continuous section $x \in CS_i(E^\odot)$ such that $y_t = x_t$.

2. For every pair $x, y \in CS_i(E^{\odot})$ of continuous sections the function
$$(s,t) \longmapsto i_{s+t}(x_s \odot y_t)$$
is continuous.

We say two embeddings i and i' have the same continuous structure, if $CS_i(E^{\odot}) = CS_{i'}(E^{\odot})$.

7.2 Example. By construction, every product system coming from a strictly continuous strict E_0–semigroup ϑ on some unital Hilbert $\mathcal{B}$–module is a continuous product system. In particular, if ϑ is an E_0–semigroup on $\mathcal{B}$, then the product system as constructed in Example 4.5 is continuous, provided that ϑ is strictly continuous (as, for instance, in Example 4.6).

However, since $\mathcal{B}^a(\mathcal{B}) = \mathcal{B}$ is unital so that the strict topology coincides with the norm topology, the E_0–semigroup $\vartheta = \mathbb{s}$ on $\mathcal{B} = \mathcal{C}_0(\mathbb{R}) + \mathbb{C}\mathbf{1}$ as in Example 4.7 is not strictly continuous. Nevertheless, the associated product system is continuous. (The continuous sections are just the continuous functions $x\colon t \mapsto x_t \in \mathcal{B}$ where $\mathcal{B} = \mathcal{B}_t$ as right module. Since functions in $\mathcal{C}_0(\mathbb{R})$ are uniformly continuous, the semigroup $\mathbb{s}$ is C_0–continuous, i.e. $t \mapsto \mathbb{s}_t(b)$ is continuous for all $b \in \mathcal{B}$. Therefore, also the functions $(s,t) \mapsto x_s \odot y_t = \mathbb{s}_t(x_s)y_t$ are continuous for all continuous sections x, y.) Evidently, this remains true for every product system associated with a C_0–continuous E_0–semigroup on a unital C^*–algebra $\mathcal{B}$.

By Definition 7.1 the mappings $t \mapsto \langle x_t, y_t \rangle$ are continuous for all $x, y \in CS_i(E^{\odot})$. By Property (1) for every $b \in \mathcal{B}$ there exists a continuous section x such that $x_0 = b$. By Property (2) for every $y \in CS_i(E^{\odot})$ the mapping $t \mapsto (0,t) \mapsto x_0 \odot y_t = by_t$ is continuous. It follows that also the section $\big(by_t\big)_{t \in \mathbb{R}_+}$ is in $CS_i(E^{\odot})$. In other words, in a continuous product system the mappings $t \mapsto \langle x_t, \bullet y_t \rangle$ are C_0–continuous for all $x, y \in CS_i(E^{\odot})$.

A natural question is, whether the units among the continuous sections are continuous in the sense of Definition 4.3. Let us recall the result from References[3,18] which asserts that a set of units is continuous, if at least one unit $\xi^{\odot}$ of them is continuous, and if the matrix elements $\langle \xi_t, \xi'_t \rangle$ and $\langle \xi'_t, \xi'_t \rangle$ for all other units $\xi'^{\odot}$ depend continuously on t.[(7.b)] The latter condition is clearly fulfilled for an arbitrary set of units which are continuous sections. It turns out that the same remains true for continuous sections, but, before we prove that we give a concise definition.

[(7.b)] As soon as we have a central $\omega^{\odot}$ unit among the continuous sections we are on the save side, because $\langle \omega_t, \omega_t \rangle$ is then continuous so that the CP-semigroup $b \mapsto \langle \omega_t, b\omega_t \rangle = \langle \omega_t, \omega_t \rangle b$ is uniformly continuous.

7.3 Definition. A continuous product system is uniformly continuous, if for all its continuous sections $x, y \in CS_i(E^{\odot})$ the mapping $t \mapsto \langle x_t, \bullet y_t \rangle$ is uniformly continuous (i.e. continuous as mapping $\mathbb{R}_+ \to \mathcal{B}(\mathcal{B})$).

Before we show in full generality that a continuous product system with a continuous unit among the continuous sections is uniformly continuous, we consider a special case (which is a slight generalization of the mentioned result from References[3,18]).

7.4 Proposition. *Let $E^{\odot}$ be the continuous product system coming from a strictly continuous strict dilation (E, ϑ, ξ) and let $\xi^{\odot} = (\xi_t)$ be the unit $\xi_t = \xi \in E_t \subset E$. Then $E^{\odot}$ is uniformly continuous, if and only if $\xi^{\odot}$ is continuous, i.e. if and only if the dilated unital CP-semigroup $T_t = \langle \xi_t, \bullet \xi_t \rangle$ is uniformly continuous.*

Proof. ξ_t embedded into E is constant, so $\xi^{\odot} \in CS_i(E^{\odot})$. If $\xi^{\odot}$ is not continuous, then $\xi^{\odot}$ is a continuous section whose matrix elements T_t are not uniformly continuous, so neither is $E^{\odot}$.

