Tensor Product Systems of CP-Semigroups on $\mathbb{C}^2$

Michael Skeide*

Lehrstuhl fr Wahrscheinlichkeitstheorie und Statistik
Brandenburgische Technische Universitt Cottbus
Postfach 10 13 44, D–03013 Cottbus, Germany
E-mail: skeide@math.tu-cottbus.de
Homepage: http://www.math.tu-cottbus.de/INSTITUT/lswas/_skeide.html

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Abstract

Tensor product systems of Hilbert modules and units for them play a crucial role in the construction of weak Markov flows for CP-semigroup T on a C^*–algebra $\mathcal{B}$. In fact, with each product system with a unit we associate a CP-semigroup and, conversely, with a CP-semigroup we associate a product system with a unit. This correspondence is unique, if we require the unit to generate the product system in a suitable sense.

The most important product systems are those associated with a time ordered Fock module, which play a similar role as symmetric Fock spaces in Arveson's theory of product systems of Hilbert spaces. We know, for instance, that the product systems of uniformly continuous semigroups is always contained in a product system associated with a time ordered Fock module.

In these notes we give a description of all generating units arising from CP-semigroups on $\mathbb{C}^2$, the diagonal subalgebra of the 2×2–matrices. In particular, we show that these units always generate the whole of a suitably chosen time ordered product system. Many of the statements remain true also for more general C^*–algebras. *En passant* we obtain a simple proof of the result by Parthasarathy ans Sunder that the exponential vectors to indicator functions are total in the symmetric Fock space $\Gamma(L^2(\mathbb{R}_+))$.

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1 Introduction

Product systems of Hilbert spaces (or short Arveson systems) are introduced in Arveson [Arv89]. They arise naturally in the study of E_0–semigroups (i.e. semigroups of unital endomorphisms) of $\mathcal{B}(H)$ where H is some Hilbert space. An Arveson system can be thought of as a family $\left(\mathfrak{H}_t\right)_{t>0}$ of Hilbert spaces which compose as

$$\mathfrak{H}_s \otimes \mathfrak{H}_t = \mathfrak{H}_{s+t}$$

in an associative way. With any E_0–semigroup on $\mathcal{B}(H)$ one can associate an Arveson system and E_0–semigroups are classified by their Arveson systems up to cocycle conjugacy. (Actually, in [Arv89] all $\mathfrak{H}_t$ are infinite dimensional and separable and the whole product system is endowed with a topology fulfilling some condition. Under these conditions Arveson [Arv90] shows a one-to-one correspondence between Arveson systems and cocycle conjugacy classes of strongly continuous normal E_0–semigroups.)

Let $\mathcal{B}$ be an arbitray unital C^*–algebra. In Bhat-Skeide [BS00] tensor product systems of Hilbert $\mathcal{B}$–$\mathcal{B}$–modules (short product systems) arise in the study of completely positive (CP-) semigroups on $\mathcal{B}$. (Recall that a Hilbert $\mathcal{B}$–module is a right $\mathcal{B}$–module with a sesquilinear $\mathcal{B}$–valued inner product, which is positive and right $\mathcal{B}$–linear in its second variable, and which is complete in the norm $\|x\| = \sqrt{\|\langle x, x\rangle\|}$. A Hilbert $\mathcal{B}$–$\mathcal{B}$–module is a Hilbert $\mathcal{B}$–module with a unital $*$–representation of $\mathcal{B}$ by bounded right module homomorphisms.) More precisely, with any CP-semigroup T on $\mathcal{B}$ we associate a family $E^\odot = \left(E_t\right)_{t\geq 0}$ of Hilbert $\mathcal{B}$–$\mathcal{B}$–modules such that

$$E_s \odot E_t = E_{s+t}$$

in an associative way (and $E_0 = \mathcal{B}$). (Recall that the inner product on a tensor product of Hilbert modules is defined, by setting $\langle x \odot y, x' \odot y'\rangle = \langle y, \langle x, x'\rangle y'\rangle$.) Also here we have that unital CP-semigroups on $\mathcal{B}$ are classified by their product systems up to a certain cocycle conjugacy class. Moreover, if the CP-semigroup is an E_0–semigroup, then, like for Arveson systems, the cocycles are unitary. Hence, we do not only recover Arveson's classification of E_0–semigroups on $\mathcal{B}(H)$, but find a generalization to E_0–semigroups on an arbitrary unital C^*–algebra $\mathcal{B}$; see [BS00] for details.

