Units for the Time Ordered Fock Module

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Abstract

We give an explicit formula for all continuous units for the product system associated with an arbitrary time ordered Fock module over a two-sided Hilbert $\mathcal{B}$–module. Like in the case of the symmetric Fock space, the units may be considered as exponential vectors to indicator function possibly renormalized by a semigroup. However, since this semigroup takes values in $\mathcal{B}$, the renormalization must be done more carefully. It turns out that the generators of the CP-semigroups on $\mathcal{B}$ associated with our units are precisely the Christensen-Evans generators. Finally, we present an example which shows that the situation changes considerably, when we drop the continuity condition.

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Let $\mathcal{B}$ be a unital C^*–algebra and let $\big(E_t\big)_{t\in\mathbb{R}_+}$ be a family of Hilbert $\mathcal{B}$–$\mathcal{B}$–modules. (Recall that a Hilbert $\mathcal{B}$–module is a right $\mathcal{B}$–module with a sesquilinear $\mathcal{B}$–valued inner product, which is positive and right $\mathcal{B}$–linear in its second variable, and which is complete in the norm $\|x\| = \sqrt{\|\langle x, x\rangle\|}$. A Hilbert $\mathcal{B}$–$\mathcal{B}$–module is a Hilbert $\mathcal{B}$–module with a unital $*$–representation of $\mathcal{B}$ by bounded right module homomorphisms.) We say the E_t form a *product system*, if there are $\mathcal{B}$–$\mathcal{B}$–linear isomorphisms (i.e. unitaries) $u_{st}\colon E_s \odot E_t \to E_{s+t}$ such that the associativity condition $u_{r(s+t)} \circ (\mathsf{id} \odot u_{st}) = u_{(r+s)t} \circ (u_{rs} \odot \mathsf{id})$ is fulfilled, and if $E_0 = \mathcal{B}$ (with the natural identification $E_t \odot \mathcal{B} = E_t = \mathcal{B} \odot E_t$). Usually, we do not refer explicitly to the family $\big(u_{st}\big)$ and write

$$E_s \odot E_t = E_{s+t}.$$

For Hilbert spaces (i.e. Hilbert $\mathbb{C}$–$\mathbb{C}$–modules) the notion of product system (including some additional technical conditions) has been introduced by Arveson [Arv89] in the study of E_0–semigroups (i.e. semigroups of normal, unital endomorphisms of $\mathcal{B}(H)$). Product systems of Hilbert modules arose naturally in [BS00] in the study of CP-semigroups (i.e. semigroups of completely positive mappings) on $\mathcal{B}$.

In both theories the notion of a unit for a product system plays an important role. A *unit* is a family $\xi^\odot = \big(\xi_t\big)_{t\in\mathbb{R}_+}$ of elements $\xi_t \in E_t$ fulfilling

$$\xi_s \odot \xi_t = \xi_{s+t}$$

and $\xi_0 = \mathbf{1}$. Units are connected with CP-semigroups on $\mathcal{B}$ by the observation that

$$\langle \xi_{s+t}, b\xi_{s+t}\rangle = \langle \xi_s \odot \xi_t, b\xi_s \odot \xi_t\rangle = \langle \xi_t, \langle \xi_s, b\xi_s\rangle \xi_t\rangle$$

from which we conclude that $T_t(b) = \langle \xi_t, b\xi_t\rangle$ defines a CP-semigroup. It is one of the main results in [BS00] that any CP-semigroup may be recovered in this way.

In the case of Arveson systems units are used to define the type of a product system and its index; see [Arv89]. For instance, type I Arveson systems are those which admit *sufficiently many* units in the sense that E_t is generated by tensor products of units to smaller times. Any type I Arveson system is isomorphic to a family $\big(\Gamma_t\big)$ of symmetric Fock spaces $\Gamma_t = \Gamma(L^2([0,t], K))$ where K is some Hilbert space, the *index*. As any Arveson system contains a unique maximal type I subsystem (namely, what is generated by its units), we take as its index the index of this type I subsystem. The units of type I Arveson systems are precisely the exponential vectors $\psi(\mathbb{I}_{[0,t]}f)$ $(f \in K)$ times a rescaling e^{ct} $(c \in \mathbb{C})$.