To show the converse, let $\xi^{\odot}$ be continuous and let $x, y \in CS_i(E^{\odot})$. Observe that $\xi = \xi \odot \xi_t$ in the factorization $E = E \odot E_t$. We find

$$\langle x_{t+\varepsilon}, by_{t+\varepsilon} \rangle - \langle x_t, by_t \rangle = \langle x_{t+\varepsilon}, by_{t+\varepsilon} \rangle - \langle \xi_\varepsilon \odot x_t, \xi_\varepsilon \odot by_t \rangle$$

$$= \langle x_{t+\varepsilon} - \xi_\varepsilon \odot x_t, by_{t+\varepsilon} \rangle + \langle \xi_\varepsilon \odot x_t, b(y_{t+\varepsilon} - \xi_\varepsilon \odot y_t) \rangle + \langle \xi_\varepsilon \odot x_t, (b\xi_\varepsilon - \xi_\varepsilon b) \odot by_t \rangle.$$

The norm of $x_{t+\varepsilon} - \xi_\varepsilon \odot x_t$ is small for ε sufficiently small, because $\xi^{\odot}$ and x are continuous sections, and similarly for $y_{t+\varepsilon} - \xi_\varepsilon \odot y_t$. Consequently, the norm of the mapping which maps b to the first plus the second summand is small. The norm of the mapping which maps b to the third summand is small, because $\|b\xi_\varepsilon - \xi_\varepsilon b\|^2 = T_\varepsilon(b^*b) - T_\varepsilon(b^*)b - b^*T_\varepsilon(b) + b^*b$ is $\|b\|^2$ times a small number. This is uniform left continuity. To see uniform right continuity replace t by $t - \varepsilon$. ■

It remains, to show that an arbitrary continuous product system with a continuous unit $\xi^{\odot}$ can be obtained, including its continuous structure, from a strictly continuous strict dilation. In a certain sense, we have to reverse the construction of a product system $E^{\odot}$ from an E_0–semigroup on a Hilbert module E. Fortunately, we have availabe the contructuction from Bhat and Skeide[6], which we describe briefly. Suppose the product system has a unital unit $\xi^{\odot}$. Then the mappings $\gamma_{(s+t)t} \colon x_t \mapsto \xi_s \odot x_t$ provide us with an inductive system of isometric embeddings $E_t \to E_{s+t}$ giving rise to an inductive limit E_∞ which is a right Hilbert module. Under the canonical embeddings $E_t \to E_\infty$ all ξ_t are mapped to the same unit vector $\xi \in E_\infty$. We verify $E_\infty \odot E_t = E_\infty$ and the associativity condition in Equation (7). Therefore, $(E_\infty, \vartheta, \xi)$ with $\vartheta_t(a) = a \odot \mathsf{id}_{E_t}$ is a strict dilation. Obviously, the product system of $(E_\infty, \vartheta, \xi)$ is $E^\odot$ and the unit $\bigl(\xi\bigr)_{t\in\mathbb{R}_+}$ gives us back the unit we started with. If the pair $(E^\odot, \xi^\odot)$ comes from a dilation, then (E, ξ) and (E_∞, ξ) are canonically isomorphic, if and only if $\lim_{t\to\infty} p_t E = E$. Such dilations are called primary.[(7.c)] Clearly, the subspace $\lim_{t\to\infty} p_t E$ is canonically isomorphic to E_∞.

If $\xi^\odot$ is a continuous unit, but not necessarily unital, then we know from Reference[3] how to normalize $\xi^\odot$ to a continuous unital unit within $E^\odot$. Moreover, if $\xi^\odot$ is among the continuous sections, then so is its normalization. (The normalization suggested in Reference[3] is unique, but even if there are several possibilities to obtain a unital unit in the subsystem generated by $\xi^\odot$, then the results below show that the continuous structure does not depend on the choice.) Henceforth, we assume that $\xi^\odot$ is unital.

Denote by $k = \bigl(k_t\bigr)_{t\in\mathbb{R}_+}$ the family of canonical embeddings $k_t \colon E_t \to E_\infty$. This provides us with a set $CS_k(E^\odot)$ of continuous sections as in Definition 7.1.

