Subsystems of Fock Need Not Be Fock: Spatial CP-Semigroups*

B.V.Rajarama Bhat, Volkmar Liebscher, and Michael Skeide

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Abstract

We show that a product subsystem of a time ordered system (that is, a product system of time ordered Fock modules), though type I, need not be isomorphic to a time ordered product system. In that way, we answer an open problem in the classification of CP-semigroups by product systems. We define spatial strongly continuous CP-semigroups on a unital C^*–algebra and characterize them as those that have a Christensen-Evans generator.

1 Introduction

Bhat and Skeide [BS00] associate with every CP-semigroup $T = (T_t)_{t\in\mathbb{R}_+}$ on a unital C^*–algebra $\mathcal{B}$ a product system $E^\odot = (E_t)_{t\in\mathbb{R}_+}$ of correspondences E_t over $\mathcal{B}$ and a unit $\xi^\odot = (\xi_t)_{t\in\mathbb{R}_+}$ for that product system, such that:

1. $T_t = \langle \xi_t, \bullet\xi_t\rangle$, and
2. $E^\odot$ is generated by $\xi^\odot$.

(Recall that ***product system*** means that $E_s \odot E_t = E_{s+t}$ in an associative way, and that ***unit*** means that the elements $\xi_t \in E_t$ compose accordingly as $\xi_s \odot \xi_t = \xi_{s+t}$. Regarding $t = 0$ we require that $E_0 = \mathcal{B}$ and $\xi_0 = \mathbf{1}$.) We refer to $E^\odot$ as the ***GNS-system*** of T and to the pair $(E^\odot, \xi^\odot)$ as its ***GNS-construction***.

Following Skeide [Ske06], we call a product system ***spatial***, if contains a unit $\omega^\odot$ that is ***central*** (that is, $b\omega_t = \omega_t b$ for all $b \in \mathcal{B}, t \in \mathbb{R}_+$) and ***unital*** (that is, $\langle \omega_t, \omega_t\rangle = \mathbf{1}$ for all $t \in \mathbb{R}_+$).

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Suppose $E^{\odot}$ is a spatial product system with another unit $\xi^{\odot}$ such that the CP-semigroup T on $\mathcal{B}$ defined by setting $T_t := \langle \xi_t, \bullet \xi_t \rangle$ and the semigroup $\langle \omega_t, \xi_t \rangle$ of elements in $\mathcal{B}$ are uniformly continuous. Then Barreto, Bhat, Liebscher and Skeide [BBLS04, Lemma 5.1.1] asserts that the generator $\mathcal{L} := \lim_{t \to 0} \frac{T_t - \mathsf{id}_{\mathcal{B}}}{t}$ of T has ***Christensen-Evans*** form, that is,

$$\mathcal{L}(b) \;=\; \mathcal{L}_0(b) + b\beta + \beta^* b$$

for a CP-map $\mathcal{L}_0$ on $\mathcal{B}$ and an element β in $\mathcal{B}$. Conversely, if a CP-semigroup has Christensen-Evans generator, then [BBLS04, Corollary5.1.3] asserts that its GNS-system is a subsystem of a product system of ***time ordered Fock modules*** (see Section 2), which is spatial. For the CP-semigroup T the "good" property of having a Christensen-Evans generator is, therefore, not so much equivalent to whether the GNS-system **is** spatial, but rather, whether it **embeds** into a spatial one. This property is so important that we would like to propose it as the definition of *spatial* CP-semigroup. However, we wish to give this definition in terms intrinsic to the CP-semigroup (Definition 3.2) and, then, show that it is equivalent to the preceding property (Theorem 3.4).