In both theories the most important objects are units . A *unit* is a family $\xi^\odot = \left(\xi_t\right)_{t\geq 0}$ of vectors $\xi_t \in E_t$ (or $\mathfrak{H}_t$) such that

$$\xi_s \odot \xi_t = \xi_{s+t}$$

(or $\xi_s \otimes \xi_t = \xi_{s+t}$). Arveson systems are classified by their supply of units. A type I Arveson system has sufficiently many units in the sense that $\mathfrak{H}_t$ is spanned by tensor products of (possibly different) units to smaller times. An Arveson system which has units, but is not of type I, is type II. An Arveson system without units is type III. Let K be a (not necessarily separable) Hilbert space, and denote by Γ_t the symmetric Fock space $\Gamma(L^2([0,t],K))$. Then $\left(\Gamma_t\right)$ is a product system via the isomorphism

$$\Gamma_s \otimes \Gamma_t \;\cong\; \Gamma(L^2([t,t+s],K)) \otimes \Gamma_t \;\cong\; \Gamma_{s+t}.$$

All type I Arveson systems are of the form $\left(\Gamma_t\right)$ for some separable K and the units are just all rescaled exponential vectors $\xi_t = e^{ct}\psi(\chi_{[0,t]}f)$ $(c \in \mathbb{C}, f \in K)$. The dimension of K (or rather K itself) for the type I Arveson subsystem of an arbitrary Arveson system is called its index; see [Arv89]. Although we know that there exist E_0–semigroups whose associated Arveson system is not of type I (see [Pow87]), until now nobody has seen yet a concrete example.

It seems natural to try the construction of a symmetric Fock module $\Gamma(L^2([0,t],F))$, where F is a Hilbert $\mathcal{B}$–$\mathcal{B}$–module and the one-particle sector $L^2([0,t],F)$ is the norm closure of the step functions on $[0,t]$ with values in F. However, it turns out that such a construction is not possible without further conditions on F; see [Ske98, AS00] for examples. On the other hand, it is well-known that the symmetric Fock space Γ_t is isomorphic to the time ordered Fock space $\mathfrak{F}^0_t$ whose n–particle sector consists of square integrable functions on $[0,t]^n$ with values in the full tensor product $K^{\otimes n}$ being zero on $(t_n,\ldots,t_1) \in [0,t]^n$, unless $t \geq t_n \geq \ldots \geq t_1 \geq 0$; see [Sch93, Bha98] for details. Here the analogue construction for Hilbert modules is possible without problems. Moreover, we know from [BS00] that the time ordered Fock modules $\mathfrak{F}^0_t$ over the one-particle sector $L^2([0,t],F)$ form a product system via the isomorphism

$$[F_s \odot G_t](s_m,\ldots,s_1,t_n,\ldots,t_1) = F_s(s_m - t,\ldots,s_1 - t) \odot G_t(t_n,\ldots,t_1)$$

where F_s is in the m–particle sector of $\mathfrak{F}^0_s$ and G_t is in the n–particle sector of $\mathfrak{F}^0_t$. Also here we have for each $x \in L^2(\mathbb{R}_+,F)$ an exponential vector $\psi(x)$ defined, by setting the component in the n–particle sector

$$\psi(x)^n(t_n,\ldots,t_1) = x(t_n) \odot \ldots \odot x(t_1) \qquad (t_n \geq \ldots \geq t_1 \geq 0).$$

Of particular interest are the exponential vectors $\psi_t(\zeta) := \psi(\chi_{[0,t]}\zeta)$ $(\zeta \in F)$. Together with their time translates they generate already the whole time ordered Fock module. Observe that under the above isomorphism $\psi_s(\zeta) \odot \psi_t(\zeta') = \psi(\chi_{[t,t+s]}\zeta + \chi_{[0,t]}\zeta')$, whereas $\psi_t(\zeta') \odot \psi_s(\zeta) = \psi(\chi_{[s,s+t]}\zeta' + \chi_{[0,s]}\zeta)$. These vectors, in general, have different lengths.

