Hilbert von Neumann Modules

versus

Concrete von Neumann Modules

Michael Skeide, May 2012

Abstract

Hilbert von Neumann modules and concrete von Neumann modules are the same thing.

Let G and H be Hilbert spaces. For $E \subset \mathcal{B}(G, H)$ denote by $[E]$ the strongly closed subspace of $\mathcal{B}(G, H)$ generated by E.

Bikram, Mukherjee, Srinivasan, and Sunder [BMSS12] say on Page 50, E is a ***von Neumann corner*** if $E = [E] \supset EE^*E$. It is ***nondegenerate*** if $\overline{\mathsf{span}}\, EG = H$, $\overline{\mathsf{span}}\, E^*H = G$. They say in [BMSS12, Definition 1.2(1)], a ***Hilbert von Neumann module*** over a von Neumann algebra $\mathcal{A}$ is a von Neumann corner E with a normal isomorphism π from $\mathcal{A}$ onto $[E^*E]$.

Let $\mathcal{B}$ be a von Neumann algebra acting nondegenerately on the Hilbert space G. Skeide [Ske06, Definition 2] says, E is a ***concrete von Neumann $\mathcal{B}$–module*** if E is strongly closed, if it is a (right) $\mathcal{B}$–submodule[1] of $\mathcal{B}(G, H)$ (that is, $E + E \subset E$ and $E\mathcal{B} \subset E$), if $E^*E \subset \mathcal{B}$ and if $\overline{\mathsf{span}}\, EG = H$.

Proposition. *Let G and H be Hilbert spaces. For the subset E of $\mathcal{B}(G, H)$ denote by $\mathcal{B}$ the strong closure in $\mathcal{B}(G)$ of the algebra generated by E^*E. Then the following are equivalent:*

1. *E is a nondegenerate von Neumann corner.*
2. *E is a Hilbert von Neumann module over $\mathcal{B}$ with $\pi = \mathsf{id}_{\mathcal{B}}$ satisfying $\overline{\mathsf{span}}\, EG = H$.*
3. *E is a concrete von Neumann $\mathcal{B}$–module.*

Moreover, if E is a Hilbert von Neumann module over $\mathcal{A}$ satisfying $\overline{\mathsf{span}}\, EG = H$, then E is a concrete von Neumann $\pi(\mathcal{A})$–module.

Proof. This is immediate from the definitions. ■

Observation. Similar statements, which we omit phrasing here, are true for ***Hilbert von Neumann bimodules*** ([BMSS12, Definition 1.2(3)]) and ***concrete von Neumann correspondences*** ([Ske06, Definition 3]).

MSC 2010: 46L07. Keywords: Von Neumann modules.

[1] In [Ske06] we omitted to repeat that a module is closed under addition.

Remark. ***Von Neumann modules*** have been introduced in Skeide [Ske00, Definition 4.4] as (pre-)Hilbert modules over von Neumann algebras for which the linking algebra is a von Neumann algebra (cf. [BMSS12, Proposition 1.1]). Immediately after ([Ske00, Proposition 4.5]), it is shown that this is equivalent, to that E is a concrete von Neumann module (of course, not calling it a concrete von Neumann module, as this definition is made not before [Ske06]). The main work is to transform the abstract (pre-)Hilbert module E over the (concrete!) von Neumann algebra $\mathcal{B} \subset \mathcal{B}(G)$ into a concrete operator module $E \subset \mathcal{B}(G, H)$. However, such a procedure is known, and it is known, too, that the result is unique up to suitable unitary equivalence; see (for instance) Rieffel [Rie74, Proposition 6.10] or Murphy [Mur97, Section 3]. All the work for deriving results (for instance, in Skeide [Ske00, Ske01]) is done with concrete von Neumann modules (respectively, Hilbert von Neumann modules). This includes, in particular, self-duality ([Ske00, Theorem 4.16], using cyclic decomposition [Ske00, Proposition 3.8], polar decomposition [Ske00, Proposition 2.10], and quasi orthonormal bases [Ske00, Theorem 4.11]; cf. [BMSS12, Proposition 1.9]). And it includes the tensor product of von Neumann correspondences ([Ske01, Equation (4.2.2) and Proposition 4.2.24], cf. [BMSS12, Section 3]). Just, a formal definition that frees us from the "burden" to transform an abstract (pre-)Hilbert module over a (concrete) von Neumann algebra into a concrete one, has not been given before [Ske06].

References

[BMSS12] P. Bikram, K. Mukherjee, R. Srinivasan, and V.S. Sunder, *Hilbert von Neumann Modules*, Commun. Stoch. Anal. **6** (2012), 49–64, (arXiv: 1102.4663).

[Mur97] G.J. Murphy, *Positive definite kernels and Hilbert C^*–modules*, Proc. Edinburgh Math. Soc. **40** (1997), 367–374.

[Rie74] M.A. Rieffel, *Morita equivalence for C^*–algebras and W^*–algebras*, J. Pure Appl. Algebra **5** (1974), 51–96.

[Ske00] M. Skeide, *Generalized matrix C^*–algebras and representations of Hilbert modules*, Mathematical Proceedings of the Royal Irish Academy **100A** (2000), 11–38, (Cottbus, Reihe Mathematik 1997/M-13).

[Ske01] ______, *Hilbert modules and applications in quantum probability*, Habilitationsschrift, Cottbus, 2001, Available at `http://web.unimol.it/skeide/`.

[Ske06] ______, *Commutants of von Neumann correspondences and duality of Eilenberg-Watts theorems by Rieffel and by Blecher*, Banach Center Publications **73** (2006), 391–408, (arXiv: math.OA/0502241).

Michael Skeide: *Dipartimento E.G.S.I., Università degli Studi del Molise, Via de Sanctis, 86100 Campobasso, Italy*, E-mail: `skeide@unimol.it`, Homepage: `http://web.unimol.it/skeide`