For product systems of Hilbert modules it is not yet clear which is the right notion of type and index. We will not discuss this here. However, it is clear what is the analogue of the symmetric Fock space. The analogue of the one-particle sector is $L^2(\mathbb{R}_+, F)$ where F

is an arbitrary Hilbert $\mathcal{B}$–$\mathcal{B}$–module and L^2 means norm completion of step functions with values in F (with obvious Hilbert $\mathcal{B}$–$\mathcal{B}$–module structure). However, the construction of a symmetric Fock module over $L^2(\mathbb{R}_+, F)$ is not possible without restrictions on F. (See [Ske98, AS00] for examples and discussion of symmetric Fock modules.) It is, however, well-known that the symmetric Fock space is isomorphic to the time ordered Fock space; see [Gui72]. The same is true for the symmetric Fock modules considered in [Ske98, AS00]. The time ordered Fock module has the advantage that it may be constructed for arbitrary F.

1 Definition. Let F be a Hilbert $\mathcal{B}$–$\mathcal{B}$–module. The *full Fock module* over $L^2(\mathbb{R}_+, F)$ is defined as

$$\mathcal{F}(L^2(\mathbb{R}_+, F)) = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+, F)^{\odot n}$$

where $L^2(\mathbb{R}_+, F)^{\odot 0} = \mathcal{B}$. By ω we denote the vacuum, i.e. the $\mathbf{1}$ in $L^2(\mathbb{R}_+, F)^{\odot 0}$. The space $L^2(\mathbb{R}_+, F)^{\odot n}$ may considered as the completion of the space of step functions on $\mathbb{R}_+^n$ with values in $F^{\odot n}$.

Let Δ_n denote the indicator function of the subset $\{(t_n, \ldots, t_1) \colon t_n \geq \ldots \geq t_1 \geq 0\}$ of $\mathbb{R}_+^n$. Then Δ_n acts on the n–particle sector as a projection via pointwise multiplication (and Δ_0 acts as identity on the vacuum). Set $\Delta = \bigoplus\limits_{n=0}^{\infty} \Delta_n$. The *time ordered Fock module* is the two-sided submodule

$$\mathrm{I\!\Gamma}(F) = \Delta \mathcal{F}(L^2(\mathbb{R}_+, F))$$

of $\mathcal{F}(L^2(\mathbb{R}_+, F))$.

By $\mathrm{I\!\Gamma}_t$ we denote those two-sided submodules of $\mathrm{I\!\Gamma}(F)$ where the functions in the n–particle sector are different from 0 only on $[0, t]^n$. Clearly, on $L^2(\mathbb{R}_+, F)$ we have a time shift $\mathfrak{s}_t$ sending $f(s)$ to $f(s-t)\mathbb{I}_{[0,\infty)}(s-t)$. This time shift extends via second quantization to a time shift $\mathcal{F}(\mathfrak{s}_t)$ on $\mathcal{F}(L^2(\mathbb{R}_+, F))$ which leaves $\mathrm{I\!\Gamma}(F)$ invariant.

2 Proposition [BS00]. *The time ordered Fock modules $\mathrm{I\!\Gamma}_t$ are a product system via the isomorphisms u_{st}, defined by setting*

$$\begin{aligned}[u_{st}(F_s \odot G_t)](s_m, \ldots, s_1, t_n, \ldots, t_1) &= [\mathcal{F}(\mathfrak{s}_t)F_s](s_m, \ldots, s_1) \odot G(t_n, \ldots, t_1) \\ &= F_s(s_m - t, \ldots, s_1 - t) \odot G(t_n, \ldots, t_1),\end{aligned}$$

for $s + t \geq s_m \geq \cdots \geq s_1 \geq t \geq t_n \geq \cdots \geq t_1 \geq 0$. We say $\mathrm{I\!\Gamma}^{\odot}(F) = \big(\mathrm{I\!\Gamma}_t\big)_{t \in \mathbb{R}_+}$ is the product system associated *with $\mathrm{I\!\Gamma}(F)$.*

3 Theorem. *Let $\xi^0 = \left(\xi^0_t\right)_{t\in\mathbb{R}_+}$ be a uniformly continuous semigroup in $\mathcal{B}$ and let $\zeta \in F$. Then $\xi^\odot = \left(\xi_t\right)_{t\in\mathbb{R}_+}$ where the component ξ^n_t of $\xi_t \in \mathbb{I}\Gamma_t$ in the n–particle sector is*

$$\xi^n_t(t_n,\ldots,t_1) = \xi^0_{t-t_n}\zeta \odot \xi^0_{t_n-t_{n-1}}\zeta \odot \ldots \odot \xi^0_{t_2-t_1}\zeta\xi^0_{t_1} \tag{1}$$