7.5 Theorem. *Let $\xi^\odot$ be a continuous (unital) unit in a continuous product system $E^\odot$. If $\xi^\odot$ is in $CS_i(E^\odot)$, then the E_0–semigroup ϑ constructed on $\mathcal{B}^a(E_\infty)$ is strictly continuous and $CS_k(E^\odot) = CS_i(E^\odot)$. In particular, the continuous structure $CS_k(E^\odot)$ of $E^\odot$ does not depend on $\xi^\odot$.*

Conversely, if $\xi^\odot \notin CS_i(E^\odot)$, then, of course, $CS_k(E^\odot) \neq CS_i(E^\odot)$.

Proof. By the preceding discussion we may suppose that $\xi^\odot$ is unital. As the case $\xi^\odot \notin CS_i(E^\odot)$ is clear, we suppose that $\xi^\odot \in CS_i(E^\odot)$.

First, we show that ϑ is strictly continuous. The following proof is an imitation of that in Reference[6] for type I systems, except that now we have to consider continuous sections which are not necessarily units. For a fixed $y \in E_\infty$ the mapping $t \mapsto \mathfrak{s}^r_t y := y \odot \xi_t$ is continuous. (Indeed, as $\mathfrak{s}^r_t$ is bounded by $\|\xi_t\| = 1$, it is sufficient to show the statement on the total subset of E_∞ consisting of elements of the form $y = k_s y_s$ for some $s \in \mathbb{R}_+, y_s \in E_s$. By definition there exists $x \in CS_i(E^\odot)$ such that $y_s = x_s$. Then (for $\varepsilon \geq 0$; $\varepsilon \leq 0$ is analogous)

$$\|k_s y_s \odot \xi_{t+\varepsilon} - k_s y_s \odot \xi_t\| \;=\; \|y_s \odot \xi_\varepsilon \odot \xi_t - \xi_\varepsilon \odot y_s \odot \xi_t\|$$

[(7.c)]If E and E_∞ are isomorphic, but not necessarily canonical, then we know from Reference[16] that the E_0–semigroups on E and E_∞ are cocycle conjugate.

$$\leq \|y_s \odot \xi_\varepsilon - \xi_\varepsilon \odot y_s\|$$

$$\leq \|i_{s+\varepsilon}(x_s \odot \xi_\varepsilon) - i_s(x_s \odot \xi_0)\| + \|i_s(\xi_0 \odot x_s) - i_{\varepsilon+s}(\xi_\varepsilon \odot x_s)\|$$

is small for ε sufficiently small.) It follows that

$$\|\vartheta_t(a)y - ay\| \leq \|\vartheta_t(a)(y - y \odot \xi_t)\| + \|ay \odot \xi_t - ay\|$$

is small for t sufficiently small. This implies, in particular, that $CS_k(E^\odot)$ is the continuous structure derived from an E_0–semigroup and, therefore, the product of sections in $CS_k(E^\odot)$ is continuous in the sense of Definition 7.1(2).

Let $x \in CS_i(E^\odot)$. We find

$$\|k_{t+\varepsilon}x_{t+\varepsilon} - k_t x_t\| = \|x_{t+\varepsilon} - \xi_\varepsilon \odot x_t\| = \|i_{t+\varepsilon}(\xi_0 \odot x_{t+\varepsilon}) - i_{t+\varepsilon}(\xi_\varepsilon \odot x_t)\|$$

which is small for ε sufficiently small so that $x \in CS_k(E^\odot)$.

Conversely, let $x \in CS_k(E^\odot)$. For fixed $t \in \mathbb{R}_+$ we may choose $y \in CS_i(E^\odot)$ such that $y_t = x_t$ for that t. Let $\varepsilon \geq 0$. Left continuity we see from

$$\|i_{t+\varepsilon}x_{t+\varepsilon} - i_t x_t\| \leq \|i_{t+\varepsilon}(x_{t+\varepsilon} - \xi_\varepsilon \odot x_t)\| + \|i_{t+\varepsilon}(\xi_\varepsilon \odot y_t) - i_t y_t\|$$

which is small for ε sufficiently small. For right continuity we observe that for small ε we have $x_t \approx \xi_\varepsilon \odot x_{t-\varepsilon}$, because $\xi^\odot, x \in CS_k(E^\odot)$, and $y_t \approx \xi_\varepsilon \odot y_{t-\varepsilon}$, because $\xi^\odot, y \in CS_i(E^\odot)$. Therefore,

$$x_{t-\varepsilon} \approx (\xi_\varepsilon^* \odot \mathsf{id}_{E_{t-\varepsilon}})x_t = (\xi_\varepsilon^* \odot \mathsf{id}_{E_{t-\varepsilon}})y_t \approx y_{t-\varepsilon}$$

and also

$$\|i_{t-\varepsilon}x_{t-\varepsilon} - i_t x_t\| \leq \|i_{t-\varepsilon}(x_{t-\varepsilon} - y_{t-\varepsilon})\| + \|i_{t-\varepsilon}y_{t-\varepsilon} - i_t y_t\|$$

is small. ■

7.6 Definition. A continuous product system $E^\odot$ is type I(II)(III), if there is a generating continuous subset $S \subset CS_i(E^\odot)$ of units (if there is a continuous unit $\xi^\odot \in CS_i(E^\odot)$) (if there is no continuous unit in $CS_i(E^\odot)$).