Like in the theory of Arveson systems, the time ordered Fock module plays a distinguished role. In [BS00] it is shown that the product system associated with an arbitrary unital uniformly continuous CP-semigroup on $\mathcal{B}$ is contained as a product subsystem of a product system of time ordered Fock modules. Given a unit $\xi^\odot$ for a product system $E^\odot$, it follows directly from the definition of the inner product of the tensor product that the mappings $T_t(b) = \langle \xi_t, b\xi_t \rangle$ define a CP-semigroup on $\mathcal{B}$. The product system $E^\odot$ is that (unique up to isomorphism) associated with the CP-semigroup T, if and only if the unit is *generating*, i.e. if the vectors

$$b_n \xi_{t_n} \odot \ldots \odot b_1 \xi_{t_1} b_0 \qquad (t_n + \ldots + t_1 = t)$$

form a total subset of E_t. Depending on the topology, there are different notions of a generating unit. Here we refer to the $\mathcal{B}$–*weak* topology wich is determined by the family $\|\langle x, \bullet \rangle\|$ of seminorms and, therefore, the module analogue of the weak topology on a Hilbert space.

In [LS01] it is pointed out that a quite large class of units for the time ordered Fock module can be described as follows. Let ξ_t^0 be a semigroup in $\mathcal{B}$ and let ζ be an element in F. Then the elements $\xi_t \in \mathcal{F}_t^0$ whose component in the 0–particle sector is ξ_t^0 and whose component ξ_t^n in the n–particle sector $(n \geq 1)$ is

$$\xi_t^n(t_n, \ldots, t_1) = \xi_{t-t_n}^0 \zeta \odot \xi_{t_n - t_{n-1}}^0 \zeta \odot \ldots \odot \xi_{t_2 - t_1}^0 \zeta \xi_{t_1}^0 \tag{1.1}$$

form a unit for $\left(\mathcal{F}_t^0\right)$. This class of units includes also CP-semigroups with unbounded generators, hence, showing that the time ordered Fock modules allows also for the description of quite a lot CP-semigroups which are only strongly continuous.

The units described by (1.1) contain two important subclasses. The first is where $\zeta = 0$. These units have only the component ξ^0 in the 0–particle sector. In fact, in order to show that a given unit is generating, it is the crucial step to show that this unit generates the vacuum unit (which corresponds to $\zeta = 0$ and $\xi_t^0 = \mathsf{id}$). The second class of unit is where ζ is arbitrary, but $\xi_t^0 = \mathsf{id}$. These are precisely the exponential vectors $\psi_t(\zeta)$.

If there is a unital unit (i.e. $\langle \xi_t, \xi_t \rangle = \mathbf{1}$ so that also T is unital), then there are other related structures arising from a product system. We only mention that for all $t \geq s$ the mapping $x_s \mapsto \xi_{t-s} \odot x_s$ is an isometric (i.e. inner product preserving) mapping $E_s \to E_t$. The family of all these mappings is an inductive system so that we may construct the inductive limit E over all E_t. On the C^*–algebra $\mathcal{B}^a(E)$ of all adjointable operators on E, we have a natural E_0–semigroup ϑ and a natural conditional expectation φ onto $\mathcal{B}$. The pair (ϑ, φ) is a dilation of T, i.e. $\varphi \circ \vartheta_t = T_t$, and the restrictions j_t of ϑ_t to $\mathcal{B}$ form a weak Markov flow in the sense of [BP94], i.e. $j_t(\mathbf{1}) j_{s+t}(b) j_t(\mathbf{1}) = j_t \circ T_s(b)$. We refer the reader to [BS00] for details.