(and, of course, ξ^0_t for $n=0$), is a unit for $\mathbb{I}\Gamma^\odot(F)$. Moreover, both the function $t \mapsto \xi_t \in \mathbb{I}\Gamma(F)$ and the CP-semigroup T with $T_t = \langle \xi_t, \bullet \xi_t\rangle$ are uniformly continuous. If the generator of ξ^0 is $\beta \in \mathcal{B}$, then the generator of T is

$$\mathcal{L}(b) = \langle \zeta, b\zeta\rangle + b\beta + \beta^* b. \tag{2}$$

Proof. ξ^0_t is bounded by $e^{t\|\beta\|}$ so that

$$\|\xi^n_t\| \leq e^{t\|\beta\|}\sqrt{\frac{t^n\,\|\zeta\|^{2n}}{n!}}. \tag{3}$$

(Recall that the *volume* of the n–simplex $t \geq t_n \geq \ldots \geq t_1 \geq 0$ is $\frac{t^n}{n!}$.) In other words, the components ξ^n_t are summable to a vector $\xi_t \in \mathbb{I}\Gamma_t$ with norm not bigger than $e^{t\left(\|\beta\|+\frac{\|\zeta\|^2}{2}\right)}$.

Of course, $(\xi_s \odot \xi_t)^k = \sum\limits_{\ell=0}^{k} \xi^{k-\ell}_s \odot \xi^\ell_t$. Evaluating at a concrete tuple $(r_k,\ldots,r_1)$, there remains (up to measure 0) only one summand, namely, that where $r_\ell \leq t$ and $r_{\ell+1} > t$. (If $r_1 > t$, then there is nothing to show.) By (1), this remaining summand equals $\xi^k_{s+t}(r_k,\ldots,r_1)$, so that $\xi^\odot$ is, indeed, a unit. From this it follows that $\|\xi_{t+dt} - \xi_t\| = \|(\xi_{dt} - \omega) \odot \xi_t\| \leq \|\xi_{dt} - \omega\|\,\|\xi_t\|$ so that $t \mapsto \xi_t$ is continuous, because

$$\|\xi_{dt} - \omega\| \;\leq\; \left\|\xi_{dt} - \xi^0_{dt}\right\| + \left\|\xi^0_{dt} - \omega\right\| \;\leq\; e^{dt\left(\|\beta\|+\frac{\|\zeta\|^2}{2}\right)} - 1 + \left\|e^{dt\beta} - \mathbf{1}\right\|.$$

For the generator we have to compute $\lim\limits_{t\to 0} \frac{\langle \xi_t, b\xi_t\rangle - b}{t}$. It is easy to see from (3) that the components ξ^n for $n \geq 2$ do not contribute. We have $\xi^0_t = \mathbf{1} + t\beta + O(t^2)$. Substituting this in ξ^1_t shows $\langle \xi_t, b\xi_t\rangle - b = t(\langle\zeta, b\zeta\rangle + b\beta + \beta^* b) + O(t^2)$. From this the form of the generator follows. ■

4 Remark. Actually, $t \mapsto \xi_t$ is an analytic function.

5 Remark. Christensen and Evans [CE79] have shown that any generator of a unital normal uniformly continuous CP-semigroup on a von Neumann algebra $\mathcal{B}$ has the form as in (2).

Notice that the operators $u_t = \xi_t\omega^*$ on $\mathbb{I}\Gamma(F)$ form a (right) cocycle with respect to the time shift endomorphism semigroup on $\mathcal{B}^a(\mathbb{I}\Gamma(F))$. If the semigroup T is unital (i.e. if $\langle \xi_t, \xi_t\rangle = \mathbf{1}$ or, in other words, if $\langle\zeta,\zeta\rangle + \frac{\beta+\beta^*}{2} = 0$), then the u_t are partial isometries and the homomorphisms $j_t\colon \mathcal{B} \to \mathcal{B}^a(\mathbb{I}\Gamma(F))$ defined by setting $j_t(b) = u^*_t b u_t$ form a weak

Markov flow in the sense of [BP94] (i.e. $j_t(\mathbf{1})j_{t+s}(b)j_t(\mathbf{1}) = j_t \circ T_s(b)$). See [BS00], where all these statements are shown for arbitrary product systems. In [BS00] the weak Markov flow j has been constructed with the help of a quantum stochastic calculus [Ske00]. In [GS99] it has been constructed (also with a calculus) for a special subclass of symmetric Fock modules. Here we see that it is possible to write the flow down directly without making use of a calculus.