A natural question in this context is whether these two properties, spatial GNS-system or embedding of the GNS-system into a spatial one, coincide. For product systems of von Neumann correspondences the main result of [BBLS04] asserts that a product system that contains the unit of a uniformly continuous CP-semigroup is spatial, automatically. (This is shown, establishing equivalence with the results by Christensen-Evans [CE79] about the form of the generators of CP-semigroups on von Neumann algebras, in fact, the Christensen-Evans form.) In Section 2 we will show by an explicit counter example that for C^*–correspondences the properties need not coincide. However, the scope of Section 2 is somewhat more ambitious:

It is well-known that a product system of Hilbert spaces in the sense of Arveson [Arv89] (***Arveson system***, for short) is isomorphic to a product systems of symmetric Fock spaces, if it is generated by its units, that is, if it is type I. Moreover, an arbitrary subsystem of a type I Arveson system is type I and, therefore, also Fock. In [BBLS04] the same is shown for von Neumann (or W^*–) correspondences if the set of generating units is continuous. The statement fails already for von Neumann modules, if the generating units fulfill weaker conditions. (The product system of the Brownian semigroup or the Ornstein-Uhlenbeck semigroup, is non-Fock, though generated by a single strongly continuous unit; see Fagnola, Liebscher and Skeide [Ske05, FLS08].) [BBLS04, Example 4.2.4] is a simple example for a type I product system of Hilbert modules that is non-Fock. However, it also is not a subsystem of a product system of Fock modules. (It is the product system of a uniformly continuous noninner automorphism group whose generator has not Christensen-Evans form.)

In Section 2 we will construct a subsystem of a product system of Fock modules that is generated by a single continuous unit but, nevertheless, is not isomorphic to another product system of Fock modules. We will show this, by establishing that the subsystem does not contain central unit vectors, so that is it not spatial. Such a product system cannot be Fock, because Fock implies spatial.

In Section 3 we show that the intrinsic definition of strongly continuous spatial CP-semigroups (given for CP-semigroups on $\mathcal{B}(H)$ by Arveson [Arv97] and elaborated further in Arveson [Arv99] and Bhat [Bha01]) and the spatial embedding property for its product system are equivalent.

We would like to mention that our definition of spatial CP-semigroup is far more general than that of Powers [Pow04] in terms of intertwining semigroups.

2 The counter example

Let F be a ***correspondence*** over a unital C^*–algebra $\mathcal{B}$ (that is, F is a right Hilbert $\mathcal{B}$–module with a nondegenerate left action of $\mathcal{B}$). Denote by $\odot$ the ***internal tensor product*** over $\mathcal{B}$. The ***full Fock module*** over $L^2(\mathbb{R}_+, F)$, the completion of the space of (right continuous) F–valued step functions, is defined as

$$\mathcal{F}(L^2(\mathbb{R}_+, F)) \;:=\; \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+, F)^{\odot n}$$

where $L^2(\mathbb{R}_+, F)^{\odot 0} = \mathcal{B}$. By $\omega := \mathbf{1} \in L^2(\mathbb{R}_+, F)^{\odot 0}$ we denote the ***vacuum***. The space $L^2(\mathbb{R}_+, F)^{\odot n}$ may be considered as the completion of the space of step functions on $\mathbb{R}_+^n$ with values in $F^{\odot n}$.

Let Δ_n denote the indicator function of the subset $\{(t_n, \ldots, t_1)\colon t_n > \ldots > t_1 \geq 0\}$ of $\mathbb{R}_+^n$. Then Δ_n acts on the n–particle sector as a projection via pointwise multiplication (and Δ_0 acts as identity on the vacuum). Set $\Delta = \bigoplus_{n=0}^{\infty} \Delta_n$. The ***time ordered Fock module*** is the subcorrespondence

$$\mathbb{I}\Gamma(F) = \Delta\mathcal{F}(L^2(\mathbb{R}_+, F))$$

of $\mathcal{F}(L^2(\mathbb{R}_+, F))$. By $\mathbb{I}\Gamma_t(F)$ we denote the subcorrespondence of those functions that are zero if the maximum time argument is $t_n \geq t \in \mathbb{R}_+$. Setting