In the remainder of these notes we investigate in detail the case $\mathcal{B} = \mathbb{C}^2$. We give an explicit form of all units obtain by (1.1). We compute explicitly all unital continuous

CP-semigroups on $\mathbb{C}^2$. Finally, we show that the product systems associated with such semigroups are always product systems of time ordered Fock modules.

2 CP-semigroups on $\mathbb{C}^2$

In this section we study in detail how the unital CP-semigroups on the diagonal subalgebra of M_2 and the associated time ordered Fock modules look like. The diagonal subalgebra is the unique unital 2–dimensional $*$–algebra. We find it convenient to identify it with the vector space $\mathbb{C}^2$ (equipped with componentwise multiplication and conjugation), rather than the diagonal matrices. In addtion to the canonical basis $e_1 = \binom{1}{0}$, $e_2 = \binom{0}{U}$ we will also use the basis $e_+ = \mathbf{1} = \binom{1}{U}$, $e_- = \binom{1}{-1}$. In the first basis it is easy to say when an element of $\mathbb{C}^2$ is positive (namely, if and only if both coordinates are positive), whereas in the second basis unital mappings have a particularly simple (triangular) matrix representation.

Let us start with an arbitrary Hilbert $\mathbb{C}^2$–$\mathbb{C}^2$–module F. Choose $\zeta \in F$, $\beta \in \mathbb{C}^2$ and consider the unit $\xi^\odot$ as defined by (1.1) with the semigroup $\xi_t^0 = e^{t\beta}$. We say $\xi^\odot$ is the unit with parameters ζ, β. In [LS01] it is shown that the CP-semigroup $T_t(b) = \langle \xi_t, b\xi_t \rangle$ has the generator $\mathcal{L}_\xi(b) = \langle \zeta, b\zeta \rangle + \beta^* b + b\beta$. As usual, the form of this generator is not determined uniquely by T. On the one hand, as $\mathbb{C}^2$ is commutative, only the sum $\beta^* + \beta$ contributes so that the imaginary part of β is arbitrary. On the other hand, a positive part in $\beta^* + \beta$ can easily be included into the inner product, by adding a direct summand $\mathbb{C}^2$ (i.e. the simplest Hilbert $\mathbb{C}^2$–$\mathbb{C}^2$–module possible) to ζ. However, we know from [BS00] that the two product subsystems generated by two such units are isomorphic.

If we have two units $\xi^\odot, \xi'^\odot$ (with parameters $\zeta, \beta, \zeta', \beta'$), then also $\langle \xi_t, b\xi'_t \rangle$ is semigroup (of course, in general, not a CP-semigroup). Like in [LS01] one shows that this semigroup has a generator which is of the form $\mathcal{L}_{\xi,\xi'}(b) = \langle \zeta, b\zeta' \rangle + \beta^* b + b\beta'$. It is not difficult to see that this can be the most general linear operator on $\mathbb{C}^2$. It is possible to give the explicit form of the semigroup $e^{t\mathcal{L}}$, in general, because $\mathcal{L}$ as an operator on $\mathbb{C}^2$ is similar either to a matrix $\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ or to a matrix $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$ whose exponentials can easily be computed. We do not work this out, because we are interested rather in the minimal tensor product system generated by a single unital unit $\xi^\odot$ with parameters ζ, β. Here computations are much more handy.