Now we show the converse of Theorem 3.

6 Theorem. *Let $\xi^\odot$ be a unit for $\mathbb{\Gamma}^\odot(F)$ such that $t \mapsto \xi_t \in \mathbb{\Gamma}(F)$ is a continuous function. Then there exist unique $\zeta \in F$ and $\beta \in \mathcal{B}$ such that ξ_t is of the form defined by* (1).

Proof. (i) ξ_t is continuous, hence, so is $\xi^0_t = \langle \omega, \xi_t \rangle$. Moreover, $\xi^0_s \xi^0_t = \xi^0_s \odot \xi^0_t = \xi^0_{s+t}$ so that $\xi^0_t = e^{t\beta}$ is a uniformly continuous semigroup in $\mathcal{B}$ with a unique generator $\beta \in \mathcal{B}$.

(ii) As observed in a special case in [Lie00], any unit is determined by its components ξ^0 and ξ^1. This follows from the fact that

$$P^{01}_{t_n - t_{n-1}} \odot \ldots \odot P^{01}_{t_2 - t_1} \odot P^{01}_{t_1} \quad \longrightarrow \quad \mathsf{id}_{\mathbb{\Gamma}_t}$$

strongly, as limit over refinement of partitions $t = t_n \geq \ldots \geq t_1 \geq t_0 = 0$, where P^{01}_t denotes the projection onto the 0– and 1–particle sector of $\mathbb{\Gamma}_t$. So we are done, if we show that ξ^1 has the desired form.

(iii) For each $f \in L^2(\mathbb{R}_+)$ we define a mapping $f^* \otimes \mathsf{id} \colon L^2(\mathbb{R}_+, F) \to F$ of norm $\|f\|$, by setting

$$(f^* \otimes \mathsf{id})x \;=\; \int \overline{f(s)} x(s)\, ds$$

on step functions and then extending it to all of $L^2(\mathbb{R}_+, F)$. (To see this, expand the step function x in terms of an ONB for $L^2(\mathbb{R}_+)$ which contains the unit vector $\frac{f}{\|f\|}$.)

Notice that $((\mathfrak{s}_t f)^* \otimes \mathsf{id})\mathfrak{s}_t x = (f^* \otimes \mathsf{id})x$. Therefore, defining the (continuous) function $\lambda(t) = (I\!\!I^*_{[0,t]} \otimes \mathsf{id})\xi^1_t$ and taking into account $\xi^1_{s+t} = \xi^0_s \odot \xi^1_t + \xi^1_s \odot \xi^0_t = e^{s\beta}\xi^1_t + \mathfrak{s}_t \xi^1_s e^{t\beta}$ we find

$$\lambda(s+t) \;=\; e^{s\beta}\lambda(t) + \lambda(s)e^{t\beta}.$$

Differentiating

$$e^{T\beta} \int_T^{T+t} e^{-s\beta}\lambda(s)\, ds \;=\; \int_0^t e^{-s\beta}\lambda(s+T)\, ds \;=\; t\lambda(T) + \int_0^t e^{-s\beta}\lambda(s)e^{T\beta}\, ds$$

with respect to T, we see that λ is (arbitrarily) continuously differentiable.

(iv) For $T \geq 0$ denote by $p_T \in \mathcal{B}^a(L^2(\mathbb{R}_+, F))$ the projection onto $L^2([0,T], F)$. Then

$$\sum_{i=1}^{2^N} 1\!\!I_{[T\frac{i-1}{2^N}, T\frac{i}{2^N}]} \tfrac{2^N}{T} (1\!\!I^*_{[T\frac{i-1}{2^N}, T\frac{i}{2^N}]} \otimes \mathsf{id})x \;\longrightarrow\; p_T x$$