$$[u_{st}(F_s^m \odot G_t^n)](s_m, \ldots, s_1, t_n, \ldots, t_1) \;:=\; F_s^m(s_m - t, \ldots, s_1 - t) \odot G_t^n(t_n, \ldots, t_1),$$

we define bilinear unitaries $u_{st}\colon \mathbb{I}\Gamma_s(F) \odot \mathbb{I}\Gamma_t(F) \to \mathbb{I}\Gamma_{s+t}(F)$ that turn the family $\mathbb{I}\Gamma^{\odot}(F) = (\mathbb{I}\Gamma_t(F))_{t\in\mathbb{R}_+}$ into a ***product system*** Hilbert modules in the sense of Bhat and Skeide [BS00], that is, the multiplication $F_sG_t := u_{st}(F_s \odot G_t)$ is associative and for $s = 0$ or $t = 0$ reduces to the left or right action of $\mathcal{B}$.

It is easy to check that for every element ζ in F, the elements ξ_t that in each n–particle sector assume the constant value $\zeta^{\odot n}$, form a ***unit*** $\xi^{\odot} = (\xi_t)_{t\in\mathbb{R}_+}$, that is, $\xi_0 = \mathbf{1}$ and $\xi_s\xi_t = \xi_{s+t}$. This unit is also continuous. (See Liebscher and Skeide [LS01] for the precise form of all continuous units.) For $\zeta = 0$ we obtain the vacuum unit $\omega^\odot = (\omega_t)_{t\in\mathbb{R}_+}$ with $\omega_t = \mathbf{1}$. The vacuum unit is ***central***, that is, $b\omega_t = \omega_t b$ for all $t \in \mathbb{R}_+, b \in \mathcal{B}$. And it consists of ***unit vectors***, that is, $\langle\omega_t, \omega_t\rangle = \mathbf{1}$ for all $t \in \mathbb{R}_+$. Therefore, if we find a subsystem that does not contain any central unit vector, then this subsystem cannot be Fock.

Let $\mathcal{B} = C_0[0,\infty) + \mathbb{C}\mathbf{1}$ denote the unital C^*–algebra of all continuous functions on $\mathbb{R}_+$ that have a limit at infinity. Define the Hilbert $\mathcal{B}$–module $F := \mathcal{B}$. We turn F into a correspondence over $\mathcal{B}$ by defining the left action

$$b.x \;:=\; \mathbb{s}_1(b)x,$$

where $\mathbb{s}_1$ is the left shift by 1, which acts as $[\mathbb{s}_1(b)](t) = b(t+1)$. Denote by $\xi^\odot$ the unit corresponding to the parameter $\zeta := \mathbf{1} \in F$.

2.1 Theorem. *The product subsystem of $\mathbb{I}\Gamma^\odot(F)$ generated by the continuous unit $\xi^\odot$ has no central unit vectors. In particular, it is not isomorphic to a time ordered system.*

Proof. $F^{\odot n}$ is $\mathcal{B}$ as Hilbert right module but with left action $b.x = \mathbb{s}_n(b)x$. No element of $F^{\odot n}$ $(n \geq 1)$ can commute with all elements of $\mathcal{B}$. Therefore, for each $t \geq 0$ the set of central elements of $\mathbb{I}\Gamma_t(F)$ is the vacuum or 0–particle sector $\mathcal{B}$. Commutative C^*–algebras do not possess proper isometries. So, the only unit vectors in $\mathcal{B}$ are unitaries. By multiplying (from the right) with the adjoint, we may assume that such a unit vector is $\mathbf{1}$.

The product subsystem of $\mathbb{I}\Gamma^\odot(F)$ generated by $\xi^\odot$ is $E^\odot = (E_t)_{t\geq 0}$ with

$$E_t \;=\; \overline{\operatorname{span}}\Big\{b_n\xi_{t_n} \odot \ldots \odot b_1\xi_{t_1}b_0 \colon n \in \mathbb{N}, t_i > 0, t_1 + \ldots + t_n = t, b_i \in \mathcal{B}\Big\}.$$

for $t > 0$. Denote by P_0 and P_1 the projection onto the vacuum component and onto the one-particle component, respectively. We are done if we show that if an element in $x_t \in E_t$ has vacuum component $P_0x_t = \mathbf{1}$, then the one-particle component $P_1x_t \in L^2([0,t), F)$ is nonzero, too.