Since we are interested in what is generated by a certain unit, we assume that F is generated by ζ. Then F decomposes into the submodules $\mathbb{C}e_i\zeta e_j$ $(i, j = 1, 2)$ some of which may be $\{0\}$. If all four spaces are non-trivial, than $\mathbb{C}e_1\zeta e_1 \oplus \mathbb{C}e_2\zeta e_2$ is isomorphic to the right Hilbert module $\mathbb{C}^2$ with natural left multiplication, whereas $\mathbb{C}e_1\zeta e_2 \oplus \mathbb{C}e_2\zeta e_1$ is isomorphic to the right Hilbert module $\mathbb{C}^2$ where, however, $\binom{z_1}{z_2}$ acts from the left via multiplication by $\binom{z_1}{z_2}_- := \binom{z_2}{z_1}$. We denote the former $\mathbb{C}^2$–$\mathbb{C}^2$–module by $\mathbb{C}^2_+$ and the latter by $\mathbb{C}^2_-$. If some of the spaces $\mathbb{C}e_i\zeta e_j$ are trivial, then F is at least contained in $\mathbb{C}^2_+ \oplus \mathbb{C}^2_-$.

The discussion about ambiguity in the choice of the parameters shows that we may include the component ζ_+ of ζ in $\mathbb{C}^2_+$ into β by adding $\frac{1}{2}\langle\zeta_+,\zeta_+\rangle$ to β without changing the semigroup T. We may, therefore, assume that $F = \mathbb{C}^2_-$. Observe that this choice corresponds to say that the completely positive part $\langle\zeta, \bullet\zeta\rangle$ of $\mathcal{L}$ is the smallest possible.

We have $\mathbb{C}^2_- \odot \mathbb{C}^2_- = \mathbb{C}^2_+$ where the canonical isomorphism is $b\odot b' \mapsto b_-b'$ and, of course, $\mathbb{C}^2_+ \odot E = E = E \odot \mathbb{C}^2_+$ for all Hilbert $\mathbb{C}^2$–$\mathbb{C}^2$–modules E. Therefore, $\mathcal{F}^0_t(\mathbb{C}^2_-) = \mathcal{F}^0_t(\mathbb{C}) \otimes \mathbb{C}^2$ as Hilbert $\mathbb{C}^2$–module. However, the left multiplication is that of $\mathbb{C}^2_+$ on $2n$–particle sectors and that of $\mathbb{C}^2_-$ on $2n+1$–particle sectors. Although $\mathcal{F}^0_t(\mathbb{C})$ is isomorphic (even as Arveson system) to $\Gamma_t(L^2(\mathbb{R}_+))$, the module structure of $\mathcal{F}^0_t(\mathbb{C}^2_-)$ is very much different from that of the symmetric Fock module $\Gamma_t(L^2(\mathbb{R}_+)) \otimes \mathbb{C}^2$ where the left multiplication is that of $\mathbb{C}^2_+$ on all n–particle sectors; see [Ske98].

Now it is very easy to write down the units for $\bigl(\mathcal{F}^0_t(\mathbb{C}^2_-)\bigr)$ explicitly.

2.1 Theorem. *Let $\xi^\odot$ be the unit for $\bigl(\mathcal{F}^0_t(\mathbb{C}^2_-)\bigr)$ with parameters $\zeta = \binom{\zeta_1}{\zeta_2}, \beta$ and set $|\zeta| = \zeta_1\zeta_2$. Then*

$$\begin{aligned} \xi^{2n}(t_{2n},\ldots,t_1) &= |\zeta|^n\, e^{t\beta} e^{(t_{2n}-t_{2n-1}+\ldots+t_2-t_1)(\beta_- - \beta)} \\ \xi^{2n+1}(t_{2n+1},\ldots,t_1) &= \zeta\, |\zeta|^n\, e^{t\beta_-} e^{(t_{2n+1}-t_{2n}+\ldots+t_1)(\beta-\beta_-)}. \end{aligned}$$

PROOF. This follows from (1.1) by making use of $b\zeta = \zeta b_-$ and $\zeta \odot \zeta = \zeta_-\zeta = |\zeta|\,\mathbf{1}$. ■

2.2 Remark. The corresponding unit for $\bigl(\mathcal{F}^0_t(\mathbb{C}^2_+)\bigr)$ would be given by $\xi^n(t_n,\ldots,t_1) = \zeta^n e^{t\beta}$. In other words, we obtain just the exponential vectors $\psi_t(\zeta) = \binom{\psi_t(\zeta_1)}{\psi_t(\zeta_2)}$ rescaled by $e^{t\beta}$. Like a single unit in an Arveson system is never generating, we find also here that none of the units is generating. Moreover, with units in this time ordered Fock module we recover only CP-semigroups of the form $T_t(b) = be^{tc}$ for some self-adjoint element $c \in \mathcal{B}$. In particular, the only unital CP-semigroup in this class is the trivial one.