The following is a simple corollary of Theorem 7.5 and Proposition 7.4

7.7 Theorem. *Non-type III continuous product systems are uniformly continuous.*

If $E^\odot$ is a non-type III product system, then, unless specified otherwise explicitly, we assume that it comes shipped with its natural continuous structure $CS_k(E^\odot)$. If $E^\odot$ is continuous with respect to this structure, then type and continuous type coincide. (If specified differently, i.e. if the continuous unit making the product system non-type III, the types need not coincide.)

7.8 Example. The product system in Example 4.7 is continuous type III, although its unit is among the continuous sections. It is an interesting problem to find examples for a continuous product system of Hilbert modules without units in $CS_i(E^\odot)$. So far, we have to content ourselves with Tsirelson's Hilbert space example. Apparently, for Hilbert modules there are different levels of type III.

Definition 7.1 excludes some interesting product systems of von Neumann modules (or even of Hilbert modules, when $\mathcal{B}$ is a von Neumann algebra or otherwise too big). For instance, the product system of the time shift $\mathbb{s}$ on the strong closure $\mathcal{B} = L^\infty(\mathbb{R}_+)$ of Example 4.7 has not a single non-zero continuous section. ($\mathbb{s}$ is only normal, but not C_0–continuous.)

Of course, a product system arising from such a natural semigroup like the time shift should belong to the objects of interest, so we have to find a definition which suits better for von Neumann modules. A first possibility is to replace everywhere in Definition 7.1 'continuous' with 'strongly continuous'. A second possibility is based on the following observation.

7.9 Proposition. *Let $E^\odot$ be a product system of Hilbert $\mathcal{B}$–$\mathcal{B}$–modules with a family $i = \big(i_t\big)_{t\in\mathbb{R}_+}$ of embeddings $i_t\colon E_t \to E$ into a unital Hilbert $\mathcal{B}$–module and define $CS_i(E^\odot)$ as in Definition 7.1. Suppose $CS_i(E^\odot)$ fulfills Condition (2) and the weaker condition*

1'. For every $t \in \mathbb{R}_+$ the subspace $CS_t = \{x_t \;(x \in CS_i(E^\odot))\}$ is dense in E_t.

Then $E^\odot$ is a continuous product system.

PROOF. Fix $t \in \mathbb{R}_+$. We have to show that the subspace CS_t is all of E_t. We recall a well-known result from Banach space theory. If W is a dense subspace of a Banach space V, then for every $v \in V$ there exists a sequence $\big(w_n\big)_{n\in\mathbb{N}}$ in W such that the series $\sum_{n=1}^\infty w_n$ converges absolutely to v. So, for $x_t \in E_t$ choose a sequence $\big(x^n\big)_{n\in\mathbb{N}}$ in $CS_i(E^\odot)$ such that the series $\sum_{n=1}^\infty x_t^n$ converges absolutely to x_t. We may assume (possibly after multiplying each section x^n by a suitable continuous numerical function) that $\|x_s^n\| \le \|x_t^n\|$ for all s, n. It follows that the series over the sections x^n converges uniformly over $s \in \mathbb{R}_+$ to a continuous section x with the correct value x_t. ■

If we replace in Definition 7.1 Condition (1) by Condition (1'), then we obtain a definition suitable for von Neumann modules, if we replace further in Condition (1') 'dense' by 'strongly dense'. In both possible definitions our example $L^\infty(\mathbb{R})$ would be continuous. Here we do not intend to decide between the possible definitions. A decission should be based on further investigation of examples and, in particular, of counter examples.

We close with a remark on continuity of units in product systems of von Neumann modules. By a recent result of Elliott[8] C_0–convergence to $\mathsf{id}_\mathcal{B}$ of a sequence of normal completely positive mappings on a von Neumann algebra implies uniform convergence. It is routine extension of this result to conclude that every normal C_0–continuous CP-semigroup is uniformly continuous. Therefore, in a continuous (in the sense of Definition 7.1) product system of von Neumann modules the set $\mathfrak{U}(E^\odot) \cap CS_i(E^\odot)$ is a continuous set of units.

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