as $N \to \infty$ for all $x \in L^2(\mathbb{R}_+, F)$. ($\sum\limits_{i=1}^{2^N} 1\!\!I_{[T\frac{i-1}{2^N}, T\frac{i}{2^N}]} \frac{2^N}{T} 1\!\!I^*_{[T\frac{i-1}{2^N}, T\frac{i}{2^N}]}$ is an increasing net of projections on $L^2(\mathbb{R}_+)$ converging strongly to the projection onto $L^2([0,T])$.) Applying this to ξ^1_T and taking into account that $(1\!\!I^*_{[s,t]} \otimes \mathsf{id})\xi^1_T = e^{(T-t)\beta}\lambda(t-s)e^{s\beta}$ for $s \leq t \leq T$, we find

$$\begin{aligned}
\xi^1_T = \lim_{N\to\infty} \sum_{i=1}^{2^N} 1\!\!I_{[T\frac{i-1}{2^N}, T\frac{i}{2^N}]} \tfrac{2^N}{T} (1\!\!I^*_{[T\frac{i-1}{2^N}, T\frac{i}{2^N}]} \otimes \mathsf{id})\xi^1_T \\
= \lim_{N\to\infty} \sum_{i=1}^{2^N} 1\!\!I_{[T\frac{i-1}{2^N}, T\frac{i}{2^N}]} \tfrac{2^N}{T} e^{T\frac{2^N-i}{2^N}\beta} \lambda\big(\tfrac{T}{2^N}\big) e^{T\frac{i-1}{2^N}\beta} \\
= \lim_{N\to\infty} \sum_{i=1}^{2^N} 1\!\!I_{[T\frac{i-1}{2^N}, T\frac{i}{2^N}]} e^{T\frac{2^N-i}{2^N}\beta} \lambda'(0) e^{T\frac{i-1}{2^N}\beta}.
\end{aligned}$$

Putting $\zeta = \lambda'(0)$, the sum converges in L^∞–norm, hence, *a fortiori* in L^2–norm of E_T, to the function $t \mapsto e^{(T-t)\beta}\zeta e^{t\beta}$, which is stated for ξ^1_T in (1). ■

7 Remark. Fixing a semigroup ξ^0 and an element ζ in F, Equation (1) gives more general units. For that it is sufficient to observe that ξ^0 is bounded by Ce^{ct} for suitable constants C, c (so that ξ^n_t are summable). The following example shows that we may not hope to generalize Theorem 6 to units which are continuous in a weaker topology only.

8 Example. Let $F = \mathcal{B} = \mathbb{B}\big(\Gamma_-\big)$ where we set $\Gamma_- = \Gamma(L^2(\mathbb{R}_-))$. Then, $\mathrm{I\Gamma}(\mathcal{B}) = \mathcal{B} \otimes \mathrm{I\Gamma}(\mathbb{C}) = \mathcal{B} \otimes \Gamma(L^2(\mathbb{R}_+))$, because $\mathcal{B} \odot \mathcal{B} = \mathcal{B}$ and $\mathrm{I\Gamma}(\mathbb{C}) \cong \Gamma(L^2(\mathbb{R}_+))$. We consider elements $b \otimes f$ in $\mathrm{I\Gamma}(\mathcal{B})$ as operators $g \mapsto bg \otimes f$ in $\mathrm{I\Gamma}^s(\mathcal{B}) := \mathbb{B}(\Gamma_-, \Gamma_- \otimes \mathrm{I\Gamma}(\mathbb{C}))$. Clearly, $\mathrm{I\Gamma}^s(\mathcal{B})$ is the strong closure of $\mathrm{I\Gamma}(\mathcal{B})$. Similarly, let $\mathrm{I\Gamma}^s_t(\mathcal{B})$ denote the strong closures $\mathbb{B}(\Gamma_-, \Gamma_- \otimes \mathrm{I\Gamma}_t(\mathbb{C}))$ of $\mathrm{I\Gamma}_t(\mathcal{B})$. It is noteworthy that the $\mathrm{I\Gamma}^s_t$ form a product system even in the algebraic sense. To see this, observe that for $x \in \mathrm{I\Gamma}^s_s(\mathcal{B}), y \in \mathrm{I\Gamma}^s_t(\mathcal{B})$ we have $x \odot y = (x \otimes \mathsf{id})y \in \mathrm{I\Gamma}^s_{s+t}$. So let $z \in \mathrm{I\Gamma}^s_{s+t}$, and let u be a unitary in $\mathrm{I\Gamma}^s_s(\mathcal{B})$. Then $(u^* \otimes \mathsf{id})z$ is in $\mathrm{I\Gamma}^s_t(\mathcal{B})$ such that $z = (u \otimes \mathsf{id})(u^* \otimes \mathsf{id})z = u \odot ((u^* \otimes \mathsf{id})z)$.