Any $x_t \in E_t$ can be approximated by expressions of the form

$$X_t \;=\; \sum_{i=1}^m b^{(i)}_{n(i)}\xi_{t^{(i)}_{n(i)}} \odot \ldots \odot b^{(i)}_1\xi_{t^{(i)}_1}b^{(i)}_0.$$

For $\varepsilon > 0$ suppose that $\|x_t - X_t\| \leq \varepsilon$. Therefore, also $\|P_0x_t - P_0X_t\| \leq \varepsilon$ and $\|P_1x_t - P_1X_t\| \leq \varepsilon$. Further, suppose that $P_0x_t = \mathbf{1}$, that is, suppose that

$$\Big\|\mathbf{1} - \sum_{i=1}^m b^{(i)}_{n(i)} \ldots b^{(i)}_1 b^{(i)}_0\Big\| \;\leq\; \varepsilon.$$

The one-particle component of an expression like $b_n\xi_{t_n} \odot \ldots \odot b_1\xi_{t_1}b_0$ is the same as the one-particle component of

$$b_n(\mathbf{1} \oplus I\!I_{[0,t_n)}\mathbf{1}) \odot \ldots \odot b_1(\mathbf{1} \oplus I\!I_{[0,t_1)}\mathbf{1})b_0,$$

where $\mathbb{I}_A$ denotes the ***indicator function*** of the set A. The one-particle component of this expression is

$$\mathbb{I}_{[t_1+\ldots t_{n-1}, t_1+\ldots t_n)} \mathbb{S}_1(b_n) b_{n-1} \ldots b_1 b_0 \; + \; \ldots \; + \; \mathbb{I}_{[t_1+t_2, t_1)} \mathbb{S}_1(b_n \ldots b_2) b_1 b_0 \; + \; \mathbb{I}_{[t_1, 0)} \mathbb{S}_1(b_n \ldots b_1) b_0.$$

From $\lim_{s\to\infty}[\mathbb{S}_1(b)c](s) = \lim_{s\to\infty} b(s+1)c(s) = \lim_{s\to\infty} b(s)c(s) = \lim_{s\to\infty}[bc](s)$ and the fact that X_t contains only finitely many summands it follows that

$$\lim_{s\to\infty}\langle P_1 X_t, P_1 X_t\rangle(s) \;=\; \lim_{s\to\infty} t\Big|\sum_{i=1}^{m} b_{n(i)}^{(i)} \ldots b_1^{(i)} b_0^{(i)}\Big|^2(s) \;=\; \lim_{s\to\infty} t\langle P_0 X_t, P_0 X_t\rangle(s).$$

The function $\langle P_0 X_t, P_0 X_t\rangle$ of s is uniformly close to $\mathbf{1}$. So, $\|P_1 X_t\|^2 \geq \lim_{s\to\infty}\langle P_1 X_t, P_1 X_t\rangle(s)$ and, therefore for also $\|P_1 x_t\|$ is bigger than a number arbitrarily close to $t \neq 0$. ■

3 Spatial CP-semigroups

Recall that a CP-map T ***dominates*** another S, if the difference $T - S$ is a CP-map, too. A CP-semigroup T ***dominates*** another S, if T_t dominates S_t for all $t \in \mathbb{R}_+$. A CP-semigroup S on a unital C^*–algebra $\mathcal{B}$ is ***elementary***, if it has the form $S_t(b) = c_t^* b c_t$ for some semigroup $c = (c_t)_{t\in\mathbb{R}_+}$ of elements c_t in $\mathcal{B}$.