Let us return to $\mathcal{F}^0_t(\mathbb{C}^2_-)$ and see which unital CP-semigroups are generated by which unital unit. Recall that $\xi^\odot$ is unital, if and only if $\beta + \beta^* = -\langle\zeta,\zeta\rangle$ and the the imaginary part of β does not influence the CP-semigroup.

Let T be a unital mapping. In the basis e_+, e_- it has the matrix representation

$$\widehat{T} = \begin{pmatrix} 1 & p \\ 0 & q \end{pmatrix}.$$

From $\binom{z_1}{z_2} = \frac{z_1+z_2}{2}\binom{1}{U} + \frac{z_1-z_2}{2}\binom{1}{-1}$ we find

$$T\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{z_1+z_2}{2}\begin{pmatrix} 1 \\ U \end{pmatrix} + \frac{z_1-z_2}{2}\left[p\begin{pmatrix} 1 \\ U \end{pmatrix} + q\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$
$$= \frac{z_1}{2}\begin{pmatrix} 1+p+q \\ U+p-q \end{pmatrix} + \frac{z_2}{2}\begin{pmatrix} 1-p-q \\ U-p+q \end{pmatrix}.$$

Hence, T is positive (which is the same as completely positive, as $\mathbb{C}^2$ is commutative), if and only if $\binom{p}{q}$ is in the square (including borders) in the $\mathbb{R}^2$–plane with corner points $(1,0), (0,1), (-1,0), (0,-1)$.

Now let T_t be a family of mappings on $\mathbb{C}^2$ having matrices $\widehat{T_t} = \begin{pmatrix} 1 & p_t \\ 0 & q_t \end{pmatrix}$ with respect to the basis e_+, e_-. In order that $T = \big(T_t\big)$ be a semigroup, p_t and q_t must solve the functional equations $p_t + p_s q_t = p_{s+t}$ and $q_s q_t = q_{s+t}$. Requiring that T_t be continuous implies, as usual, differentialbility of p_t and q_t. Using this, we find $q_t = e^{-ct}$ and $p_t = \alpha(1-e^{-ct})$ with complex constants c and α. In order that T_t be positive we find $c \geq 0$ and $-1 \leq \alpha \leq 1$. These conditions are necessary and sufficient. The corresponding CP-semigroup is

$$T_t\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{z_1+z_2}{2}\begin{pmatrix} 1 \\ U \end{pmatrix} + \frac{z_1-z_2}{2}\left[\alpha(1-e^{-ct})\begin{pmatrix} 1 \\ U \end{pmatrix} + e^{-ct}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right].$$

The generator is

$$\mathcal{L}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{z_1-z_2}{2}c\begin{pmatrix} \alpha-1 \\ \alpha+1 \end{pmatrix}. \tag{2.1}$$

On the other hand, the generator of the CP-semigroup generated by the unital unit $\xi^{\odot}$ is

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = b \longmapsto \langle\zeta, b\zeta\rangle - \langle\zeta,\zeta\rangle b = \langle\zeta,\zeta\rangle(b_- - b) = (z_1 - z_2)\begin{pmatrix} -\left|\zeta_1\right|^2 \\ \left|\zeta_2\right|^2 \end{pmatrix}.$$

Equating this to (2.1), we find $c = |\zeta_1|^2 + |\zeta_2|^2$ and $\alpha = \frac{|\zeta_2|^2 - |\zeta_1|^2}{|\zeta_1|^2 + |\zeta_2|^2}$.