Replacing everywhere the members $\mathrm{I\Gamma}_t(\mathbb{C})$ of the product system $\mathrm{I\Gamma}^\odot(\mathbb{C})$ by the members Γ_t of the isomorphic product system $\big(\Gamma_t\big)$ (and keeping the notation $\mathrm{I\Gamma}^s_t(\mathcal{B})$), we see that units correspond to cocycles of type (H) as introduced in [Lie00]. An example is the second quantized mirrored shift $\xi_t = \Gamma(\tau_t) \in \mathbb{B}\big(\Gamma(L^2(\mathbb{R}_-), \Gamma(L^2(\mathbb{R}_- \cup [0,t]))\big) = \mathrm{I\Gamma}^s_t(\mathcal{B})$, where

$$\tau_t f(s) = \begin{cases} f(s-t) & s < 0 \\ f(-s) & 0 \le s \le t. \end{cases}$$

A calculation shows that for an exponential vector $\psi(f)$ to a continuous function f

$$\zeta\psi(f) \;=\; \lim_{r\searrow 0} \xi^1_r(0)\psi(f) \;=\; \psi(f)f(0)$$

is only a distribution (a nonclosable operator, a boundary condition). Clearly, $\zeta \notin \mathcal{B}$.

9 Remark. For a step function $x\colon \mathbb{R}_+ \to F$ denote by $\psi(x) = \Delta \sum_{n=0}^{\infty} x^{\odot n}$ the *exponential vector* to x. One easily verifies that

$$\psi(x) \odot \psi(y) \;=\; \psi(\mathfrak{s}_t(x) + y)$$

for step functions $x \in L^2(\mathbb{R}_+, F)$ and $y \in L^2([0,t], F)$. In particular, if $x = \sum_{i=1}^{n} x_i \mathbb{I}_{[t_{i-1},t_i]}$ $(x_i \in F, 0 = t_0 < t_1 \ldots < t_n)$, then $\psi(x) = \psi(x_n \mathbb{I}_{[0,t_n-t_{n-1}]}) \odot \ldots \odot \psi(x_1 \mathbb{I}_{[0,t_1-t_0]})$. (Observe that we cannot guarantee existence of $\psi(x)$ for all $x \in L^2(\mathbb{R}_+, F)$. However, at least, if $t \mapsto \|x(t)\|$ makes sense as a square integrable function, then $\psi(x)$ and all inner products with other exponential vectors of this type exist.)

Choosing in (1) parameters $\xi^0 = \mathbf{1}$ and $\zeta \in F$, we find find $\xi_t = \psi(\zeta \mathbb{I}_{[0,t]})$. On the time ordered Fock space the units are precisely these *exponential units* times a scaling with the semigroup e^{ct} in $\mathbb{C}$. We see that in the time ordered Fock module we have a similar situation. Here we obtain all (continuous) units by rescaling with the semigroup ξ^0_t. However, due to non-commutativity the rescaling is more complicated. We cannot just multiply by the semigroup from the right (or from the left). Equation (1) tells us that we have to *distribute* the action of the semigroup over the envolved time intervals. Notice, that like in a type I Arveson system our time ordered Fock module is generated already by the exponential units.

Addendum: Meanwhile, we know from [Ske01b] that all spatial product systems generated (in the sense of [BS00]) by their continuous units are time ordered Fock modules. (A product system is spatial, if it contains a unital central unit $\omega^{\odot}$, i.e. the elements of $\mathcal{B}$ commute with all ω_t. Clearly, this central unit plays the role of the vacuum in the Fock module.) It is also clear that the space F is determined by the product system up to isomorphism and plays the role of Arveson's index in the scalar case. Finally, in [BBLS00] we show that product systems of von Neumann modules generated (strongly) by their continuous units are always spatial. This result is equivalent to [CE79].

For an integrated presentation of all these aspects (without [Ske01b]) we refer the reader to [Ske01a]. A revision of [Ske01a] (to be published) contains also [Ske01b].

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