A semigroup c^* such that T dominates the elementary CP-semigroup $b \mapsto c_t^* b c_t$ is what Arveson [Arv97] called a ***unit*** for T in the case $\mathcal{B} = \mathcal{B}(H)$. Without continuity conditions, every CP-semigroup dominates an elementary CP-semigroup, namely, the 0–semigroup which is $\mathsf{id}_{\mathcal{B}}$ for $t = 0$ and 0 otherwise. Depending on the context, there are several topologies around in which a CP-semigroup can be continuous with respect to time $t \in \mathbb{R}_+$. The uniform (or norm) topology, the strong and weak topologies of operators on the Banach space $\mathcal{B}$, and pointwise versions of all the operator topologies when $\mathcal{B} \subset \mathcal{B}(H)$ is a concrete operator algebra, for instance, if $\mathcal{B}$ is a von Neumann algebra.

In Arveson's definition, a unit for a CP-semigroup on $\mathcal{B}(H)$ is required pointwise continuous in the strong operator topology of $\mathcal{B}(H)$. The usual topology used for CP-semigroups on a C^*–algebra is the strong topology of operators on the Banach space $\mathcal{B}$. It is well-known that a weakly continuous semigroup is also strongly continuous; see, for instance, Bratteli and Robinson [BR87, Corollary 3.1.8]. For semigroups $c = (c_t)_{t\in\mathbb{R}_+}$ in $\mathcal{B}$, in absence of a strong topology on $\mathcal{B}$ or a predual $\mathcal{B}_*$, the only obvious topology apart from the norm topology is the weak topology. However, if $t \mapsto c_t$ is weakly continuous, then also the semigroup $b \mapsto bc_t$ of operators on $\mathcal{B}$ is weakly, hence, strongly continuous. However, strong continuity for that semigroup means that, in particular for $b = \mathbf{1}$, the map $t \mapsto c_t$ is norm continuous.

3.1 Remark. This is not a contradiction to the existence weakly continuous unitary groups on a Hilbert space H. Here weakly continuous refers to the weak operator topology of $\mathcal{B}(H)$ that is much weaker than the weak topology of $\mathcal{B}(H)$.

3.2 Definition. A strongly continuous CP-semigroup is ***spatial***, if it admits a continuous unit.

To prove the following theorem, we need to recall a few definitions and facts from [BBLS04, Ske03]. A ***kernel*** on a set S with values in the set $\mathcal{B}(\mathcal{A},\mathcal{B})$ of bounded mappings from $\mathcal{A}$ to $\mathcal{B}$ is just a map $\mathfrak{K}\colon S\times S\to\mathcal{B}(\mathcal{A},\mathcal{B})$. The kernel $\mathfrak{K}$ is ***completely positive definite*** (or CPD), if

$$\sum_{i,j} b_i^*\mathfrak{K}^{s_i,s_j}(a_i^*a_j)b_j \;\geq\; 0$$

for all choices of finitely many $s_i\in S, a_i\in\mathcal{A}, b_i\in\mathcal{B}$. If $\mathcal{A}=\mathcal{B}$, we say $\mathfrak{K}$ is a kernel ***on*** $\mathcal{B}$. A ***CPD-semigroup*** on $\mathcal{B}$ is a family $\mathfrak{T}=(\mathfrak{T}_t)_{t\in\mathbb{R}_+}$ of CPD-kernels on $\mathcal{B}$, such that for all $s,s'\in S$ the maps $\mathfrak{T}_t^{s,s'}$ form semigroups on $\mathcal{B}$. The CPD-semigroup is ***continuous*** in a certain topology, if every semigroup $\mathfrak{T}_t^{s,s'}$ is continuous in that topology. By [BBLS04, Theorem 4.3.5] for every CPD-semigroup on a unital C^*–algebra $\mathcal{B}$ there is a product system $E^\odot$ and a family $(\xi^{s\odot})_{s\in S}$ of units for $E^\odot$, such that