Of course, $c = 0$ (i.e. $\zeta = 0$) yields the trivial CP-semigroup independently of α. Different choices for $c > 0$ correspond to a time scaling. Here we have to distinguish the two essentially different cases where $|\zeta| = 0$ (i.e. $\alpha = \pm 1$) and where $|\zeta| \neq 0$ (i.e. $|\alpha| < 1$).

In the case $|\zeta| = 0$ only the components ξ^0 and ξ^1 of the unit $\xi^{\odot}$ are different from 0. The case $\alpha = -1$ is analyzed in detail in [BS00] and the case $\alpha = 1$ follows from this, because, in general, a sign change of α just corresponds to flip of the components in $\mathbb{C}^2$. Since $\zeta \odot \zeta = 0$ in this case, we find that the time ordered Fock module over the (one-dimensional) module $\mathbb{C}^2\zeta\mathbb{C}^2 = \mathbb{C}\zeta$ consists only of its 0– and 1–particle sector and, indeed, is generated by the unit. The CP-semigroup maybe considered as the *unitization* of a contractive CP-semigroup on $\mathbb{C}$. Such a truncated Fock module is related to boolean independence [Wal73]; see [Ske00] for details.

Now we come to the case $|\zeta| \neq 0$, whence ζ is invertible. It is our goal to show that also in this case any unit is generating for $\big(\mathcal{F}^0_t(\mathbb{C}^2_-)\big)$ at least in the $\mathbb{C}^2$–weak topology. In this way we show that the product system associated with the corresponding CP-semigroup is dense in $\big(\mathcal{F}^0_t(\mathbb{C}^2_-)\big)$ in this topology.

Our strategy is inspired very much by [Arv89, Section 6]. In [Arv89] the argument is based on the simple properties of semigroups on $\mathbb{C}$. Here we need some basic properties of semigroups on $\mathbb{C}^2$. As the first part of this section shows, these case is already considerably more complicated. We remark that many of the following results are true also for more general C^*–algebras than $\mathbb{C}^2$.

Let us repeat the following well-known result on semigroups.

2.3 Lemma. *Let T, S be two semigroups on a Banach space B with bounded generators $\mathcal{L}, \mathcal{M}$, respectively, and let $0 \leq \varkappa \leq 1$. Then for all $t \geq 0$ the limit*

$$\widehat{TS}_t = \lim_{n\to\infty} \big(T_{\frac{t\varkappa}{n}} S_{\frac{t(1-\varkappa)}{n}}\big)^n$$

exists in norm. Moreover, the mappings $\widehat{TS}_t$ form a semigroup with bounded generator $\varkappa\mathcal{L} + (1-\varkappa)\mathcal{M}$.

2.4 Corollary. *Let $\xi^\odot, \xi'^\odot$ be two units for $\big(\mathcal{F}^0_t(\mathbb{C}^2_-)\big)$ with parameters $\zeta, \beta, \zeta', \beta'$, respectively, and let $0 \leq \varkappa \leq 1$. Then for all $t \geq 0$ the limit*

$$\widehat{\xi\xi'}_t = \lim_{n\to\infty} \big(\xi_{\frac{t\varkappa}{n}} \odot \xi'_{\frac{t(1-\varkappa)}{n}}\big)^{\odot n}$$

exists $\mathbb{C}^2$–weakly in $\mathcal{F}^0_t(\mathbb{C}^2_-)$ and coincides with the unit with parameters

$$\varkappa\zeta + (1-\varkappa)\zeta' \quad \textit{and} \quad \varkappa\beta + (1-\varkappa)\beta'.$$