$$\mathfrak{T}_t^{s,s'} \;=\; \langle\xi_t^s,\bullet\xi_t^{s'}\rangle$$

for all $s,s'\in S$. The subsystem of $E^\odot$ generated by all these units is unique in an obvious way, and we refer to it as the ***GNS-system*** of $\mathfrak{T}$. If $\mathfrak{T}$ is strongly continuous, then the GNS-system of $\mathfrak{T}$ is ***continuous*** in the sense of Skeide [Ske03]. We do not repeat the complete definition, but recall only what is relevant to us. Roughly speaking, a continuous product system has enough continuous sections $(x_t)_{t\in\mathbb{R}_+},(y_t)_{t\in\mathbb{R}_+}$ for which the map $t\mapsto\langle x_t,\bullet y_t\rangle$ is strongly continuous (and which behave "nicely" under tensor product). The fact that $E^\odot$ is generated by the units of a strongly continuous CPD-semigroup is enough to assure continuity of $E^\odot$. (This follows by putting together [Ske03, Theorem 7.5] and its improvement Skeide [Ske07, Theorem 3.3].)

3.3 Definition. A continuous product system is ***spatial***, if among its continuous sections there is a unital central unit.

3.4 Theorem. *A strongly continuous CP-semigroup is spatial, if and only if its GNS-system can be embedded into a continuous spatial product system.*

Proof. "$\Longleftarrow$." Let T be a strongly continuous CP-semigroup. Suppose $E^\odot$ is a continuous product system that contains a unit $\xi^\odot$ such that $T_t=\langle\xi_t,\bullet\xi_t\rangle$ and a continuous central unital unit $\omega^\odot$. Then $c_t:=\langle\omega_t,\xi_t\rangle$ is a semigroup of elements in $\mathcal{B}$; see [BBLS04, Section 5.1]. As $t\mapsto\langle\omega_t,\bullet\xi_t\rangle$ is strongly continuous, $t\mapsto c_t$ is uniformly continuous. Define the bilinear projection $q_t:=\mathsf{id}_t-\omega_t\omega_t^*\in\mathcal{B}^{a,bil}(E_t)$. By

$$T_t(b)-c_t^*bc_t \;=\; \langle\xi_t,b\xi_t\rangle-\langle\xi_t,\omega_t\rangle b\langle\omega_t,\xi_t\rangle \;=\; \langle\xi_t,q_tb\xi_t\rangle \;=\; \langle(q_t\xi_t),b(q_t\xi_t)\rangle,$$

we see that $T_t - c_t^* \bullet c_t$ is completely positive for all $t \in \mathbb{R}_+$.

"$\Longrightarrow$." Let $c = (c_t)_{t\in\mathbb{R}_+}$ be a unit for the strongly continuous CP-semigroup T. Then the strongly continuous semigroup $\mathfrak{T}_t$ of kernels on $\{0,1\}$ with values in $\mathcal{B}(\mathcal{B})$ defined by setting

$$\begin{pmatrix} \mathfrak{T}_t^{0,0} & \mathfrak{T}_t^{0,1} \\ \mathfrak{T}_t^{1,0} & \mathfrak{T}_t^{1,1} \end{pmatrix} := \begin{pmatrix} \mathsf{id}_{\mathcal{B}} & c_t^*\bullet \\ \bullet c_t & T_t \end{pmatrix} := \begin{pmatrix} 0 & 0 \\ 0 & T_t - c_t^* \bullet c_t \end{pmatrix} + \begin{pmatrix} \mathsf{id}_{\mathcal{B}} & c_t^*\bullet \\ \bullet c_t & c_t^* \bullet c_t \end{pmatrix}$$

is completely positive definite. (Indeed, c is a unit for T, so the first summand is CPD, and the second is a simple example of what would be called an ***elementary*** CPD-semigroup and, obviously, CPD.) The GNS-system of $\mathfrak{T}$ is, then, a spatial continuous product system with unital central unit $\xi^{0\odot}$ containing the GNS-system of T_t as the subsystem generated by the unit $\xi^{1\odot}$. ■

3.5 Remark. Theorem 2.1 tells us that we may **not** replace Definition 3.3 with the property that T has a spatial product system. Theorem 3.4 tells us that we **may** replace Definition 3.3 with the property that the product system of T imbeds into a spatial product system. The clarification of these facts was the main scope of these notes.

By Skeide [Ske03, Theorem 7.7], a continuous product system that has a ***uniformly continuous*** unit $\xi^\odot$ (that is, the CP-semigroup $\langle \xi_t, \bullet\xi_t\rangle$ is uniformly continuous), is ***uniformly continuous*** (that is, for all continuous sections $(x_t)_{t\in\mathbb{R}_+}, (y_t)_{t\in\mathbb{R}_+}$ of $E^\odot$ the map $t \mapsto \langle x_t, \bullet y_t\rangle$ is uniformly continuous). In particular, all continuous units are uniformly continuous.

3.6 Corollary. *Every spatial strongly continuous CP-semigroup is uniformly continuous. The other way round, every strongly continuous CP-semigroup that is not uniformly continuous, is nonspatial, too.*

Proof. The central unital unit $\omega^\odot$ induces the trivial CP-semigroup which is uniformly continuous. ■

The opposite statement need not be true. In fact, the generator of [BBLS04, Example 4.2.4] has not Christensen-Evans form. By the following sharp version of the corollary, this means it is a counter example.

3.7 Corollary. *A strongly continuous CP-semigroup is spatial, if and only it has a Christensen-Evans generator.*

Proof. This summarizes some of the discussions in the introduction, Corollary 3.6, and Theorem 3.4. If the semigroup is spatial, then by Corollary 3.6 it is uniformly continuous. Therefore, by [BBLS04, Lemma 5.1.1] it has Christensen-Evans generator. On the other hand, by [BBLS04, Corollary 5.1.3], the product system of a CP-semigroup with Christensen-Evans generator embeds into a time ordered product system, so that, by Theorem 3.4, it is spatial. ■

3.8 Remark. Following Definition 3.2, both Corollaries are intrinsic statements about strongly continuous CP-semigroups. However, we think it would not be easily possible to prove these statements without reference to product systems. These results continue a whole series of intrinsic statements about CP- or CPD-semigroups that have comparably simple proofs in terms of their GNS-system; see also Liebscher and Skeide [LS08, Remark 3.6].

3.9 Remark. In Corollary 3.6 we have seen that spatial strongly continuous CP-semigroups on a unital C^*–algebra $\mathcal{B}$ are uniformly continuous. This is so due to the fact that that there is no semigroup $c = (c_t)_{t\in\mathbb{R}_+}$ in $\mathcal{B}$ continous in any of the natural topologies of $\mathcal{B}$, that would not be uniformly continuous. For a von Neumann algebra $\mathcal{B}$, from the beginning, it is not reasonable to consider CP-semigroups that are strongly continuous. A result due to Elliott [Ell00] asserts that such a semigroup would be uniformly continuous. But in weaker topologies where also units c^* need no longer be uniformly continuous, there will be much richer classes of spatial CP-semigroups. Actually, practically all known explicit examples of CP-semigroup on $\mathcal{B}(H)$ are spatial in this sense, when continuity is with respect to the strong operator topology. We do not state the obvious modification of Definition 3.2 to the strongly (operator) continuous case, because a theory of strongly continuous product systems of von Neumann correspondences ([Ske08]) is still under development.

References

[Arv89] W. Arveson, *Continuous analogues of Fock space*, Mem. Amer. Math. Soc., no. 409, American Mathematical Society, 1989.

[Arv97] ______, *The index of a quantum dynamical semigroup*, J. Funct. Anal. **146** (1997), 557–588.

[Arv99] ______, *On the index and dilations of completely positive semigroups*, Int. J. Math. **10** (1999), 791–823.