Proof. Clearly, the sequence is bounded. Therefore, it is sufficient to check convergence on a generating subset of $\mathcal{F}^0_t(\mathbb{C}^2_-)$. As generating subset we choose vectors of the form $b_k\Xi^k_{t_k}\odot\ldots\odot b_1\Xi_{t_1}b_0$ $(t_k+\ldots+t_1 = t)$ where Ξ^ℓ_t are (possibly different) units. Convergence of inner products with such elements follows by careful applications of Lemma 2.3. Moreover, the limits of the inner products coincide with the inner products with the unit having parameters $\varkappa\zeta + (1-\varkappa)\zeta', \varkappa\beta + (1-\varkappa)\beta'$. Therefore, the limit exists in $\mathcal{F}^0_t(\mathbb{C}^2_-)$ and has the stated form. ■

2.5 Lemma. *Let $\xi^\odot$ be a unit for $\big(\mathcal{F}^0_t(\mathbb{C}^2_-)\big)$ with parameters ζ, β and let $\beta' \in \mathcal{B}$. Then for all $t \geq 0$ the limit*

$$\xi'_t = \lim_{n\to\infty} \big(\xi_{\frac{t}{n}} e^{\frac{t\beta'}{n}}\big)^{\odot n}$$

exists $\mathbb{C}^2$–weakly in $\mathcal{F}^0_t(\mathbb{C}^2_-)$ and coincides with the unit with parameters $\zeta, \beta + \beta'$.

PROOF. Precisely, as in Corollary 2.4. ■

In particular, we see that the $\mathbb{C}^2$–weak closure of what is generated by a unit with parameters ζ, β contains the unit with $\beta = 0$ (i.e. the exponential vectors $\psi_t(\zeta)$).

Until here the results are true for time ordered Fock modules over arbitrary two-sided Hilbert modules. Now we refer to the special structure of $\mathbb{C}^2$. Define $P_t \colon \mathcal{F}_t^0(\mathbb{C}^2_-) \to \mathcal{F}_t^0(\mathbb{C}^2_-)$ by setting $P_t x = e_1 x e_1 + e_2 x e_2$. Notice that P_t is the projection onto the direct sum over all $2n$–particle sectors.

2.6 Lemma. *Let $\xi^\odot$ be an arbitrary unit for $\big(\mathcal{F}_t^0(\mathbb{C}^2_-)\big)$. Then*

$$\lim_{n\to\infty} \big(P_{\frac{t}{n}} \xi_{\frac{t}{n}}\big)^{\odot n} = \xi_t^0.$$

PROOF. Let P_t^{01} denote the projection onto the direct sum of the 0– and the 1–particle sector. Then as in the symmetric Fock space, we have $\lim\limits_{n\to\infty} \big(P^{01}_{\frac{t}{n}}\big)^{\odot n} = \mathsf{id}_{\mathcal{F}_t^0(\mathbb{C}^2_-)}$ in the strong topology. Now the result follows from the fact that $P_t P_t^{01}$ is the projection onto the vaccum. ■

Applying this to an exponential vector and taking into account also Lemma 2.5 we find that any unit with parameters ζ, β generates besides the exponential vectors $\psi_t(\zeta)$ also the vacuum. Taking the vacuum as $\xi'^\odot$ in Corollary 2.4, we find that also the exponential vectors $\psi_t(\varkappa\zeta)$ is generated by $\xi^\odot$. Putting these units piecewise together, we find that $\xi^\odot$ generates all exponential vectors to step functions on $[0, t]$ which take values in the convex combinations of ζ and 0. If $|\zeta| \neq 0$, then, clearly, these form a total subset of $\mathcal{F}_t^0(\mathbb{C}^2_-)$ for each t. Summarizing, we find our main result.

2.7 Theorem. *All units $\xi^\odot$ for $\big(\mathcal{F}_t^0(\mathbb{C}^2_-)\big)$ with parameters ζ, β $(|\zeta| \neq 0)$ are generating.*

2.8 Remark. The preceding discussion gives a new simple proof of the result by Parthasarathy und Sunder [PS98] that exponential vectors in the symmetric Fock space $\Gamma(L^2(0, 1))$ to indicator functions are total. See also Bhat [Bha01].

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