[BBLS04] S.D. Barreto, B.V.R. Bhat, V. Liebscher, and M. Skeide, *Type I product systems of Hilbert modules*, J. Funct. Anal. **212** (2004), 121–181, (Preprint, Cottbus 2001).

[Bha01] B.V.R. Bhat, *Cocycles of CCR-flows*, Mem. Amer. Math. Soc., no. 709, American Mathematical Society, 2001.

[BR87] O. Bratteli and D.W. Robinson, *Operator algebras and quantum statistical mechanics I (2nd ed.)*, Texts and Monographs in Physics, Springer, 1987.

[BS00] B.V.R. Bhat and M. Skeide, *Tensor product systems of Hilbert modules and dilations of completely positive semigroups*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **3** (2000), 519–575, (Rome, Volterra-Preprint 1999/0370).

[CE79] E. Christensen and D.E. Evans, *Cohomology of operator algebras and quantum dynamical semigroups*, J. London Math. Soc. **20** (1979), 358–368.

[Ell00] G.A. Elliott, *On the convergence of a sequence of completely positive maps to the identity*, J. Austral. Math. Soc. Ser. A **68** (2000), 340–348.

[FLS08] F. Fagnola, V. Liebscher, and M. Skeide, *Product systems of Ornstein-Uhlenbeck processes (tentative title)*, Preprint, in preparation, 2008.

[LS01] V. Liebscher and M. Skeide, *Units for the time ordered Fock module*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **4** (2001), 545–551, (Rome, Volterra-Preprint 2000/0411).

[LS08] ______, *Constructing units in product systems*, Proc. Amer. Math. Soc. **136** (2008), 989–997, electronically Nov 2007, (arXiv: math.OA/0510677).

[Pow04] R.T. Powers, *Addition of spatial E_0–semigroups*, Operator algebras, quantization, and noncommutative geometry, Contemporary Mathematics, no. 365, American Mathematical Society, 2004, pp. 281–298.

[Ske03] M. Skeide, *Dilation theory and continuous tensor product systems of Hilbert modules*, Quantum Probability and Infinite Dimensional Analysis (W. Freudenberg, ed.), Quantum Probability and White Noise Analysis, no. XV, World Scientific, 2003, Preprint, Cottbus 2001, pp. 215–242.

[Ske05] ______, *Lévy processes and tensor product systems of Hilbert modules*, Quantum Probability and Infinite Dimensional Analysis — From Foundations to Applications (M. Schürmann and U. Franz, eds.), Quantum Probability and White Noise Analysis, no. XVIII, World Scientific, 2005, pp. 492–503.

[Ske06] ______, *The index of (white) noises and their product systems*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **9** (2006), 617–655, (Rome, Volterra-Preprint 2001/0458, arXiv: math.OA/0601228).

[Ske07] ______, *E_0–semigroups for continuous product systems*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **10** (2007), 381–395, (arXiv: math.OA/0607132).

[Ske08] ______, *Dilations of product sytems and commutants of von Neumann modules*, Preprint, in preparation, 2008.

B.V. Rajarama Bhat: *Statistics and Mathematics Unit, Indian Statistical Institute Bangalore, R. V. College Post, Bangalore 560059, India*, E-mail: `bhat@isibang.ac.in`,
Homepage: `http://www.isibang.ac.in/Smubang/BHAT/`

Volkmar Liebscher: *Institut für Mathematik und Informatik, Ernst-Moritz-Arndt-Universität Greifswald, 17487 Greifswald, Germany*,
E-mail: `volkmar.liebscher@uni-greifswald.de`,
Homepage: `http://www.math-inf.uni-greifswald.de/biomathematik/liebscher/`

Michael Skeide: *Dipartimento S.E.G.e S., Università degli Studi del Molise, Via de Sanctis, 86100 Campobasso, Italy*, E-mail: `skeide@math.tu-cottbus.de`,
Homepage: `http://www.math.tu-cottbus.de/INSTITUT/lswas/_skeide.html`